CHAPTER - IV

RELATED FIXED POINT THEOREMS ON TWO MENER SPACES

This chapter presents some results connecting the fixed points of two mappings on two different Menger spaces. We generalize some of the previous known results on related fixed points in two metric and two Menger spaces.

This chapter is divided into the following two sections:

1. Introduction

2. Related fixed points.
4.1 Introduction.

B. Fisher [59,60] has investigated the conditions for the existence of a relation connecting the fixed points of two mappings in two different metric spaces. The following are the main results in this direction.

Theorem F₁ [59]. Let \((X, d)\) and \((Y, \bullet)\) be complete metric spaces. If \(T\) is a mapping from \(X\) to \(Y\) and \(S\) is a mapping from \(Y\) to \(X\) satisfying
\[
\begin{align*}
\text{d}(Sv, STu) &\bullet h \max \{ \bullet(v, Tu), d(u, Sv), d(u, STu) \} \\
\bullet(Tu, TSv) &\bullet h \max \{ d(u, Sv), \bullet(v, Tu), \bullet(v, TSv) \}
\end{align*}
\]
for all \(u \in X\), \(v \in Y\) and \(h \in [0, 1)\) then \(ST\) has a unique fixed point \(z\) in \(X\) and \(TS\) has a unique fixed point \(w\) in \(Y\). Further, \(Tz = w\) and \(Sw = z\).

Theorem F₂ [60]. Let \((X, d)\) and \((Y, \bullet)\) be complete metric spaces. If \(T\) is a continuous mapping of \(X\) into \(Y\) and \(S\) is a mapping of \(Y\) into \(X\) satisfying
\[
\begin{align*}
\text{d}(STu, STu') &\bullet h \max \{ d(u, u'), d(u, STu), d(u', STu'), \bullet(Tu, Tu') \} \\
\bullet(TSv, TSv') &\bullet h \max \{ \bullet(v, v'), \bullet(v, TSv'), \bullet(v', TSv'), d(Sv, Sv') \}
\end{align*}
\]
for all u, u' in X and v, v' in Y where 0 • h < 1, then ST has a unique fixed point z in X and TS has a unique fixed point w in Y. Further, Tz = w and Sw = z.

Later on, Jain and Sahu [82], Namdeo and Fisher [111], Namdeo, Tiwari, Fisher and Tas [112] and others have contributed to this related fixed point theory.

The probabilistic generalizations of Theorem F₁ and Theorem F₂ have been established by B.D. Pant [115].

The purpose of this chapter is to generalize and modify some results of B.D. Pant [op.cit.] under different contraction conditions in two Menger spaces.

In the following F and G denote the distribution functions related to Menger spaces X and Y respectively.

4.2 Related fixed points.

**Theorem 4.2.1.** Let (X, F, t) and (Y, G, t) be two complete Menger spaces, where t is a continuous t-norm with t(x, x) • x for all x in [0,1]. If S is a mapping from Y to X and T is a continuous mapping from X to Y satisfying
(4.2.1) \[ F_{STp,STp'}(hx) \geq \min\{F_{p,STp}(x), F_{p',STp'}(x), \]
\[ F_{p,STp'}(\alpha x), G_{T^p, T^p'}(x)\} \]

(4.2.2) \[ G_{TSq,TSq'}(hx) \geq \min\{G_{q,TSq}(x), G_{q',TSq'}(x), \]
\[ G_{q,TSq'}(\alpha x), F_{Sq,Sq'}(x)\} \]

for all \( p, p' \) in \( X \); \( q, q' \) in \( Y \), \( 0 < h < 1 \) and \( 1 < \cdot \cdot \cdot 2 \); then \( ST \) has a unique fixed point \( z \) in \( X \) and \( TS \) has a unique fixed point \( w \) in \( Y \).

Further, \( Tz = w \) and \( Sw = z \).

**Proof.** Let \( p_0 \) be an arbitrary point in \( X \). We construct sequences \( \{p_n\}_{n \in \mathbb{N}} \) and \( \{q_n\}_{n \in \mathbb{N}} \) in \( X \) and \( Y \) respectively in the following way:

(4.2.3) \[ (ST)^n p_0 = p_n, \ T(ST)^n p_0 = q_n \quad \text{for} \quad n = 1, 2, \ldots \]

From (4.2.1),

\[ F_{p_n, p_{n+1}}(x) \geq \min\{F_{p_n - 1, p_n}(x/h), F_{p_n, p_{n+1}}(x/h), \]
\[ F_{p_n - 1, p_{n+1}}(\alpha x/h), G_{q_n, q_{n+1}}(x/h)\}. \]

Taking \( \cdot = 1+h \), we have

\[ F_{p_n, p_{n+1}}(x) \geq \min\{F_{p_n - 1, p_n}(x/h), F_{p_n, p_{n+1}}(x), \]
\[ p_n, p_{n+1} \]
\[
G \left( q_n, q_{n+1} \right) \frac{(x/h)}{}
\]
since
\[
F_{p_{n-1}, p_n, p_n+1} \left( (1+h)x/h \right) \geq \min \{ F_{p_{n-1}, p_n, p_n+1} \left( x/h \right), F_{p_n, p_n+1} \left( x/h \right) \}.
\]
Hence,
\[
F_{p_n, p_n+1} \left( x/h \right) \geq \min \{ F_{p_{n-1}, p_n, p_n+1} \left( x/h \right), G_{q_n, q_{n+1}} \left( x/h \right) \}.
\]
Similarly, from (4.2.2)
\[
G_{q_n, q_{n+1}} \left( x/h \right) \geq \min \{ G_{q_{n-1}, q_n, q_{n+1}} \left( x/h \right), F_{p_{n-1}, p_n, q_0, q_1, q_2} \left( x/h \right) \}.
\]
Therefore by induction
\[
F_{p_n, p_n+1} \left( x/h^n \right) \geq \min \{ F_{p_{n-1}, q_0, q_1, q_2} \left( x/h^n \right), G_{q_n, q_{n+1}} \left( x/h^n \right) \}
\]
and
\[
G_{q_n, q_{n+1}} \left( x/h^n \right) \geq \min \{ F_{p_n, q_0, q_1, q_2} \left( x/h^{n-1} \right), G_{q_n, q_{n+1}} \left( x/h^{n-1} \right) \}
\]
for \( n = 1, 2, ... \).
Since \( h < 1 \),
\[
F_{p_n, p_n+1} \left( x/h^n \right) \to 1 \text{ and } G_{q_n, q_{n+1}} \left( x/h^n \right) \to 1 \text{ as } n \to \infty.
\]
Thus as in [40] (see also Lemma 3 [29]), \( \{ p_n \} \) and \( \{ q_n \} \) are Cauchy sequences. Let \( z \) and \( w \) be their respective limits. Then continuity of \( T \) together with (4.2.3) implies
\[
\lim_{n \to \infty} q_n = \lim_{n \to \infty} T p_n = T z = w
\]
Since \( \lim_{n \to \infty} p_n = z \) and \( \lim_{n \to \infty} q_n = Tz \), there exists an integer \( N = N(x, \bullet) \) such that for \( x > 0, 0 < \bullet < 1 \),

\[
(4.2.5) \quad F_{z, p_n + 1} \left( \frac{1 - h}{2h} x \right) > 1 - \bullet, \\
F_{p_n, p_n + 1} \left( \frac{1 - h}{2h} x \right) > 1 - \bullet \quad \text{and} \\
G_{Tz, q_n + 1} \left( \frac{1 - h}{2h} x \right) > 1 - \bullet \quad \text{for } n \cdot N.
\]

Now from (4.2.1), taking \( \bullet = 1+h \)

\[
F_{STz, p_n + 1}^{STz, ST(ST)^n p_0} (x) = F_{STz, ST(ST)^n p_0} (x) \\
\geq \min \{ F_{z, STz} (x/h), F_{p_n, p_n + 1} (x/h), \\
F_{z, p_n + 1} \left( \frac{1 + h}{h} x \right), G_{Tz, q_n + 1} (x/h) \}.
\]

Since \( F_{z, STz} (x/h) \geq \min \{ F_{z, p_n + 1} \left( \frac{1 - h}{2h} x \right), F_{p_n + 1, STz} \left( \frac{1 + h}{2h} x \right) \} \)

and in view of \( \frac{1 + h}{h} x > \frac{1 - h}{2h} x, \frac{x}{h} > \left( \frac{1 - h}{h} \right) \frac{x}{h} \) above inequality implies that

\[
F_{STz, p_n + 1} (x) \geq \min \{ F_{z, p_n + 1} \left( \frac{1 - h}{2h} x \right), F_{STz, p_n + 1} \left( \frac{1 + h}{2h} x \right), \\
F_{p_n + 1, STz} \left( \frac{1 + h}{2h} x \right) \}.
\]
Therefore from (4.2.5) and in view of \( \frac{1}{2h} x > x \), we have
\[
F_{p_n, p_{n+1} (\frac{1-h}{2h} x), G_{Tz, q_{n+1} (\frac{1-h}{2h} x)}}.
\]

Thus, \( F_{STz, p_{n+1}} (x) > 1 - \) for \( n \in \mathbb{N} \)
implies that \( STz = z \).

Hence from (4.2.4), we have
\[
STz = Sw = z.
\]

To prove the uniqueness of \( z \) as a fixed point of \( ST \), let \( y \) be another fixed point of \( ST \), then from (4.2.1) and (4.2.2), for any \( x > 0 \),
\[
F_{z, y} (x) = F_{STz, STy} (x) \geq \min \{ F_{z, STz} (x/h), F_{w, STy} (x/h),
\]
\[
F_{z, STy} (\alpha x/h), G_{Tz, Ty} (x/h) \}.
\]
\[
= \min \{ 1, 1, F_{z, y} (\alpha x/h), G_{Tz, Ty} (x/h) \} \]
\[
= G_{Tz, Ty} (x/h) \text{ since } x < \alpha x / h \]
\[
= G_{TSTz, TSTy} (x/h) \}
\]
\[ \geq \min\{G_{T_z, TSTz}(x/h^2), G_{T_y, TSTy}(x/h^2)\} \]

\[ G_{T_z, TSTy}(\alpha x/h^2), F_{STz, STy}(x/h^2) \} \]

\[ \geq \min\{G_{T_z, Tz}(x/h^2), G_{T_y, Ty}(x/h^2)\} \]

\[ G_{T_z, Ty}(x/h^2), F_{z, y}(x/h^2) \} \]

\[ = F_{z, y}(x/h^2), \text{ which is a contradiction.} \]

Therefore \( z \) is a unique fixed point of \( ST \).

Similarly, \( w \) can be shown to be the unique fixed point of \( TS \).

**Remark 4.2.1.** Replacing conditions (4.2.1) and (4.2.2) by (4.2.1)' and (4.2.2)' respectively as follows, we obtain Theorem 3 of Pant [115].

(4.2.1)' \[ F_{STp, STp'}((hx) \geq \min\{F_{p, p'}(x), F_{p, STp}(x), \]

\[ F_{p', STp'}(x), F_{p, STp'}(2x), \]

\[ F_{p', STp}(2x), G_{T_p, Tp'}(x)\} \]

(4.2.2)' \[ G_{TSq, TSq'}((hx) \geq \min\{G_{q, q'}(x), G_{q, TSq}(x), \]

\[ G_{q', TSq'}(x), G_{q, TSq'}(2x), \]
Theorem 4.2.2. Let \((X, F, t)\) be complete Menger space, where \(t\) is a continuous \(t\)-norm with \(t(x, x) \cdot x\) for all \(x\) in \([0, 1]\). If \(S\) and \(T\) are mappings from \(X\) to itself, with \(T\) is continuous and satisfying

\[
F_{ABu,STv}(kx) \geq t\{F_{Iu,Jv}(x), F_{STv,Jv}(x),
F_{ABu,Iu}(x), F_{ABu,Jv}(2x),
F_{STv,Iu}(x)\}.
\]

\[
(4.2.7) F_{TSp,TSq}(hx) \geq \min\{F_{p,TSp}(x), F_{q,TSq}(x),
F_{p,TSq}(\alpha x), F_{Sp,Sq}(x)\}
\]

for all \(p, q\) in \(X\), \(0 < h < 1\) and \(1 < \cdot \cdot \cdot 2\); then \(ST\) has a unique fixed point \(z\) and \(TS\) has a unique fixed point \(w\). Further, \(Tz = w\) and \(Sw = z\) and if \(z = w\), then \(z\) is the unique common fixed point of \(S\) and \(T\).

Proof. The existence of \(z\) and \(w\) follows from Theorem 4.2.1. If \(z = w\), then \(z\) is necessarily a common fixed point of \(S\) and \(T\).
Suppose that S and T have another fixed point y. Then from either of (4.2.6) and (4.2.7), for any \( x > 0 \),

\[
F_{z,y}(x) = F_{STz,STy}(x) \geq \min\{F_{z,STz}(x/h), F_{y,STy}(x/h), F_{z,y}(\alpha x/h), F_{STz,Ty}(x/h)\} = \min\{F_{z,z}(x/h), F_{y,y}(x/h), F_{z,y}(\alpha x/h), F_{z,y}(x/h)\} = F_{z,y}(x/h),
\]

which is a contradiction.

Hence, S and T have a unique common fixed point

**Corollary 4.2.1.** Let \((X, F, t)\) be complete Menger space, where \( t \) is a continuous \( t \)-norm with \( t(x, x) \cdot x \) for all \( x \) in \([0, 1]\). If S is a continuous self map on \( X \) such that,

\[
F_{Sp,Sq}(hx) \geq \min\{F_{p,Sp}(x), F_{q,Sq}(x), F_{p,Sq}(\alpha x)\}
\]

for all \( p, q \in X, h \in (0, 1), \bullet \in (1, 2) \); then, S has a unique fixed point. Setting, either \( S = I \) (identity map) or \( T = I \) in (4.2.7), its proof follows from Theorem 4.2.1.

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