CHAPTER - VI

FIXED POINT THEOREMS IN
RANDOM NORMED SPACES

In this chapter, first we prove a common fixed point theorem for three maps $A$, $S$ and $T$ in random normed spaces (RN-spaces) using the notion of compatibility of mappings $(A, S)$ and $(A, T)$. We further establish a theorem for four maps under a different contraction condition which extend a result of K.P. Chamola in RN-spaces. This chapter is divided into following two sections:

1. Preliminaries

2. Common fixed point theorems for three and four maps.
6.1 Preliminaries.

The concept of a random normed space (RN-space) was introduced by Serstnev [120] as a probabilistic generalization of the concept of normed linear spaces. The study of fixed point theorems in RN-spaces was initiated by Bocsan [16] and further results have been established in [20], [36], [67] etc. In the present chapter we prove some common fixed point theorems for three and four mappings in random normed spaces. First we prove a common fixed point theorem for three maps A, S and T in RN-spaces by using the notion of compatibility of pairs of mappings (A, S) and (A, T). We further establish a theorem for four maps under a different contraction condition which extend and generalize a result of K.P. Chamola [20] in RN-spaces.

Definition 6.1.1 [20]. Let $D$ denote the set of all distribution functions. A random normed space is a triplet $(X, F, T)$ consisting of a real or complex linear space $X$, the mapping $F : X \to D$ and a continuous $T$-norm $t$ [131], satisfying the following conditions in which $F_u$ denotes the distribution function $F(u)$:

(RN-1) $F_u(0) = 0$ for all $u \in X$;
\( (R\text{-}N-2) \quad F_u(x) = H(x) \text{ iff } u = 0 \text{ where } \\
H(x) = \begin{cases} 
1, & x > 0 \\
0, & x \leq 0.
\end{cases} 
\)

\( (R\text{-}N-3) \quad F_{qu}(x) = F_u\left( \frac{x}{q} \right) \text{ where } q \text{ is a non-zero scalar and } \\
x \in \mathbb{R}. 
\)

\( (R\text{-}N-4) \quad F_{u+v}(x+y) \geq T\{F_u(x), F_v(y)\} \text{ for all } u, v \text{ in } X \text{ and } \\
x, y \geq 0. 
\)

\( (R\text{-}N-5) \quad T(x, y) \cdot \max \{x+y-1, 0\} \text{ for all } x, y \in [0, 1]. 
\)

\( F_u(x) \) is interpreted as the probability of the norm of \( u \) being less than \( x. \)

For detailed topological preliminaries of RN-spaces, we refer to Bocsan [16].

**Definition 6.1.2 [29].** Let \( \phi : \mathbb{R}^+ \star \mathbb{R}^+ \) be a strictly increasing function such that \( \phi(0) = 0 \) and \( \lim_{t \to \infty} \phi(t) = +\infty. \)

Define a function \( \star : \mathbb{R}^+ \star \mathbb{R}^+ \) by
\[ t = 0 \]
\[ \inf\{s > 0: \phi(s) > t\}, \quad t > 0. \]

It is easy to prove that \( \cdot \) is a continuous and non-decreasing function.

**Definition 6.1.3 [29]**. A function \( \phi: \mathbb{R}^+ \times \mathbb{R}^+ \) is said to satisfy the condition (\( \cdot \)) if it is strictly increasing and left-continuous function such that \( \phi(0) = 0 \),

\[
\lim_{t \to \infty} \phi(t) = \infty \quad \text{and} \quad \sum_{n=0}^{\infty} \phi^n(t) < \infty \quad \text{for all} \ t > 0.
\]

**Lemma 6.1.1 [29]**. Let \( \phi: \mathbb{R}^+ \times \mathbb{R}^+ \) satisfy condition (\( \cdot \)) and let \( \cdot \) be defined by (A). Then we have the following:

(i) \( \phi(t) \leq t \) for all \( t > 0 \)

(ii) \( \phi(\psi(t)) \leq t \) and \( \psi(\phi(t)) = t \) for all \( t \cdot 0 \),

(iii) \( \psi(t) \geq t \) for all \( t \cdot 0 \),

(iv) \( \lim_{n \to \infty} \psi^n(t) = +\infty \) for all \( t > 0 \).

Now we give the following definitions of compatible maps in \( \mathbb{R}^N \)-spaces.
Definition 6.1.4. Two self-maps $S$ and $T$ on a RN-space $(X, F)$ are said to be compatible if
\[
\lim_{n \to \infty} F_{STx_n} - TSx_n(t) = 1 \text{ for all } t > 0,
\]
whenever sequence $\{x_n\}$ is a sequence such that $Sx_n, Tx_n \to z$ for some $z$ in $X$.

For proving our result we shall need the following Lemma which extend and modify the definition of a Cauchy sequence in RN-space.

Proof of this Lemma is implicated in ([29], Lemma 3).

Lemma 6.1.2. Let $(X, F, T)$ be an RN-space, where $T$ is a continuous $t$-norm. If a sequence $\{y_n\}$ in $X$ satisfies the following condition:

\[
F_{y_n - y_{n+1}}(t) \geq F_{y_0 - y_1}(\psi^n(t)) \text{ for } n \in \mathbb{N}, \text{ and } t > 0,
\]

where $\phi$ is a function satisfying the condition $(\bullet)$ and $\bullet$ is defined by (A). Then the sequence $\{y_n\}$ is a Cauchy sequence in $X$. 
6.2 Common fixed point theorems for three and four maps.

**Theorem 6.2.1.** Let $A$, $S$, and $T$ be self-maps of a complete random normed space $(X, F, T)$ where $T$ is a continuous t-norm with $T = \min$ satisfying the following conditions:

(6.2.1) $A(X) \subset S(X) \cdot T(X)$,
(6.2.2) $S$ and $T$ are continuous,
(6.2.3) $(A, S)$ and $(A, T)$ are compatible pair of maps,
(6.2.4) $F_{Au - Av}(\psi(t)) \geq \min\{F_{Tu - Sv}(t), F_{Au - Tu}(t), F_{Av - Sv}(t)\}$ for all $t \in [0, \cdot)$, where $\phi: [0, \cdot) \times [0, \cdot)$ satisfies condition $(\cdot \cdot)$.

Then $A$, $S$, $T$ have a unique common fixed point in $X$.

**Proof.** Let $x_0$ be an arbitrary point in $X$. Since $A(X) \subset S(T) \cdot T(X)$, construct a sequence $\{x_n\}$ in $X$ such that

$Ax_{2n - 2} = Sx_{2n - 1}$ and $Ax_{2n - 1} = Tx_{2n}$ for all $n \in \mathbb{N}$.

First, we prove that the sequence $\{Ax_n\}$ is a Cauchy sequence.

Let $x_1, x_2 \in X$, then by Lemma 6.1.1 and (6.2.4),

$$F_{Ax_{2n} - Ax_{2n-1}}(t) \geq F_{Ax_{2n} - Ax_{2n-1}}(\psi(t))$$
\[ \geq F_{T_2 - S_1}(\psi(t)), \]
\[ F_{A_1 - T_2}(\psi(t)), F_{A_2 - T_2}(\psi(t)) \]
\[ = \min\{F_{A_1 - A_0}(\psi(t)), F_{A_1 - A_0}(\psi(t)) \} \]
\[ \geq \min\{F_{A_1 - A_0}(\psi(t)), F_{A_2 - A_1}(\psi(t)) \} \]

where \( \cdot (t) \) is defined by (A).

Using above argument repeatedly, we have

\[ F_{A_2 - A_1}(t) \geq \min\{F_{A_1 - A_0}(\psi(t)), F_{A_1 - A_0}(\psi^2(t)), \]
\[ F_{A_2 - A_1}(\psi^2(t)) \} \]
\[ = \min\{F_{A_1 - A_0}(\psi(t)), F_{A_2 - A_1}(\psi^2(t)) \} \]
\[ \cdot \ldots \]
\[ \geq \min\{F_{A_1 - A_0}(\psi(t)), F_{A_2 - A_1}(\psi^n(t)) \}. \]

Letting \( n \cdot \cdot \cdot \), we have
\[ F_{Ax_2 - Ax_1}(t) \geq F_{Ax_1 - Ax_0}(\psi(t)), \quad t > 0. \]

Taking this procedure repeatedly, we obtain

\[
F_{Ax_{n+1} - Ax_n}(t) \geq F_{Ax_n - Ax_{n-1}}(\psi(t)) \\
\quad \quad \quad \quad \vdots \\
\geq F_{Ax_1 - Ax_0}(\psi^n(t)) \text{ for all } n \in \mathbb{N} \text{ and } t > 0.
\]

Thus by Lemma 6.1.2, \{Ax_n\} is a Cauchy sequence. Since X is complete, we can assume that

\[
\lim_{n \to \infty} Ax_n = z \in X.
\]

Therefore \{Sx_{2n-1}\} and \{Tx_{2n}\} also converge to z.

Since S is continuous, \(SAx_{2n-2} \to Sz\). Similarly \(Tx_{2n} \to z\) implies \(STx_{2n} \to Sz\).

Using compatibility of A and S and from (RN-4), we have

\[
F_{ASx_{2n-1} - Sz}(t) \geq \min\{F_{ASx_{2n-1} - SAx_{2n}}(t/2), F_{SAx_{2n} - Sz}(t/2)\}
\]

Therefore,

\[
F_{ASx_{2n-1} - Sz}(t) \to 1 \text{ as } n \to \infty.
\]
Thus, \( ASx_{2n-1} \rightarrow Sz \) as \( n \rightarrow \infty \).

Similarly \( ATx_{2n} \rightarrow Tz \) as \( n \rightarrow \infty \).

From (6.2.4) and continuity of \( T \), we have

\[
F_{ASx_{2n-1} - ATx_{2n}}(\phi(t)) \geq \min\{F_{TSx_{2n-1} - STx_{2n}}(t), F_{ASx_{2n-1} - TSx_{2n}-1}(t), F_{ATx_{2n} - STx_{2n}}(t)\}
\]

Letting \( n \rightarrow \infty \) and using property (RN-3), it follows that

\[
F_{Sz - Tz}(\phi(t)) \geq F_{Sz - Tz}(t) \text{ for } t > 0.
\]

Hence we must have \( Sz = Tz \) since \( \phi(t) < t \).

Again, from (6.2.4),

\[
F_{Az - ATx_{2n}}(\phi(t)) \geq \min\{F_{Tz - STx_{2n}}(t), F_{Az - Tz}(t), F_{ATx_{2n} - STx_{2n}}(t)\}
\]

Taking \( n \rightarrow \infty \), above inequality implies that \( Az = Tz \).

Thus \( Az = Sz = Tz \).

Now, we shall prove that \( z = Az \).
From (6.2.4), for \( t > 0 \),

\[
F_{Ax_{2n}} - Az \phi(t) \geq \min\{F_{Tx_{2n}} - Sz(t),
\]

\[
F_{Ax_{2n}} - Tz(t), F_{Az - Sz(t)}\}
\]

Taking limit as \( n \to \infty \), we get,

\( Az = z \). thus \( z \) is a common fixed point of \( A, S \) and \( T \).

For uniqueness, let \( w \) be another common fixed point of \( A, S \) and \( T \). Then

\[
F_{z - w} \phi(t) = F_{Az - Aw} (\phi(t))
\]

\[
\geq \min\{F_{Tz - Sw(t)}, F_{Az - Tz(t)}, F_{Aw - Sw(t)}\}
\]

\[
= \min\{F_{z - w(t)}, F_{z - z(t)}, F_{w - w(t)}\}
\]

\[
= F_{z - w(t)} > F_{z - w} \phi(t)
\]

Hence \( z = w \). This completes the proof.

Chamola [20] proved a common fixed point theorem for a pair of self-mappings satisfying a new contraction condition in an RN-space as follows.

**Theorem C.** Let \( (X, F, T) \) be a complete random normed space with \( T = \min \{x, y\} \) for every \( x, y \in [0, 1] \) and \( f, g \) be two mappings from \( X \) to itself such that for \( k \in (0, 1) \)
\[ \{F_{fu - gv}(kx)\}^2 \geq \min\{F_{u - fu}(x), F_{v - gv}(x), F_{u - gv}(2x), F_{v - fu}(x), F_{u - gv}(2x), F_{v - fu}(x), F_{v - gv}(2x)\} \]

holds for all \( u, v \in X \) and \( x \cdot 0 \). Then \( f \) and \( g \) have a common fixed point.

In this section we generalize and extend Theorem C for four maps under a different contraction condition. For proving our result we shall need the following Lemma due to Chamola [20].

**Lemma 6.2.1.** Let \( \{y_n\} \) be a sequence in an RN-space \((X, F, T)\) where \( T \) is continuous \( t \)-norm and satisfies \( T(x, x) \cdot x \) for every \( x \in [0, 1] \). If there exists a constant \( k \in (0, 1) \) such that

\[
F_{y_n - y_{n+1}}(kx) \geq F_{y_{n-1} - y_n}(x) \quad \text{for all } n,
\]

then \( \{y_n\} \) is a Cauchy sequence in \( X \).

**Theorem 6.2.2.** Let \( A, B, S \) and \( T \) be self-maps of a complete RN-space \((X, F, T)\) where \( T \) is a continuous \( t \)-norm and satisfies

\[ T(x, x) \cdot x \quad \text{for every } x \in [0, 1] \]

with “\( T = \min \)” such that
\[\begin{align*}
(6.2.5) & \quad A(X) \subset T(X), B(X) \subset S(X) \\
(6.2.6) & \quad A \text{ is continuous and } A \text{ commutes with each of } B, S \text{ and } T, \\
& \quad \text{or } B \text{ is continuous and } B \text{ commutes with each of } A, S \text{ and } T, \\
& \quad \text{or } S \text{ is continuous and } S \text{ commutes with each of } A, B \text{ and } T, \\
& \quad \text{or } T \text{ is continuous and } T \text{ commutes with each of } A, B \text{ and } S, \\
(6.2.7) & \quad \text{for all } x, y \in X, k \in (0, 1), t > 0, \\
& \quad \{F_{Ax - By}^{(kt)}\}^2 \geq \min\{ (F_{Sx - Ty}^{(t)})^2, F_{Sx - Ty}^{(t)}F_{Sx - Ax}^{(t)}, F_{Sx - Ty}^{(t)}F_{Sx - By}^{(2t)}, \\
& \quad F_{Sx - Ty}^{(t)}F_{Ty - By}^{(t)}, F_{Sx - Ty}^{(t)}F_{Sx - Ax}^{(t)}, F_{Ty - By}^{(t)}, F_{Sx - Ax}^{(t)}F_{Sx - By}^{(2t)}, \\
& \quad F_{Sx - Ax}^{(t)}F_{Sx - By}^{(2t)}, F_{Sx - Ax}^{(t)}F_{Ty - Ax}^{(t)}, F_{Ty - By}^{(t)}F_{Sx - By}^{(2t)}, F_{Ty - By}^{(t)}F_{Ty - Ax}^{(t)}, \\
& \quad F_{Sx - By}^{(2t)}F_{Ty - Ax}^{(t)}\}. \\
\end{align*}\]

Then \(A, B, S\) and \(T\) have a unique common fixed point.
Proof. Let $x_0$ be an arbitrary point in $X$. Since $A(X) \subset T(X)$ and $B(X) \subset S(X)$, there exist sequences $\{x_n\}$ and $\{y_n\}$ in $X$ such that,

$$y_{2n} = T_{x_{2n+1}} = A_{2n}$$

$$y_{2n+1} = S_{x_{2n+2}} = B_{2n+1}$$

for all $n \in \mathbb{N}$.

We shall prove that $F \frac{y_{2n} - y_{2n+1}}{y_{2n} - y_{2n+1}} \geq F \frac{y_{2n} - y_{2n+1}}{y_{2n} - y_{2n+1}}$ (6.2.8)

Suppose that $F \frac{y_{2n} - y_{2n+1}}{y_{2n} - y_{2n+1}} < F \frac{y_{2n} - y_{2n+1}}{y_{2n} - y_{2n+1}}$.

From (6.2.7), we have

$$\{F \frac{y_{2n} - y_{2n+1}}{y_{2n} - y_{2n+1}}\}^2 = \{F \frac{A_{2n} - B_{2n+1}}{y_{2n} - y_{2n+1}}\}^2$$

$$\geq \min\{F \frac{y_{2n} - y_{2n+1}}{y_{2n} - y_{2n+1}}\}^2,$$
\[
F_{y_{2n} - 1 - y_{2n}} (t)F_{y_{2n} - 1 - y_{2n} + 1} (2t),
\]
\[
F_{y_{2n} - 1 - y_{2n}} (t)F_{y_{2n} + y_{2n}} (t),
\]
\[
F_{y_{2n} - y_{2n} + 1 - y_{2n}} (2t),
\]
\[
F_{y_{2n} - y_{2n} + 1 - y_{2n} + 1} (t),
\]
\[
F_{y_{2n} - y_{2n} + 1 - y_{2n}} (t),
\]
\[
F_{y_{2n} - 1 - y_{2n} + 1} (2t)F_{y_{2n} + y_{2n}} (t).\]

Since \( F_{y_{2n} - 1 - y_{2n} + 1} (2t) \geq \min\{F_{y_{2n} - y_{2n} + 1} (t), F_{y_{2n} - y_{2n}} (t)\} \), then from the above inequality we have

\[
\{F_{y_{2n} - y_{2n} + 1} (kt)\}^2 \geq \min\{(F_{y_{2n} - y_{2n} + 1} (t))\}^2,
\]
\[
F_{y_{2n} - 1 - y_{2n}} (t)F_{y_{2n} - y_{2n} + 1} (t),
\]
\[
F_{y_{2n} - 1 - y_{2n}} (t)F_{y_{2n} - y_{2n} + 1} (t)\}
\]
\[
> (F_{y_{2n} - y_{2n} + 1} (kt))^2 \text{ (using (6.2.8) and observing that } F_{y_{2n} - y_{2n} + 1} (kt) < F_{y_{2n} - y_{2n} + 1} (t)).
\]
Thus, we have a contradiction. Therefore

\[
F_{y_{2n}} - y_{2n+1} 
\leq F_{y_{2n-1} - y_{2n}} (kt) \quad \text{for all } n \in \mathbb{N}.
\]

In general, \( F_{y_n} - y_{n+1} \leq F_{y_{n-1} - y_n} (kt) \) for all \( n \in \mathbb{N} \).

Thus, by Lemma (6.2.1), \( \{y_n\} \) is a Cauchy sequence and converges to a point \( z \) in \( X \).

Hence sequences \( \{AX_{2n}\}, \{TX_{2n+1}\}, \{BX_{2n+1}\} \) and \( \{SX_{2n+2}\} \) converges to \( z \).

Suppose that \( A \) is continuous and \( AB = BA, AS = SA, AT = TA \).

Then

\[
ABX_{2n+1} \rightarrow Az \Rightarrow BAX_{2n+1} \rightarrow Az \text{ as } n \rightarrow \infty.
\]

Therefore \( Az = Bz \).

Similarly, \( Az = Sz \) and \( Az = Tz \).

(6.2.9) Therefore \( Az = Bz = Sz = Tz \).

Now, we prove that \( Bz = z \).

From (6.2.7),

\[
\{F_{y_{2n}} - Bz (kt)\}^2 = \{F_{Ax_{2n}} - Bz (kt)\}^2
\]
\[ \geq \min \{(F_{y_{2n-1}} - T_z(t))^2, \]

\[ F_{y_{2n-1}} - T_z(t) \cdot F_{y_{2n-1} - y_{2n}}, \]

\[ F_{y_{2n-1} - T_z(t)} \cdot F_{T_z - Bz}(t), \]

\[ F_{y_{2n-1} - T_z(t)} \cdot F_{T_z - y_{2n}}(t), \]

\[ F_{y_{2n-1} - y_{2n}}(t) \cdot F_{T_z - Bz}(t), \]

\[ F_{y_{2n-1} - y_{2n}}(t) \cdot F_{T_z - y_{2n}}(t), \]

\[ F_{T_z - Bz}(t) \cdot F_{y_{2n-1} - Bz}(2t), \]

\[ F_{T_z - Bz}(t) \cdot F_{T_z - y_{2n}}(t), \]

\[ F_{y_{2n-1} - Bz}(2t) \cdot F_{T_z - y_{2n}}(t). \]

Using property (RN-4), we have

\[ \{F_{y_{2n} - Bz(kt)}\}^2 \geq \min \{(F_{y_{2n-1} - T_z(t)})^2, \]
\[
\begin{align*}
F_{y_{2n-1}} - Tz(t)F_{y_{2n-1}} - y_{2n} \\
F_{y_{2n-1}} - Tz(t)F_{Tz - Bz(t)} \\
F_{y_{2n-1}} - Tz(t) \min\{F_{y_{2n-1}} - Tz(t), F_{Tz - Bz(t)}\}, \\
F_{y_{2n-1}} - y_{2n} - Tz(t)F_{Tz - y_{2n}} \\
F_{y_{2n-1}} - y_{2n} - Tz(t)F_{Tz - Bz(t)}, \\
F_{y_{2n-1}} - y_{2n} \min\{F_{y_{2n-1}} - Tz(t), F_{Tz - Bz(t)}\}, \\
F_{y_{2n-1}} - y_{2n} - Tz(t)F_{Tz - y_{2n}} \\
F_{Tz - Bz(t)} \min\{F_{y_{2n-1}} - Tz(t), F_{Tz - Bz(t)}\}, \\
F_{Tz - Bz(t)} F_{Tz - y_{2n}}, \\
\min\{F_{y_{2n-1}} - Tz(t), F_{Tz - Bz(t)}\}, F_{Tz - y_{2n}}.
\end{align*}
\]

In view of the equation (6.2.9), letting \(n \cdot \cdot \cdot\), from above inequality, we have
\[ \{F_{z - Bz}(kt)\}^2 \geq \min\{ (F_{z - Bz}(t))^2, F_{z - Bz}(t), 1\} \]

\[ = \{F_{z - Bz}(t)\}^2 \text{ implying thereby} \]

\[ F_{z - Bz}(kt) \geq F_{z - Bz}(t) \]. Therefore we must have \( Bz = z \).

Hence \( z \) is a common fixed point of \( A, B, S \) and \( T \).

For uniqueness, let \( w \) be another common fixed point of \( A, B, S \) and \( T \).

Then from (6.2.7)

\[ \{F_{z - w}(kt)\}^2 = \{F_{Az - Bw}(kt)\}^2 \]

\[ \geq \min\{ (F_{Sz - Tw}(t))^2, F_{Sz - Tw}(t), F_{Sz - Az}(t), \]

\[ F_{Sz - Tw}(t), F_{Tw - Bw}(t), \]

\[ F_{Sz - Tw}(t), F_{Sz - Bw}(2t), \]

\[ F_{Sz - Tw}(t), F_{Tw - Az}(t), \]

\[ F_{Sz - Az}(t), F_{Tw - Bw}(t), \]

\[ F_{Sz - Az}(t), F_{Sz - Bw}(2t), \]

\[ F_{Sz - Az}(t), F_{Tw - Az}(t), \]

\[ F_{Tw - Bw}(t), F_{Sz - Bw}(2t), \]
\[ F_{Tw - Bw} (t) F_{Tw - Az} (t), \]

\[ F_{Sz - Bw} (2t) F_{Tw - Az} (t) \}

\[ \geq \min \{ (F_{z - w} (t))^2 , F_{z - w} (t) \} \]

\[ = \{ F_{z - w} (t) \}^2 \quad \text{which implies that} \]

\[ z = w. \text{ Hence } z \text{ is a unique common fixed point of } A, B, S \text{ and } T. \text{ In the similar way we can discuss other cases also.} \]