CHAPTER 3

EQUITABLE COLORING OF CORONA PRODUCT OF GRAPHS $K_n$, $C_n$ AND $P_n$

3.1 INTRODUCTION

In this chapter, let us consider the equitable coloring of some corona products $G \circ H$ of two graphs $G$ and $H$. In particular, deciding the colorability of $G \circ H$ is NP-complete even if $G$ is 4-regular and $H$ is $K_2$. Next, exact values or upper bounds on the equitable chromatic number $\chi_e(G \circ H)$, where $G$ is equitably 4-colorable graph and $H$ is a complete graph, a cycle, or a path are given. In this way equitable Coloring Conjecture for corona products of these classes of graphs are confirmed.

3.2 PRELIMINARIES

The notion of equitable colorability was introduced by Meyer [48]. However, an earlier work of Hajnal and Szemerédi [25] showed that a graph $G$ with degree $\Delta$ is equitably $k$-colorable, if $k \geq \Delta(G) + 1$. Mydlarz and Szemerediádi [49] found a polynomial-time algorithm for such a coloring. Recently, Kierstead and Kostochka [39] gave a short proof of this bound and presented another polynomial-time algorithm. In 1973, Meyer [48] formulated the following conjecture:

Conjecture 3.2.1 (Equitable Coloring Conjecture (ECC)). For any con-
nected graph $G$, other than complete graph or odd cycle, $\chi_e(G) \leq \Delta(G)$.

This conjecture has been verified for all graphs on six or fewer vertices. Lih and Wu [44] proved that the Equitable Coloring Conjecture is true for all bipartite graphs. Wang and Zhang [55] considered a broader class of graphs, namely $r$-partite graphs. They proved that Meyer’s conjecture is true for complete graphs from this class. Also, the conjecture was confirmed for outerplanar graphs [60] and planar graphs with maximum degree at least 13 [58].

A straightforward reduction from graph coloring to equitable coloring may be proven by adding sufficiently many isolated vertices to a graph, showing that it is NP-complete to test whether a graph has an equitable coloring with a given number of colors (greater than two). Bodlaender and Fomin [2] showed that equitable coloring can be solved to optimality in polynomial time for trees (previously known due to Chen and Lih [8]) and outerplanar graphs. A polynomial time algorithm is also known for equitable coloring of split graphs [6].

The corona of two graphs $G$ and $H$ is the graph $G \circ H$ formed from one copy of $G$ and $|V(G)|$ copies of $H$ where the $i$th vertex of $G$ is adjacent to every vertex in the $i$th copy of $H$. Such type of graph products was introduced by Frucht and Harary in 1970 [17]. For example, the corona $P_n \circ K_1$ is a comb graph. Another corona graph, namely $L(K_4) \circ K_2$, where $L(G)$ is a line graph of graph $G$, is depicted in Figure 3.1(b)

The remainder of this chapter is organized as follows. Let us start the next section from a theorem concerning the complexity of equitable coloring of coronas. It turns out that the problem is NP-hard even for the corona of line graphs of cubic graphs (i.e. 3-regular) and $K_2$. In Section 3.4 corona products of graphs $G$ with $\chi_e(G) \leq 4$ and cycles are considered. In Section 3.5 corona products of those graphs $G$ and paths are studied. In this way let us establish a new class of graphs that can be colored optimally in polynomial time and confirm the ECC conjecture. Finally, in Section 3.6 our results are summarized and also holds good for bipartite graphs.
3.3 NP-COMPLETENESS PROOF

The problem of equitable vertex coloring of corona graphs is NP-hard.

**Theorem 3.3.1.** The problem of deciding if \( \chi_\leq(G \circ K_2) \leq 3 \) is NP-complete even if \( G \) is the line graph of a cubic graph.

*Proof.* Our reduction comes from the problem of edge coloring. Let us know from [29] that the latter problem is NP-complete even if graph is cubic.

Let \( G_3 \) be a cubic graph such that \( G = L(G_3) \), where \( L(G_3) \) denotes the line graph of \( G_3 \). It is well known that the equitable chromatic index \( \chi'_\leq(G_3) = 3 \) or 4. This is so because if \( \chi'(G_3) = 3 \) then each color class constitutes a perfect matching, and if \( \chi'(G_3) = 4 \), then any 4-edge-coloring can be transformed to an equitable 4-coloring by using a path recoloring technique.

Let us assume that \( \chi'_\leq(G_3) = 3 \). It follows that \( \chi_\leq(G) = 3 \). To claim that the graph obtained from \( G \), namely the corona \( G \circ K_2 \), is also equitably 3-colorable. Indeed, given 3-coloring of \( G \), let us color vertices in copies of \( K_2 \) within two colors allowed. In such a coloring the color classes have the same cardinality, thus the coloring is equitable.

Of course, if the edges of \( G_3 \) cannot be colored with 3 colors, graph \( G \circ K_2 \) is not equitably 3-colorable, since its chromatic number is greater than 3. So, \( \chi_\leq(G \circ K_2) = 3 \) if and only if \( \chi'(G_3) = 3 \). An example of the reduction is given in Fig. 3.1.

Let us notice that \( G \circ K_2 \) can be obtained from \( G \) in polynomial time. As the verification of the equitability of a 3-coloring of \( G \circ K_2 \) is in NP, the thesis of the theorem follows.

Although the problem of equitable coloring is NP-hard for coronas, it is not hard to give an upper bound for \( \chi_\leq(G \circ H) \), where graph \( H \) is any graph on \( n \) vertices and graph \( G \) is properly colored with at most \( n + 1 \) colors. The bound is
an easy consequence of the following proposition and the fact that if $G$ and $G'$ are simple graphs on the same set of vertices and $E(G) \subseteq E(G')$, then $\chi_=(G) \leq \chi_=(G')$.

**Proposition 3.3.2.** If $G$ is a simple graph with $\chi(G) \leq n + 1$, then $\chi_=(G \circ K_n) = n + 1$.

**Proof.** Let $V(G) = \{u_1, u_2, \ldots, u_m\}$ and $V(K_n) = \{v_1, v_2, \ldots, v_n\}$. By the definition of corona graph, each vertex of $G$ is adjacent to every vertex of a copy of $K_n$. Since $\chi(G) \leq n + 1$, color vertices $u_1, u_2, \ldots, u_m$ of $G$ with $n + 1$ colors. Graph $G \circ K_n$ contains $m$ cliques of order $n + 1$, $K_{n+1}^i, i = 1, \ldots, m$, so it is impossible to use fewer colors than $n + 1$.

Now, in every clique $K_{n+1}^i, i = 1, \ldots, m$, one of the vertices is colored. Extending this coloring onto other vertices of the clique. In every case, use $n + 1$ different colors. In the whole graph $G \circ K_n$ every color $j, j = 1, \ldots, n + 1$ is used exactly $m$ times. Thus our coloring is equitable.

The degree of the corona $\Delta(G \circ K_n)$ is equal to $\Delta(G) + n$. This means that ECC is true for corona graph $G \circ K_n$, provided that $G$ contains at least one edge.
3.4 EQUITABLE COLORING OF CORONA PRODUCT OF CYCLE

The first class of graphs that is considered in full detail are cycles. Let us begin with the following

**Theorem 3.4.1.** Let $G$ be an equitably 3-colorable graph on $n \geq 2$ vertices and let $k \geq 2$ be a positive integer. If $k = 2$ or $3|n$, then $\chi(G \circ C_{2k}) = 3$.

**Proof.** Suppose $G$ has been equitably colored with 3 colors so that the cardinalities of color classes are arranged in non-increasing way. Let us order the vertices of $G$: $v_1, v_2, \ldots, v_n$ in such a way that vertex $v_i$ is colored with color $i \mod 3$, and let us assume color 3 instead of 0. In the following consider the two cases.

**Case 1:** $k = 2$.

Let us color the consecutive copies of $C_4$. In each case use two available colors alternately (one color is forbidden). After the $i$th step (after coloring the $i$th copy of $C_4$) one of the following situations holds:

1. $i \mod 3 = 1$
   
   The color 1 appears $5\lfloor i/3 \rfloor + 1$ times, the color 2 appears $5\lfloor i/3 \rfloor + 2$ times and color 3 appears $5\lfloor i/3 \rfloor + 2$ times.

2. $i \mod 3 = 2$
   
   The color 1 appears $5\lfloor i/3 \rfloor + 3$ times, the color 2 appears $5\lfloor i/3 \rfloor + 3$ times and color 3 appears $5\lfloor i/3 \rfloor + 4$ times.

3. $i \mod 3 = 0$
   
   The color 1 appears $5\lfloor i/3 \rfloor + 5$ times, the color 2 appears $5\lfloor i/3 \rfloor + 5$ times and color 3 appears $5\lfloor i/3 \rfloor + 5$ times.

In every case the coloring is equitable.
Case 2: $n$ is divisible by three.

Let us color the consecutive copies of $C_{2k}$. In each case, use two available colors alternately (one color is forbidden). In the coloring of every three copies of $C_{2k}$ with appropriate vertices of graph $G$, use each of three colors exactly $2k + 1$ times. This means that the coloring of the whole $G \circ C_{2k}$ is equitable.

To complete the proof note that $\chi_=(G \circ C_{2k})$ cannot be less than 3, because the corona graph contains a triangle. \hfill \Box

It turns out that the above theorem includes all cases in which three colors are enough. Now, let us assume that $G$ is equitably 4-colorable.

**Theorem 3.4.2.** Let $G$ be an equitably 4-colorable graph on $n \geq 2$ vertices, and let $k \geq 3$ be a positive integer. If $3 \nmid n$, then $\chi_=(G \circ C_{2k}) = 4$.

**Proof.** Suppose graph $G$ has been colored equitably with 4 colors and the cardinalities of color classes are arranged in non-increasing way. Let us order the vertices of $G$: $v_1, v_2, \ldots, v_n$ in such a way that vertex $v_i$ is colored with color $i \mod 4$, and let us use color 4 instead of 0. Let us consider two cases.

Case 1: $n$ is even.

Let us color the consecutive copies of $C_{2k}$ in the following way. Every time two colors are used. If vertex in $G$ is colored with 1 or 4, then color appropriate copy of $C_{2k}$ with colors 2 and 3, alternately. In other cases color a copy of $C_{2k}$ with color 1 and 4. Since $n$ is even, there are only two cases.

1. $n \mod 4 = 2$

   Each color is used the same number of times, namely $\lfloor n/4 \rfloor (2k + 1)$ times.

2. $n \mod 4 = 0$

   Each of colors 1 and 2 is used $\lfloor n/4 \rfloor (2k + 1) + k + 1$ times, and each of colors 3 and 4 is used $\lfloor n/4 \rfloor (2k + 1) + k$ times.
This means that our coloring is equitable.

Case 2: \( n \) is odd.

1. \( n \mod 4 = 1 \)

Let us start with graph \( G \) on 5 vertices. Let us color the vertices in the consecutive copies of \( C_{2k} \) in corona \( G \circ C_{2k} \) as follows:

- color the 1st and 5th copy using \( k \) times color 2, \( \lceil k/2 \rceil \) times color 3, \( \lfloor k/2 \rfloor \) times color 4
- color the 2nd copy using \( \lfloor k/2 \rfloor \) times color 1, \( k \) times color 3 and \( \lceil k/2 \rceil \) times color 4
- color the 3rd copy using \( k \) times color 1 and \( k \) times color 4
- color the 4th copy using \( k \) times color 1, \( \lceil k/2 \rceil \) times color 2 and \( \lfloor k/2 \rfloor \) times color 3.

In such a coloring of \( G \circ C_{2k} \) one must have used:

- \( 2 + 2k + \lceil k/2 \rceil \) times color 1
- \( 1 + 2k + \lceil k/2 \rceil \) times color 2
- \( 1 + k + 2\lfloor k/2 \rfloor + \lceil k/2 \rceil = 1 + 2k + \lceil k/2 \rceil \) times color 3
- \( 1 + k + 2\lfloor k/2 \rfloor + \lceil k/2 \rceil = 1 + 2k + \lceil k/2 \rceil \) times color 4

The difference between the number of appearances of each color does not exceed 1, so our coloring is equitable.

In cases when \( n \geq 13 \) (recall that the cases when \( n \) is divisible by 3 are excluded) color first \( 4(\lfloor (n-1)/4 \rfloor - 1) \) copies of \( C_{2k} \) using in the \( i \)th copy \( k \) times color \( (i \mod 4 + 1) \mod 4 \) and \( k \) times color \( (i \mod 4 + 2) \mod 4 \). In this part each color is used the same number of times. Finally, the last five copies can be colored in the way given above. Graph \( G \circ C_{2k} \) is colored equitably.
Let us start with graph $G$ on 7 vertices. Let us color the vertices in the consecutive copies of $C_{2k}$ in graph $G \circ C_{2k}$ as follows:

- the $i$th copy, $1 \leq i \leq 4$, is colored using $k$ times the color $(i \mod 4 + 1) \mod 4$ and $k$ times the color $(i \mod 4 + 2) \mod 4$
- the 5th copy is colored using $k$ times color 2, $\lfloor k/2 \rfloor$ times color 3, $\lceil k/2 \rceil$ times color 4
- the 6th copy is colored using $k$ times color 1 and $k$ times color 3
- the 7th copy is colored using $\lceil k/2 \rceil$ times color 1, $\lfloor k/2 \rfloor$ times color 2 and $k$ times color 4

In such a coloring of $G \circ C_{2k}$ each of colors: 1, 2, and 3 is used $2+3k+\lceil k/2 \rceil$ times while color 4 is used $1+3k+\lfloor k/2 \rfloor$ times. The coloring is equitable.

In cases when $n \geq 11$ color first $4(\lfloor (n-3)/4 \rfloor - 1)$ copies of $C_{2k}$ using in the $i$th copy $k$ times color $(i \mod 4 + 1) \mod 4$ and $k$ times color $(i \mod 4 + 2) \mod 4$. In this part each color is used the same number of times. Finally, coloring the remaining seven copies in the way given above. Graph $G \circ C_{2k}$ is colored equitably.

Now, let us prove that less than 4 colors are not possible. Since graph $G \circ C_{2k}$ includes cycle $C_3$, so $\chi=(G \circ C_{2k}) \geq 3$. Let us consider whether our graph can be colored with three colors. Let us assume that it can be. Then the size of the smallest color class is equal to $\lfloor (2k+1)n/3 \rfloor$ while the size of the largest independence set is equal to $\lceil n/3 \rceil + (n - \lceil n/3 \rceil)k$. Since $n$ is not divisible by three, then

$$\lfloor (2k+1)n/3 \rfloor > \lceil n/3 \rceil + (n - \lceil n/3 \rceil)k.$$  

Consequently, equitable 3-coloring does not exist, and the thesis of the theorem follows. $\square$

Now, consider cycles with odd number of vertices. It is easy to see that the
corona of graph \( G \) and odd cycle needs at least four colors. It turns out that this number suffices for graphs under consideration.

**Theorem 3.4.3.** Let \( G \) be equitably 4-colorable graph on \( n \geq 2 \) vertices, and let \( k \) be a positive integer. Then \( \chi = (G \circ C_{2k+1}) = 4 \).

**Proof.** Suppose \( G \) has been equitably colored with 4 colors so that the cardinalities of color classes are arranged in non increasing way. Let us order the vertices of \( G: v_1, v_2, \ldots, v_n \) in such a way that vertex \( v_i \) is colored with color \( i \mod 4 \), and, as previously, let us use color 4 instead of 0. The following two cases are consider.

Case 1: \( n \) is even.

1. \( n \mod 4 = 0 \)

Let us start with graph \( G \) on 4 vertices. Color the vertices in the consecutive copies of \( C_{2k+1} \) in corona \( G \circ C_{2k+1} \) as follows:

- the 1st copy is colored using \( k \) times color 2, \( k \) times color 3, 1 time color 4
- the 2nd copy is colored using \( k \) times color 1, \( k \) times color 4, 1 time color 3
- the 3rd copy is colored using \( k \) times color 1, \( k \) times color 4, 1 time color 2
- the 4th copy is colored using \( k \) times color 2, \( k \) times color 3, 1 time color 1

In such a coloring of \( G \circ C_{2k+1} \) every color is used exactly \( 2k + 2 \) times. Hence, the coloring is equitable.

In cases when \( n \geq 8 \) coloring every four copies in the way given above. Graph \( G \circ C_{2k+1} \) is colored equitably.
2. \( n \mod 4 = 2 \)

Let us start with graph \( G \) on 2 vertices. Coloring the vertices in the consecutive copies of \( C_{2k+1} \) in corona \( G \circ C_{2k+1} \) as follows:

- the 1st copy is colored using \( k \) times color 2, \( k \) times color 3, 1 time color 4
- the 2nd copy is colored using \( k \) times color 1, \( k \) times color 4, 1 time color 3

In such a coloring of \( G \circ C_{2k+1} \) every colored is used \( k + 1 \) times.

In cases when \( n \geq 6 \) coloring the vertices of the first \( 4\lfloor (n - 2)/4 \rfloor \) copies of \( C_{2k+1} \) in the same way as one have colored the corresponding vertices in Case 1(i). In this coloring, every color appears the same number of times. Finally, coloring the last two copies in the way given above. Graph \( G \circ C_{2k+1} \) is colored equitably.

Case 2: \( n \) is odd.

1. \( n \mod 4 = 1 \)

Let us start with graph \( G \) on 5 vertices. Coloring the vertices in the consecutive copies of \( C_{2k+1} \) in graph \( G \circ C_{2k+1} \) as follows:

- the 1st and 5th copy are colored using \( k \) times color 2, \( \lfloor k/2 \rfloor \) times color 3, \( \lceil k/2 \rceil + 1 \) times color 4
- the 2nd copy is colored using \( \lceil k/2 \rceil + 1 \) times color 1, \( k \) times color 3 and \( \lfloor k/2 \rfloor \) times color 4
- the 3rd copy is colored using \( k \) times color 1, \( k \) times color 4 and 1 time color 2
- the 4th copy is colored using \( k \) times color 1, \( \lfloor k/2 \rfloor \) times color 2 and \( \lceil k/2 \rceil + 1 \) times color 3

In such a coloring of \( G \circ C_{2k+1} \) one must have used:
• $3 + 2k + \lfloor k/2 \rfloor$ times color 1
• $2 + 2k + \lfloor k/2 \rfloor$ times color 2
• $2 + k + 2\lceil k/2 \rceil + \lfloor k/2 \rfloor$ times color 3
• $3 + k + 2\lfloor k/2 \rfloor + \lceil k/2 \rceil$ times color 4

The difference between the number of appearance of each pair of colors does not exceed 1, so our coloring is equitable.

In cases when $n \geq 9$ coloring the vertices of the first $4((n-1)/4-1)$ copies of $C_{2k+1}$ in the same way as one have colored the corresponding vertices in Case 1(i). In this coloring, every color appears the same number of times. Finally, coloring the remaining five copies in the way given above. Graph $G \circ C_{2k+1}$ is colored equitably.

2. $n \mod 4 = 3$

Let us start with a graph $G$ on 3 vertices. Coloring the vertices in the consecutive copies of $C_{2k+1}$ in graph $G \circ C_{2k+1}$ as follows:

• the 1st copy is colored using $k$ times color 2, $\lfloor k/2 \rfloor + 1$ times color 3, $\lceil k/2 \rceil$ times color 4
• the 2nd copy is colored using $k$ times color 1, $k$ times color 3 and 1 time color 4
• the 3rd copy is colored using $\lceil k/2 \rceil$ times color 1, $\lfloor k/2 \rfloor + 1$ times color 2 and $k$ times color 4

In such a coloring of $G \circ C_{2k+1}$ one must have used:

• $1 + k + \lfloor k/2 \rfloor$ times color 1
• $2 + k + \lfloor k/2 \rfloor$ times color 2
• $2 + k + \lfloor k/2 \rfloor$ times color 3
• $1 + k + \lfloor k/2 \rfloor$ times color 4

The difference between the numbers of appearance of each pair of colors does not exceed 1, so our coloring is equitable.
In cases when \( n \geq 7 \) coloring the vertices of the first \( 4\lfloor (n - 3)/4 \rfloor \) copies of \( C_{2k+1} \) in the same way as one have colored the corresponding vertices in Case1(i). In this coloring, every color appears the same number of times. Finally, coloring the remaining three copies in the way given above. Hence, corona \( G \circ C_{2k+1} \) is colored equitably.

For any \( k \), \( C_{2k+1} \) is colored in the proper way with minimum three colors. Since each vertex of \( G \) is adjacent to every vertex of the corresponding copy of \( C_{2k+1} \), this vertex needs an additional color. Hence, the minimum number of colors used for equitable coloring of \( G \circ C_{2k+1} \) is 4.

To complete our considerations, let us consider our results for corona of one vertex graph \( G \) and cycles. Since \( K_1 \circ C_m = W_{m+1} \) for \( m \geq 3 \), the color assigned to the universal vertex (the vertex of \( K_1 \)) cannot be used more times than one and any other color can be used at most twice. Thus, for \( m \geq 3 \) we have:

\[
\chi(K_1 \circ C_m) = \begin{cases} 
3 & \text{for } m = 4, \\
\lceil \frac{m}{2} \rceil + 1 & \text{for } m > 4
\end{cases}
\]

Note that \( \Delta(G \circ C_m) = \Delta(G) + m, \ m \geq 3 \). Hence, the above results confirm Equitable Coloring Conjecture for such graphs.

### 3.5 EQUITABLE COLORING OF CORONA PRODUCT OF PATH

First, let us start with simple results for corona of one vertex graph \( G \) and paths. Since \( K_1 \circ P_m, \ m \geq 3, \) is a fan with one universal vertex, using similar approach as above, one must have:

\[
\chi(K_1 \circ P_m) = \lceil \frac{m}{2} \rceil + 1.
\]
Similarly, as it was in the case of even cycles, each corona \( G \circ P_m, m \geq 2 \) needs at least three colors for equitable coloring. In the way similar to that used in the proof of Theorem 3.4.1 let us prove

**Proposition 3.5.1.** Let \( G \) be an equitably 3-colorable graph on \( n \geq 2 \) vertices and let \( m \geq 2 \) be a positive integer. If \( 3 \| n \) or \( m = 4 \), then \( \chi_e(G \circ P_m) = 3 \).

These are not the only coronas of paths that need 3 colors for equitable coloring.

**Theorem 3.5.2.** Let \( G \) be an equitably 3-colorable graph on \( n \geq 2 \) vertices and let \( m \geq 2 \) be a positive integer. If \( m = 2, 3, 5 \), then \( \chi_e(G \circ P_m) = 3 \).

**Proof.** Suppose \( G \) has been equitably colored with 3 colors so that the cardinalities of color classes are arranged in non increasing way. Let us order the vertices of \( G: v_1, v_2, \ldots, v_n \) in such a way that vertex \( v_i \) is colored with color \( i \mod 3 \), and let us assume color 3 instead of 0. Consider the following three cases.

Case 1: \( m = 2 \).

Let us color the consecutive copies of \( P_2 \). In each case, use two available colors, alternately (one color is forbidden). After coloring the \( i \)th copy of \( P_2 \) one of the following situation holds:

1. \( i \mod 3 = 1 \)
   
   Color 1 appears \( 3\lfloor i/3 \rfloor + 1 \) times, color 2 appears \( 3\lfloor i/3 \rfloor + 1 \) times and color 3 appears \( 3\lfloor i/3 \rfloor + 1 \) times.

2. \( i \mod 3 = 2 \)
   
   Color 1 appears \( 3\lfloor i/3 \rfloor + 2 \) times, color 2 appears \( 3\lfloor i/3 \rfloor + 2 \) times and color 3 appears \( 3\lfloor i/3 \rfloor + 2 \) times.

3. \( i \mod 3 = 0 \)
   
   Color 1 appears \( 3\lfloor i/3 \rfloor \) times, color 2 appears \( 3\lfloor i/3 \rfloor \) times and color 3 appears \( 3\lfloor i/3 \rfloor \) times.
Case 2: \( m = 3 \).

Let us color the consecutive copies of \( P_3 \). In each case, use two available colors, alternately (one color is forbidden). After the \( i \)th step one of the following situations holds:

1. \( i \mod 3 = 1 \)
   Color 1 appears \( 4\lfloor i/3 \rfloor + 1 \) times, color 2 appears \( 4\lfloor i/3 \rfloor + 2 \) times and color 3 appears \( 4\lfloor i/3 \rfloor + 1 \) times.

2. \( i \mod 3 = 2 \)
   Color 1 appears \( 4\lfloor i/3 \rfloor + 2 \) times, color 2 appears \( 4\lfloor i/3 \rfloor + 3 \) times and color 3 appears \( 4\lfloor i/3 \rfloor + 3 \) times.

3. \( i \mod 3 = 0 \)
   Color 1 appears \( 4\lfloor i/3 \rfloor \) times, color 2 appears \( 4\lfloor i/3 \rfloor \) times and color 3 appears \( 4\lfloor i/3 \rfloor \) times.

Case 3: \( m = 5 \).

Let us color the consecutive copies of \( P_5 \). In each case, use two available colors, alternately (one color is forbidden). Depending on the cardinality of \( V(G) \) one can have the following situations (the case of \( n \mod 3 = 0 \) was considered in Theorem 3.5.1 and can be omitted now):

1. \( n \mod 3 = 1 \)
   Let us start from the case when \( n = 4 \). Coloring the \( i \)th, \( i = 1, 2, 3, 4 \), copy of \( P_5 \) using \( k_1 \) times color \( ((i \mod 3) + 1) \mod 3 \) and \( k_2 \) times color \( ((i \mod 3) + 2) \mod 3 \), where \( k_1 = 3 \) and \( k_2 = 2 \), if \( i = 1 \) and \( k_1 = 2 \) and \( k_2 = 3 \), otherwise. Every color is used the same number of times. For \( n \geq 7 \), first let us color \( 3\lfloor n/3 \rfloor \) copies of \( P_5 \) using the same number of each color - similarly as in the case when \( 3|n \) (see. Proposition 3.5.1). The last four copies are colored in the way described above.
2. $n \, \text{mod} \, 3 = 2$

Let us start from the case when $n = 2$. Coloring the $i$th, $i = 1, 2$, copy of $P_5$ using $k_1$ times color $((i \, \text{mod} \, 3) + 1) \, \text{mod} \, 3$ and $k_2$ times color $((i \, \text{mod} \, 3) + 2) \, \text{mod} \, 3$, where $k_1 = 3$, $k_2 = 2$, if $i = 1$ and $k_1 = 2$, $k_2 = 3$, if $i = 2$. Every color is used four times. For $n \geq 5$, first let us color $3 \lfloor n/3 \rfloor$ copies of $P_5$ using the same number of each color - similarly as in the case when $3 \vert n$ (see. Proposition 3.5.1). The last two copies are colored in the way described above.

In every case the coloring is equitable. □

In the remaining cases of $G \circ P_m$ one have to use four colors. In the way similar to that given in the proof of Theorems 3.4.2. Now let us prove the following corollary.

**Corollary 3.5.3.** Let $G$ be an equitably 4-colorable graph on $n \geq 2$ vertices and let $m \geq 6$ be a positive integer. If $3 \nmid n$ then

$$\chi_\ast (G \circ P_m) \leq 4.$$ 

In fact, this number is equal to 4.

**Theorem 3.5.4.** Let $G$ be an equitably 4-colorable graph on $n \geq 2$ vertices and let $m \geq 6$ be positive integer. If $3 \nmid n$ then

$$\chi_\ast (G \circ P_m) = 4.$$ 

**Proof.** It suffices to show that $\chi_\ast (G \circ P_m) \neq 3$. Suppose to the contrary that our graph can be colored with 3 colors. Then the size of the smallest color class is equal to $\lfloor (m + 1)n/3 \rfloor$ while the size of the largest independence set is equal to $\lceil n/3 \rceil + (n - \lceil n/3 \rceil) \lfloor m/2 \rfloor$. Since $n$ is not divisible by three and $m \geq 6$, then

$$\lfloor (m + 1)n/3 \rfloor > \lceil n/3 \rceil + (n - \lceil n/3 \rceil) \lfloor m/2 \rfloor.$$ 

An equitable 3-coloring does not exist. This contradiction concludes the proof. □
Let us notice that $\Delta(G \circ P_m) = \Delta(G) + m, n \geq 2$. So, again the ECC conjecture holds for such graph products.

### 3.6 FINAL REMARKS

In the chapter, NP-completeness proof for equitable coloring of corona graphs are given also a polynomal time solution for equitable coloring is established for some special cases of such products. Of course, the complexity of equitable coloring of $G \circ H$ depends on the complexity of equitable 3- or 4-coloring of graph $G$, which is generally NP-hard. However, the following graphs: broken spoke wheels, reels and some graph products admit equitable 3-coloring in polynomial time, and so do the corresponding coronas [19, 20, 46]. In addition to this Equitable Coloring Conjecture for these graphs have been confirmed. Our results are summarized in Table 3.1.

<table>
<thead>
<tr>
<th>$G$</th>
<th>$H$</th>
<th>even cycles $C_{2k}$</th>
<th>odd cycles</th>
<th>paths $P_k$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>$k = 2$</td>
<td>$k \geq 3$</td>
<td>$2 \leq k \leq 5$</td>
</tr>
<tr>
<td>equitably 3-colorable graph $G$ on $n \geq 2$ vertices</td>
<td>$3 \nmid n$</td>
<td>3</td>
<td>3</td>
<td>3</td>
</tr>
<tr>
<td>equitably 4-colorable graph $G$ on $n \geq 2$ vertices</td>
<td>$3 \nmid n$</td>
<td>3</td>
<td>4</td>
<td>4</td>
</tr>
</tbody>
</table>

Table 3.1: Values of the equitable chromatic number of coronas $G \circ H$.

Finally, note that some of our methods can be extended to equitable coloring of other classes of graphs. For example the algorithm given in the proof of Theorem 3.4.1 is also good for coloring bipartite graphs.
Theorem 3.6.1. Let $G$ be an equitably 3-colorable graph on $n \geq 2$ vertices and let $H$ be a bipartite graph. If $3|n$ then $\chi_e(G \circ H) \leq 3$. Moreover, if $H$ has at least one edge then $\chi_e(G \circ H) = 3$.

Proof. Let us color the graph $G$ equitably with 3 colors. After that let us order vertices of $G$: $v_1, v_2, \ldots, v_n$ in such a way that vertex $v_i$ is colored with color $i \mod 3$ and, as previously, use color 3 instead of color 0.

Let us assume that our bipartite graphs $H = H(V_1, V_2)$. Coloring the $i$th, $i = 1, \ldots, n$, copy of bipartite graph $H$ using $|V_1(H)|$ times color $((i \mod 3) + 1) \mod 3$ and $|V_2(H)|$ times color $((i \mod 3) + 2) \mod 3$.

In the coloring each of three colors is used exactly $1 + |V_1(H)| + |V_2(H)|$ times. This means that our coloring is equitable. □