Chapter 3

Stability analysis for neutral type
stochastic neural networks with
polytopic and linear fractional
uncertainties

3.1 Introduction

In certain physical systems, mathematical models are described by some functional
differential equations of neutral type. Functional DE of neutral type depends on the
delays of state and state derivative, see for example [10, 65, 66, 69, 70, 71, 73, 74,
112]. Practically, such phenomenon always appear in studies of automatic control,
chemical reactors, distributed networks, dynamic process including steam and water
pipes, population ecology, heat exchanges, microwave oscillators, systems of turbojet
engine, lossless transmission lines, vibrating masses attached to an elastic bar and so
on. Hence, stability and stabilization problems for neutral time-delay systems have
been considered in the recent years [31, 32, 36, 57, 75, 96, 106]. In addition, stability
of neutral SNNs with time-varying delays have been reported in [50, 68, 80].

In practice, uncertainty in mathematical modelling is unavoidable because it is very
difficult to obtain an exact mathematical model due to environmental noise, uncertain
or slowly varying parameters, etc., see for example [47]. Mainly, in this chapter two type
of uncertainties known as polytopic and linear fractional uncertainties are discussed.
First, the polytopic uncertainties can arise when the uncertain matrix in norm-bounded
uncertainties provide some prior known structures of uncertainties. Therefore the poly­
topic type uncertainty can be regarded as an important class of parameter uncertainty.
In the recent trends, more interesting results have been proposed for the robust stabil­
ity and stabilization of DNNs. Control systems with polytopic uncertainties have been
studied in [20, 35, 47, 72, 77] through LMI-based approaches.

Further, a new type of uncertainty namely linear fractional form have been con­
sidered in [44, 52, 53, 54, 111] which can include the norm-bounded uncertainties as a
special case. As is well known, it is usually fractional uncertainties are more general
than the norm-bounded uncertainties. Comparing with norm-bounded uncertainties,
formulation of linear fractional uncertainties gives less conservative results. Hence
it generalizes the norm-bounded uncertainties. Formulation of linear fractional un­
certainties are often important when (i) parameters or parasitic processes that are
not completely known, (ii) due to limitations of mathematical tools to approximate a
practical system, (iii) some control systems are required to operate within a range of
different operating conditions. To capture these uncertain factors, it is often possible
to identify a bounding set such that all the possible uncertainties fall within this set
and yet it is not too difficult to analyze mathematically in [29].

Based on the above discussions, this chapter focuses the problem of neutral-type
SNNs with interval time-varying delays described by neutral differential equations.
In this chapter, the delay-dependent robust stability criteria is studied for neutral
SNNs under polytopic and linear fractional uncertainties with constraint that lower
and upper bounds of the delay interval are assumed to be known. Based on a suitable
Lyapunov-Krasovskii functional and Itô’s differential rule, some delay-dependent stability conditions are derived for the considered neutral SNNs in terms of LMIs, which can be easily calculated by Matlab LMI control toolbox. The feasibility and superiority of the proposed results are demonstrated by means of numerical examples.

3.2 Delay-interval dependent robust stability criteria for neutral stochastic neural networks with polytopic and linear fractional uncertainties

3.2.1 Problem description and preliminaries

In this section, consider the following uncertain neutral SNNs with time-varying delays:

\[
\begin{align*}
\dot{x}(t) - D x(t - \tau(t)) &= \left[ -Ax(t) + W_0 f(x(t)) + W_1 f(x(t - \tau(t))) \right] dt \\
&\quad + \left[ Cx(t) + Dx(t - \tau(t)) + W_2 f(x(t)) + W_3 f(x(t - \tau(t))) \right] dw(t),
\end{align*}
\]

where \( D \) is delayed connected weight matrix and each activation function \( f_q \) is bounded, continuously differentiable with \( f_q(0) = 0 \) and satisfies the Lipschitz condition

\[
|f_q(x_1) - f_q(x_2)| \leq l_q |x_1 - x_2|, \quad \forall \ x_1, x_2 \in \mathbb{R}, \ x_1 \neq x_2, \ q = 1, \ldots, n,
\]

we can have

\[
|f_q(x_q)| \leq l_q |x_q|, \quad \forall \ x_q \in \mathbb{R}, \ q = 1, \ldots, n.
\]

The time-varying delays \( \tau(t) \) satisfy

\[
0 \leq h_1 \leq \tau(t) < h_2, \quad \dot{\tau}(t) \leq \mu,
\]

where \( h_1, h_2 \) and \( \mu \) are constants.
Consider robust stability of the system described by (3.1) and (3.3) subject to polytopic uncertainty. For polytopic uncertainty, the matrices $A$, $W_0$, $W_1$, $C$, $D$, $W_2$ and $W_3$ in (3.1) can be expressed as

$$
\begin{bmatrix}
    A & W_0 & W_1 & D & W_2 & W_3
\end{bmatrix} = \sum_{i=1}^{r} \lambda_i \begin{bmatrix}
    A^{(i)} & W_0^{(i)} & W_1^{(i)} & C^{(i)} & D^{(i)} & W_2^{(i)} & W_3^{(i)}
\end{bmatrix},
$$

where $\sum_{i=1}^{r} \lambda_i = 1$, $0 \leq \lambda_i \leq 1$.

Next, address the linear fractional norm-bounded uncertainty. Suppose that matrices $A$, $W_0$, $W_1$, $C$, $D$, $W_2$ and $W_3$ have parameter perturbations $\Delta A(t)$, $\Delta W_0(t)$, $\Delta W_1(t)$, $\Delta C(t)$, $\Delta D(t)$, $\Delta W_2(t)$ and $\Delta W_3(t)$ which are in the form

$$
\begin{bmatrix}
    \Delta A(t) & \Delta W_0(t) & \Delta W_1(t) & \Delta C(t) & \Delta D(t) & \Delta W_2(t) & \Delta W_3(t)
\end{bmatrix} = HA(t) \begin{bmatrix}
    T_1 & T_2 & T_3 & T_4 & T_5 & T_6 & T_7
\end{bmatrix},
$$

where $T_1$, $T_2$, $T_3$, $T_4$, $T_5$, $T_6$ and $T_7$ are given known matrices. The class of parametric uncertainties $A(t)$ that satisfies

$$A(t) = [I - F(t)J]^{-1}F(t),
$$

is said to be admissible, where $J$ is a known matrix satisfying

$$I - JJ^T > 0,
$$

and $F(t)$ is uncertain matrix satisfying

$$F^T(t)F(t) \leq I.
$$

### 3.2.2 Robust stability criteria

In this section, the stability criteria is derived by using the Lyapunov method in terms of LMIs. Now, defining two new state variables for the neutral SNNs (3.1),

$$y(t) = -A(t)x(t) + W_0(t)f(x(t)) + W_1(t)f(x(t - \tau(t))),$$

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and
\[ g(t) = C(t)x(t) + D(t)x(t - \tau(t)) + W_2(t)f(x(t)) + W_3(t)f(x(t - \tau(t))), \]

the neutral SNNs (3.1) can be written as
\[ d\left[ x(t) - \bar{D}x(t - \tau(t)) \right] = y(t)dt + g(t)d\omega(t). \]

The following theorem gives the mean square asymptotic stability results for neutral SNNs (3.1) without uncertainty.

**Theorem 3.2.1** For given scalars \( h_2 > h_1 \geq 0 \) and \( \mu \), the equilibrium point of neutral SNNs (3.1) without uncertainty is asymptotically stable in the mean square if there exist symmetric matrices \( P > 0, \quad R_v > 0 \quad (v = 1, 2, 3, 4, 5), \quad Q_b > 0 \quad (b = 1, 2, 3, 4) \), for any matrices \( N_k, M_k \) and \( S_k \quad (k = 1, \ldots, 10) \), further there exist diagonal matrices \( K_1 \geq 0 \) and \( K_2 \geq 0 \) such that the following LMI is feasible:

\[ \Phi = \begin{bmatrix} \hat{\Omega} & N \\ * & -Q_2 \end{bmatrix} < 0, \quad (3.9) \]

where \( \hat{\Omega} = \left( \Omega_{\alpha,\beta} \right)_{10 \times 10} \) with

\[ \Omega_{1,1} = R_1 + R_2 + R_3 + R_4 + h_2Q_3 + (h_2 - h_1)Q_4 - M_1A - A^T M_1^T + S_1C + C^T S_1^T + N_1 + N_1^T, \quad \Omega_{1,2} = -N_1\bar{D} - N_2 - A^T M_2^T + S_1D + C^T S_2^T, \]

\[ \Omega_{1,3} = N_1\bar{D} + N_3^T - A^T M_3^T + C^T S_3^T, \quad \Omega_{1,4} = N_4 - A^T M_4^T + C^T S_4^T, \quad \Omega_{1,5} = N_5^T 
\]

\[ - A^T M_5^T + C^T S_5^T, \quad \Omega_{1,6} = P + N_6^T - M_1 - A^T M_6^T + C^T S_6^T, \quad \Omega_{1,7} = N_7^T - S_1 
\]

\[ - A^T M_7^T + C^T S_7^T, \quad \Omega_{1,8} = N_8^T + M_1W_0 - A^T M_8^T + S_1W_2 + C^T S_8^T + LK_1, \]

\[ \Omega_{1,9} = N_9^T + M_1W_1 - A^T M_9^T + S_1W_3 + C^T S_9^T, \quad \Omega_{1,10} = -N_1 + N_{10}^T - A^T M_{10}^T + C^T S_{10}^T, \]

\[ \Omega_{2,2} = -(1 - \mu)R_1 - N_2\bar{D} - \bar{D}^T N_2^T - N_2 - N_2^T + S_2D + C^T S_2^T, \quad \Omega_{2,3} = N_2\bar{D} - \bar{D}^T N_3^T 
\]

\[ - N_3^T + D^T S_3^T, \quad \Omega_{2,4} = -\bar{D}^T N_4^T - N_4^T + D^T S_4^T, \quad \Omega_{2,5} = -\bar{D}^T N_5^T - N_5^T + D^T S_5^T, \]

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\begin{align*}
\Omega_{2,6} &= -\tilde{D}^T P - \tilde{D}^T N_6^T - N_6^T - M_2 + D^T S_6^T, \\
\Omega_{2,7} &= -\tilde{D}^T N_7^T - N_7^T - S_2 + D^T S_7^T, \\
\Omega_{2,8} &= -\tilde{D}^T N_8^T - N_8^T + M_2 W_0 + S_2 W_2 + D^T S_8^T, \\
\Omega_{2,9} &= -\tilde{D}^T N_9^T - N_9^T + M_2 W_0 \\
&\quad + S_2 W_3 + D^T S_9^T + LK_2, \\
\Omega_{2,10} &= -N_2 - \tilde{D}^T N_{10}^T - N_{10}^T + D^T S_{10}^T, \\
\Omega_{3,3} &= -(1 - 2\mu) R_2 + N_3 \tilde{D} + \tilde{D}^T N_3^T, \\
\Omega_{3,4} &= \tilde{D}^T N_4^T, \\
\Omega_{3,5} &= \tilde{D}^T N_5^T, \\
\Omega_{3,6} &= \tilde{D}^T N_6^T - M_3, \\
\Omega_{3,7} &= \tilde{D}^T N_7^T - S_3, \\
\Omega_{3,8} &= \tilde{D}^T N_8^T + M_3 W_0 + S_3 W_2, \\
\Omega_{3,9} &= \tilde{D}^T N_9^T + M_3 W_1 + S_3 W_3, \\
\Omega_{3,10} &= -N_3 + \tilde{D}^T N_{10}^T, \\
\Omega_{4,4} &= -R_3, \\
\Omega_{4,5} &= 0, \\
\Omega_{4,6} &= -M_4, \\
\Omega_{4,7} &= -S_4, \\
\Omega_{4,8} &= M_4 W_0 + S_4 W_2, \\
\Omega_{4,9} &= M_4 W_1 + S_4 W_3, \\
\Omega_{4,10} &= -N_4, \\
\Omega_{5,5} &= -R_4, \\
\Omega_{5,6} &= -M_5, \\
\Omega_{5,7} &= -S_5, \\
\Omega_{5,8} &= M_5 W_0 + S_5 W_2, \\
\Omega_{5,9} &= M_5 W_1 + S_5 W_3, \\
\Omega_{5,10} &= -N_5, \\
\Omega_{6,6} &= -M_6 - M_6^T + h_2 Q_1, \\
\Omega_{6,7} &= -M_7^T - S_6, \\
\Omega_{6,8} &= M_6 W_0 - M_8^T + S_6 W_2, \\
\Omega_{6,9} &= M_6 W_1 - M_9^T + S_6 W_3, \\
\Omega_{6,10} &= -N_6 - M_{10}^T, \\
\Omega_{7,7} &= P - S_7 - S_7^T + h_2 Q_2, \\
\Omega_{7,8} &= M_7 W_0 + S_7 W_2 - S_8^T, \\
\Omega_{7,9} &= M_7 W_1 + S_7 W_3 - S_9^T, \\
\Omega_{7,10} &= -N_7 - S_{10}^T, \\
\Omega_{8,8} &= R_5 + M_8 W_0 + W_0^T M_8 \\
&\quad + S_8 W_2 + W_2^T S_8^T - 2K_1, \\
\Omega_{8,9} &= M_8 W_1 + W_0^T M_9^T + S_8 W_3 + W_2^T S_9^T, \\
\Omega_{8,10} &= -N_8 + W_0^T M_{10}^T + W_2^T S_{10}^T, \\
\Omega_{9,9} &= -(1 - \mu) R_5 + M_9 W_1 + W_1^T M_9^T + S_9 W_3 \\
&\quad + W_3^T S_{10}^T - 2K_2, \\
\Omega_{9,10} &= -N_9 + W_1^T M_{10}^T + W_3^T S_{10}^T, \\
\Omega_{10,10} &= -N_{10} - N_{10}^T - \frac{1}{h_2} Q_1, \\
N &= \begin{bmatrix} N_1^T & N_2^T & N_3^T & N_4^T & N_5^T & N_6^T & N_7^T & N_8^T & N_9^T & N_{10}^T \end{bmatrix}^T, \\
M &= \begin{bmatrix} M_1^T & M_2^T & M_3^T & M_4^T & M_5^T & M_6^T & M_7^T & M_8^T & M_9^T & M_{10}^T \end{bmatrix}^T, \\
S &= \begin{bmatrix} S_1^T & S_2^T & S_3^T & S_4^T & S_5^T & S_6^T & S_7^T & S_8^T & S_9^T & S_{10}^T \end{bmatrix}^T.
\end{align*}

**Proof:** Consider the Lyapunov-Krasovskii functional

\[ V(x_t, t) = V_1(x_t, t) + V_2(x_t, t) + V_3(x_t, t), \quad (3.10) \]
where

\[ V_1(x_t, t) = [x(t) - \bar{D}x(t - \tau(t))]^T P[x(t) - \bar{D}x(t - \tau(t))], \]

\[ V_2(x_t, t) = \int_{t-\tau(t)}^{t} x^T(s)R_1x(s)ds + \int_{t-2\tau(t)}^{t} x^T(s)R_2x(s)ds + \int_{t-h_1}^{t} x^T(s)R_3x(s)ds \]
\[ + \int_{t-h_2}^{t} x^T(s)R_4x(s)ds + \int_{t-\tau(t)}^{t} f^T(x(s))R_5f(x(s))ds, \]

\[ V_3(x_t, t) = \int_{-h_2}^{t} y^T(s)Q_1y(s)ds + \int_{-h_2}^{t} \int_{t+\theta}^{t} g^T(s)Q_2g(s)dsd\theta \]
\[ + \int_{-h_2}^{t} \int_{t+\theta}^{t} x^T(s)Q_3x(s)dsd\theta + \int_{-h_2}^{t} \int_{t+\theta}^{t} x^T(s)Q_4x(s)dsd\theta. \]

Then, it can be obtained by Itô's formula that

\[ dV(x_t, t) = \mathcal{L}V(x_t, t)dt + 2[x(t) - \bar{D}x(t - \tau(t))]^T P g(t)dw(t), \]

(3.11)

where

\[ \mathcal{L}V_1(x_t, t) = 2[x(t) - \bar{D}x(t - \tau(t))]^T P y(t) + g^T(t)P g(t), \]

\[ \mathcal{L}V_2(x_t, t) \leq x^T(t)R_1x(t) - (1 - \mu)x^T(t - \tau(t))R_1x(t - \tau(t)) + x^T(t)R_2x(t) \]
\[ - (1 - 2\mu)x^T(t - 2\tau(t))R_2x(t - 2\tau(t)) + x^T(t)R_3x(t) - x^T(t - h_1)R_3x(t - h_1) \]
\[ + x^T(t)R_4x(t) - x^T(t - h_2)R_4x(t - h_2) + f^T(x(t))R_5f(x(t)) \]
\[ - (1 - \mu)f^T(x(t - \tau(t)))R_5f(x(t - \tau(t))), \]

\[ \mathcal{L}V_3(x_t, t) = h_2y^T(t)Q_1y(t) - \int_{t-\tau(t)}^{t} y^T(s)Q_1y(s)ds + h_2g^T(t)Q_2g(t) \]
\[ - \int_{t-\tau(t)}^{t} g^T(s)Q_2g(s)ds + h_2x^T(t)Q_3x(t) - \int_{t-h_2}^{t} x^T(s)Q_3x(s)ds \]
\[ + (h_2 - h_1)x^T(t)Q_4x(t) - \int_{t-h_2}^{t-h_1} x^T(s)Q_4x(s)ds. \]
Using Lemma 1.16.11, one can obtain

\[ \mathcal{L}V_3(x_1, t) \leq h_2 g^T(t)Q_1 y(t) - \frac{1}{h_2} \left( \int_{t-h_2}^t y^T(s)ds \right) Q_1 \left( \int_{t-h_2}^t y(s)ds \right) + h_2 g^T(t)Q_2 g(t) - \int_{t-h_2}^t g^T(s)Q_2 g(s)ds + h_2 x^T(t)Q_3 x(t) - \int_{t-h_2}^t x^T(s)Q_3 x(s)ds + (h_2 - h_1)x^T(t)Q_4 x(t) - \int_{t-h_2}^{t-h_1} x^T(s)Q_4 x(s)ds. \]

It is obvious from (3.2) under restriction of lower bound to zero, it follows that

\[ f_q(x_q(t))[f_q(x_q(t) - l_qx_q(t))] \leq 0, \quad q = 1, \ldots, n, \quad (3.12) \]

\[ f_q(x_q(t - \tau(t)))[f_q(x_q((t - \tau(t))) - l_qx_q(t - \tau(t)))] \leq 0. \quad (3.13) \]

So, for any \( K_1 = diag\{k_{11}, k_{21}, \ldots, k_{n1}\} \geq 0 \) and \( K_2 = diag\{k_{12}, k_{22}, \ldots, k_{n2}\} \geq 0 \), from (3.12) and (3.13), it follows that

\[ 0 \leq -2 \sum_{q=1}^n k_{q1}f_q(x_q(t))[f_q(x_q(t) - l_qx_q(t))] - 2 \sum_{q=1}^n w_{q2}f_q(x_q(t - \tau(t))) \]

\[ \times [f_q(x_q(t - \tau(t)) - l_qx_q(t - \tau(t)))] \]

\[ = 2x^T(t)LW_1f(x(t)) - 2f^T(x(t))W_1f(x(t)) + 2x^T(t - \tau(t))Lk_2f(x(t - \tau(t))) \]

\[ - 2f^T(x(t - \tau(t)))W_2f(x(t - \tau(t))), \quad (3.14) \]

where \( L = diag\{l_1, l_2, \ldots, l_n\} \). From (2.12), the following equalities hold for any matrices \( N, M \) and \( S \).

\[ 2\xi^T(t)N \left[ x(t) - \bar{D}x(t - \tau(t)) - x(t - \tau(t)) + \bar{D}(t - 2\tau(t)) \right] \]

\[ - \int_{t-\tau(t)}^t y(s)ds - \int_{t-\tau(t)}^t g(s)dw(s) \] \[ = 0, \quad (3.15) \]

\[ 2\xi^T(t)M \left[ - Ax(t) + W_0f(x(t)) + W_1f(x(t - \tau(t)) - y(t) \right] = 0, \quad (3.16) \]

\[ 2\xi^T(t)S \left[ Cx(t) + Dx(t - \tau(t)) + W_2f(x(t)) + W_3f(x(t - \tau(t))) - g(t) \right] = 0. \quad (3.17) \]
From (3.15), it can be easily seen that

\[-2\xi^T(t)N\int_{t-\tau(t)}^tg(s)dw(s) \leq \xi^T(t)NQ_2^{-1}N^T\xi(t)\]

\[+ \left(\int_{t-\tau(t)}^tg(s)dw(s)\right)^TQ_2\left(\int_{t-\tau(t)}^tg(s)dw(s)\right). \quad (3.18)\]

Adding the right sides of (3.14)-(3.18) into (3.11), it follows that

\[dV(x_t,t) \leq \xi^T(t)\Phi\xi(t)dt + 2[x(t) - \bar{D}x(t - \tau(t))]^TPg(t)dw(t), \quad (3.19)\]

where

\[\Phi = \hat{Q} + NQ_2^{-1}N^T,\]

\[\xi^T(t) = \begin{bmatrix} x^T(t) & x^T(t - \tau(t)) & x^T(t - 2\tau(t)) & x^T(t - \tau(t) - h_1) & x^T(t - \tau(t) - h_2) & y^T(t) & g^T(t) \\ f^T(x(t)) & f^T(x(t - \tau(t))) & \int_{t-\tau(t)}^ty^T(s)ds \end{bmatrix}.\]

From (3.9), there exists a scalar \(\alpha > 0\) such that

\[\Phi + \text{diag}\{\alpha I_n, 0, 0, 0, 0, 0, 0, 0, 0, 0\} < 0. \quad (3.20)\]

Since

\[\mathbb{E}\left\{\int_{t-\tau(t)}^tg^T(s)dw(s)Q_2\int_{t-\tau(t)}^tg^T(s)dw(s)\right\} = \mathbb{E}\left\{\int_{t-\tau(t)}^tg^T(s)Q_2g(s)ds\right\}.\]

By taking the mathematical expectation on both sides of (3.19) and considering (3.20), one can obtain

\[\mathbb{E}\left[\frac{dV(x_t,t)}{dt}\right] \leq \mathbb{E}(\xi^T(t)\Phi\xi(t)) \leq -\alpha\mathbb{E}|x(t)|^2.\]

Thus if \(\Phi < 0\), the neutral SNNs (3.1) is asymptotically stable in the mean square. The proof is completed.

**Remark 3.2.2** The conditions in Theorem 3.2.1 are formulated in terms of LMIs and can be easily solved by using Matlab LMI control toolbox. It is worth to note that by
applying convex optimization algorithms, we can conclude that the maximum allowable upper bound of the time-varying delay, that is, $h_2$ guarantees the feasibility of the presented LMIs. To find maximum allowable upper bound of the given DNNs the following algorithm is very useful.

Step 1: The given SNNs of neutral type system can be converted into convex programming problem (CPP),

$$\begin{aligned}
\text{Max} & \quad h_2 \\
\text{subject to the LMI constraints} & \quad \Phi < 0, \\
\text{symmetric matrices} & \quad P > 0, \quad R_i > 0 \ (i = 1, 2, 3, 4, 5), \quad Q_l > 0 \ (l = 1, 2, 3, 4), \\
\text{real matrices} & \quad N_k, M_k, S_k \ (k = 1, \ldots, 10) \text{ of appropriate dimensions} \\
\text{and diagonal matrices} & \quad K_1 \geq 0 \quad K_2 \geq 0.
\end{aligned}$$

Step 2: Interior point algorithm solves CPP which is better than simplex algorithm when the problems are large in general.

Step 3: Matlab LMI control toolbox was developed by interior point algorithms. It solves only positive definite unknown decision variables.

Step 4: CPP is solved by Matlab LMI control toolbox to get the required feasible solution by means of unknown decision variables.

Step 5: The obtained feasible solution of CPP is solution for stability of SNNs of neutral type system is one and the same.

Step 6: We achieved the feasible solution of SNNs of neutral type system by means of solving CPP reported in step 1.

Using this Algorithm, we can find the maximum allowable upper bound of the time-varying delay for SNNs of neutral type system.

**Remark 3.2.3** The reduced conservatism of Theorem 3.2.1 benefits from the construction of the new Lyapunov-Krasovskii functional in (3.10) and introducing some free-weighting matrices. It can be easily seen that the results discussed and derived in this section is quite different from the most of existing results in the literature in
the following perspective theoretical stability analysis of neutral SNNs with the same time-varying neutral and discrete interval delays which is much more complicated, especially, for the case where the neutral delay is time-varying. In this section, the proposed design is dependent to same time-varying neutral and discrete interval delays, which makes the treatment in this section is more general with less conservative when compared to the recent existing results [80].

**Theorem 3.2.4** For given scalars $h_2 > h_1 \geq 0$ and $\mu$, the equilibrium point of neutral SNNs (3.1) subject to polytopic uncertainty (3.4) is asymptotically stable in the mean square if there exist symmetric matrices $P > 0$, $R_{v}^{(i)} > 0 (v = 1, 2, 3, 4, 5)$, $Q_b > 0$ ($b = 1, 2, 3, 4$), for any matrices $N_k$, $M_k$ and $S_k$ ($k = 1, \ldots, 10$), further there exist diagonal matrices $K_1 \geq 0$ and $K_2 \geq 0$ such that the following LMI is feasible:

$$
\Phi_l = \begin{bmatrix}
\Omega_l & N \\
* & -Q_2
\end{bmatrix} < 0,
$$

for $l = 1, 2, \ldots, r$, where $\Omega_l = (\tilde{\Omega}_{l,6})_{10 \times 10}$ with

$$
\begin{align*}
\tilde{\Omega}_{1,1} &= R_1^{(i)} + R_2^{(i)} + R_3^{(i)} + R_4^{(i)} + h_2 Q_3 + (h_2 - h_1) Q_4 - M_1 A^{(i)} - A^{(i)T} M_1^T + S_1 C^{(i)} \\
&\quad + C^{(i)T} S_1^T + N_1 + N_1^T, \\
\tilde{\Omega}_{1,2} &= -N_1 \tilde{D} - N_1 + N_2^T - A^{(i)T} M_2^T + S_1 D^{(i)} + C^{(i)T} S_2^T, \\
\tilde{\Omega}_{1,3} &= N_1 \tilde{D} + N_3^T - A^{(i)T} M_3^T + C^{(i)T} S_3^T, \\
\tilde{\Omega}_{1,4} &= N_4 - A^{(i)T} M_4^T + C^{(i)T} S_4^T, \\
\tilde{\Omega}_{1,5} &= N_5^T - A^{(i)T} M_5^T + C^{(i)T} S_5^T, \\
\tilde{\Omega}_{1,6} &= P + N_6^T - M_1 - A^{(i)T} M_6^T + C^{(i)T} S_6^T, \\
\tilde{\Omega}_{1,7} &= N_7^T - S_1 - A^{(i)T} M_7^T + C^{(i)T} S_7^T, \\
\tilde{\Omega}_{1,8} &= N_8^T + M_1 W_0^{(i)} - A^{(i)T} M_8^T + S_1 W_1^{(i)} \\
&\quad + C^{(i)T} S_8^T + LK_1, \\
\tilde{\Omega}_{1,9} &= N_9^T + M_1 W_0^{(i)} - A^{(i)T} M_9^T + S_1 W_3^{(i)} + C^{(i)T} S_9^T, \\
\tilde{\Omega}_{1,10} &= -N_1 + N_1^T - A^{(i)T} M_{10}^T + C^{(i)T} S_{10}^T, \\
\tilde{\Omega}_{2,2} &= -(1 - \mu) R_1^{(i)} - N_2 \tilde{D} - \tilde{D}^T N_2^T - N_2 \\
&\quad - N_2^T + S_2 D^{(i)} + D^{(i)T} S_2^T, \\
\tilde{\Omega}_{2,3} &= N_2 \tilde{D} - \tilde{D}^T N_3^T - N_3^T + D^{(i)T} S_3^T, \\
\tilde{\Omega}_{2,4} &= -\tilde{D}^T N_4^T - N_4^T + D^{(i)T} S_4^T, \\
\tilde{\Omega}_{2,5} &= -\tilde{D}^T N_5^T - N_5^T + D^{(i)T} S_5^T, \\
\tilde{\Omega}_{2,6} &= -\tilde{D}^T P \\
&\quad - \tilde{D}^T N_6^T - N_6^T - M_2 + D^{(i)T} S_6^T, \\
\tilde{\Omega}_{2,7} &= -\tilde{D}^T N_7^T - N_7^T - S_2 + D^{(i)T} S_7^T,
\end{align*}
$$

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\[ \begin{align*}
\bar{\Omega}_{2,8} &= -\bar{D}^T N_8^T - N_8^T + M_2 W_0^{(l)} + S_2 W_2^{(l)} + D^{(l)T} S_8^T, \\
\bar{\Omega}_{2,9} &= -\bar{D}^T N_9^T - N_9^T \\
+ M_3 W_1^{(l)} + S_2 W_3^{(l)} + D^{(l)T} S_7^T + \bar{L} K_2, \\
\bar{\Omega}_{2,10} &= -N_2 - \bar{D}^T N_{10}^T - N_{10}^T + D^{(l)T} S_{10}^T, \\
\bar{\Omega}_{3,3} &= -(1 - 2\mu) R_2^{(l)} + N_3 \bar{D} + \bar{D}^T N_3^T, \\
\bar{\Omega}_{3,4} &= \bar{D}^T N_4^T, \\
\bar{\Omega}_{3,5} &= \bar{D}^T N_5^T, \\
\bar{\Omega}_{3,6} &= \bar{D}^T N_6^T - M_3, \\
\bar{\Omega}_{3,7} &= \bar{D}^T N_7^T - S_3, \\
\bar{\Omega}_{3,8} &= \bar{D}^T N_8^T + M_3 W_0^{(l)} + S_3 W_2^{(l)}, \\
\bar{\Omega}_{3,9} &= \bar{D}^T N_9^T + M_3 W_0^{(l)} + S_3 W_3^{(l)}, \\
\bar{\Omega}_{3,10} &= -N_3 + \bar{D}^T N_{10}^T, \\
\Omega_{4,4} &= -R_3^{(l)}, \\
\bar{\Omega}_{4,5} &= 0, \\
\bar{\Omega}_{4,6} &= -M_4, \\
\bar{\Omega}_{4,7} &= -S_4, \\
\bar{\Omega}_{4,8} &= M_4 W_0^{(l)} + S_4 W_2^{(l)}, \\
\bar{\Omega}_{4,9} &= M_4 W_1^{(l)} + S_4 W_3^{(l)}, \\
\bar{\Omega}_{4,10} &= -N_4, \\
\Omega_{5,5} &= -R_4^{(l)}, \\
\Omega_{5,6} &= -M_5, \\
\Omega_{5,7} &= -S_5, \\
\Omega_{5,8} &= M_5 W_0^{(l)} + S_5 W_2^{(l)}, \\
\bar{\Omega}_{5,9} &= M_5 W_1^{(l)} + S_5 W_3^{(l)}, \\
\bar{\Omega}_{5,10} &= -N_5, \\
\Omega_{6,6} &= -M_6 - M_6^T + h_2 Q_1, \\
\bar{\Omega}_{6,7} &= -M_7^T - S_6, \\
\bar{\Omega}_{6,8} &= M_6 W_0^{(l)} - M_6^T + S_6 W_2^{(l)}, \\
\bar{\Omega}_{6,9} &= M_6 W_1^{(l)} - M_6^T + S_6 W_3^{(l)}, \\
\bar{\Omega}_{6,10} &= -N_6 - M_6^T, \\
\bar{\Omega}_{7,7} &= P - S_7 - S_7^T + h_2 Q_2, \\
\Omega_{7,8} &= M_7 W_0^{(l)} + S_7 W_2^{(l)} - S_7^T, \\
\Omega_{7,9} &= M_7 W_1^{(l)} + S_7 W_3^{(l)} - S_7^T, \\
\Omega_{7,10} &= -N_7 - M_7^T, \\
S_7 W_3^{(l)} - S_7^T, \\
\Omega_{8,8} &= R_5 + M_8 W_0^{(l)} + W_0^{(l)T} M_8 + S_8 W_2^{(l)} + W_2^{(l)T} S_8^T - 2K_1, \\
\bar{\Omega}_{8,9} &= M_8 W_1^{(l)} + W_0^{(l)T} M_9^T + S_8 W_3^{(l)} + W_2^{(l)T} S_9^T, \\
\bar{\Omega}_{8,10} &= -N_8 + W_0^{(l)T} M_{10} + W_2^{(l)T} S_{10}^T, \\
\bar{\Omega}_{9,9} &= -(1 - \mu) R_6^{(l)} + M_9 W_1^{(l)} + W_1^{(l)T} M_9^T + S_9 W_3^{(l)} + W_3^{(l)T} S_9^T - 2K_2, \\
\bar{\Omega}_{9,10} &= -N_9 + W_1^{(l)T} M_{10} + W_3^{(l)T} S_{10}^T, \\
\bar{\Omega}_{10,10} &= -N_{10} - N_{10}^T - \frac{1}{h_2} Q_1.
\end{align*}\]

**Proof:** By Schur complement lemma 1.16.10, the matrix inequality (3.35) implies
\[ \sum_{i=1}^{r} \lambda_i \bar{\Phi}_i < 0 \text{ or } \Phi < 0, \]
where \( A = \sum_{i=1}^{r} \lambda_i A^{(l)}, W_0 = \sum_{i=1}^{r} \lambda_i W_0^{(l)}, W_1 = \sum_{i=1}^{r} \lambda_i W_1^{(l)}, C = \sum_{i=1}^{r} \lambda_i C^{(l)}, D = \sum_{i=1}^{r} \lambda_i D^{(l)}, W_2 = \sum_{i=1}^{r} \lambda_i W_2^{(l)}, W_3 = \sum_{i=1}^{r} \lambda_i W_3^{(l)}, R_v = \sum_{i=1}^{r} \lambda_i R_v^{(l)} \) (\( v = 1, 2, 3, 4, 5 \)). This completes the proof as it is similar to the proof of Theorem 3.2.1.

**Theorem 3.2.5** For given scalars \( h_2 > h_1 \geq 0 \) and \( \mu \), the equilibrium point of neutral SNNs (3.1) subject to linear fractional norm-bounded uncertainty (3.5) is robustly asymptotically stable in the mean square if there exist scalars \( \epsilon_1 > 0, \epsilon_2 > 0 \), matrices \( P > 0, R_v > 0 \) (\( v = 1, 2, 3, 4, 5 \)), and scalars \( Q_b > 0 \) (\( b = 1, 2, 3, 4 \)), for any matrices.
Further there exist diagonal matrices $K_1 \geq 0$ and $K_2 \geq 0$ such that the following LMI is feasible:

$$
\begin{bmatrix}
\Phi & \Theta_1 & \epsilon_1 I^T & \Theta_2 & \epsilon_2 I^T \\
* & -\epsilon_1 I & \epsilon_1 J^T & 0 & 0 \\
* & * & -\epsilon_1 I & 0 & 0 \\
* & * & * & -\epsilon_2 I & \epsilon_2 J^T \\
* & * & * & * & -\epsilon_2 I
\end{bmatrix} < 0,
$$

(3.22)

where $\Phi$ is defined in (3.9).

**Proof:** Assume that inequality (3.22) holds. From (3.22), it can be easily obtained that

$$
\Psi = \begin{bmatrix}
\Phi & \Theta_1 & \epsilon_1 I^T & \Theta_2 & \epsilon_2 I^T \\
* & -\epsilon_1 I & \epsilon_1 J^T & 0 & 0 \\
* & * & -\epsilon_1 I & 0 & 0 \\
* & * & * & -\epsilon_2 I & \epsilon_2 J^T \\
* & * & * & * & -\epsilon_2 I
\end{bmatrix} < 0,
$$

$$
\Theta_1 = \begin{bmatrix}
M_1 H & M_2 H & M_3 H & M_4 H & M_5 H & M_6 H & M_7 H & M_8 H & M_9 H & M_{10} H & 0
\end{bmatrix}^T,
$$

$$
\Gamma_1 = \begin{bmatrix}
-T_1 & 0 & 0 & 0 & 0 & 0 & 0 & T_2 & T_3 & 0 & 0
\end{bmatrix},
$$

$$
\Theta_2 = \begin{bmatrix}
S_1 H & S_2 H & S_3 H & S_4 H & S_5 H & S_6 H & S_7 H & S_8 H & S_9 H & S_{10} H & 0
\end{bmatrix}^T,
$$

$$
\Gamma_2 = \begin{bmatrix}
T_4 & T_5 & 0 & 0 & 0 & 0 & 0 & T_6 & T_7 & 0 & 0
\end{bmatrix}.
$$

Thus, $\Psi = \Phi + \Theta_1 \Delta(t) \Gamma_1 + \Gamma_1^T \Delta(t) \Theta_1^T + \Theta_2 \Delta(t) \Gamma_2 + \Gamma_2^T \Delta(t) \Theta_2^T < 0$ holds according to Lemma 1.16.13. It can be verified that $\Psi$ is exactly the same $\Phi$ of (3.9) when $A$, $W_0$, $W_1$, $C$, $D$, $W_2$ and $W_3$ are replaced by $A + H\Delta(t)T_1$, $W_0 + H\Delta(t)T_2$, $W_1 + H\Delta(t)T_3$, $C + H\Delta(t)T_4$, $D + H\Delta(t)T_5$, $W_2 + H\Delta(t)T_6$ and $W_3 + H\Delta(t)T_7$, respectively.

**Remark 3.2.6** It is clear to see that if we set $J = 0$, the linear fractional norm-bounded uncertainty reduces to a routine norm-bounded uncertainty. It is interesting to point out that in testing robust stability of an uncertain neutral SNNs, one has the impression that a less conservative result can be obtained by using the criterion.
formulated for a polytopic uncertainty than that of using the criterion derived for a norm-bounded uncertainty provided that the uncertainty can be handled as polytopic uncertainty and norm-bounded uncertainty respectively.

**Remark 3.2.7** In [4], authors discussed delay-interval dependent robust stability criteria for SNNs with linear fractional uncertainties and constructed descriptor form matrix in the Lyapunov-Krasovskii functional. In this section, we have constructed Lyapunov-Krasovskii functional depends on the delay of state and state derivative which is neutral type time-delay. This concept attempts at the first time to dealt with neutral SNNs involving polytopic and linear fractional uncertainties.

### 3.2.3 Numerical examples

In this section, five numerical examples are given to show the effectiveness of established theories.

**Example 3.2.1** Consider the neutral SNNs (3.1) and (3.4) with following matrices

\[
A^1 = \begin{bmatrix} 4 & 0 \\ 0 & 7 \end{bmatrix}, \quad A^2 = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}, \quad W_0^1 = \begin{bmatrix} 0.3 & -0.5 \\ 0.1 & 0.4 \end{bmatrix}, \quad W_0^2 = \begin{bmatrix} 0.2 & -0.2 \\ 0.1 & 0.6 \end{bmatrix},
\]
\[
W_1^1 = \begin{bmatrix} 0.4 & 0.2 \\ 0.1 & 0.3 \end{bmatrix}, \quad W_1^2 = \begin{bmatrix} 0.3 & 0.1 \\ 0.1 & 0.12 \end{bmatrix}, \quad C^1 = \begin{bmatrix} -0.3 & 0 \\ 0 & 0.3 \end{bmatrix}, \quad C^2 = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.1 \end{bmatrix},
\]
\[
D^1 = \begin{bmatrix} 0.7 & -0.1 \\ 0.5 & 0.1 \end{bmatrix}, \quad D^2 = \begin{bmatrix} 0.5 & -0.1 \\ 0.5 & 0.12 \end{bmatrix}, \quad W_2^1 = \begin{bmatrix} 0.2 & 0.6 \\ 0.5 & 0.2 \end{bmatrix}, \quad W_2^2 = \begin{bmatrix} 0.02 & 0.3 \\ 0.4 & 0.2 \end{bmatrix},
\]
\[
W_3^1 = \begin{bmatrix} 0.3 & -0.6 \\ 0 & 0.1 \end{bmatrix}, \quad W_3^2 = \begin{bmatrix} -0.3 & 0.6 \\ 0.2 & 0.1 \end{bmatrix}, \quad \bar{D} = \begin{bmatrix} -0.2 & 0 \\ 0 & 0.3 \end{bmatrix}, \quad L = \begin{bmatrix} 0.2 & 0 \\ 0 & 0.2 \end{bmatrix}.
\]

By Theorem 3.2.4 and Matlab LMI control toolbox given LMI is feasible for any finite constant allowable upper bound \( h_2 \) when \( h_1 = 100 \), and \( \mu = 0 \). Now let \( \mu = 3 \), \( h_1 = 0.5 \) and \( h_2 = 0.8 \) by applying Theorem 3.2.4, the following feasible solution
guarantees the robust stability of the NNs (3.1).

\[ P = \begin{bmatrix} 0.1663 & 0.3298 \\ 0.3298 & 1.3364 \end{bmatrix}, \quad R_1^1 = \begin{bmatrix} 0.0002 & 0.0004 \\ 0.0004 & 0.0018 \end{bmatrix}, \quad R_2^1 = 10^{-4} \times \begin{bmatrix} 0.0625 & 0.1691 \\ 0.1691 & 0.9844 \end{bmatrix}, \]

\[ R_3^1 = \begin{bmatrix} 0.0081 & 0.0180 \\ 0.0180 & 0.1175 \end{bmatrix}, \quad R_4^1 = \begin{bmatrix} 0.0081 & 0.0180 \\ 0.0180 & 0.1175 \end{bmatrix}, \quad R_5^1 = \begin{bmatrix} 1.5489 & 0.1090 \\ 0.1090 & 0.1297 \end{bmatrix}, \]

\[ R_1^2 = \begin{bmatrix} 0.0007 & 0.0019 \\ 0.0019 & 0.0064 \end{bmatrix}, \quad R_2^2 = 10^{-4} \times \begin{bmatrix} 0.0171 & -0.0139 \\ -0.0139 & 0.1353 \end{bmatrix}, \]

\[ R_3^2 = \begin{bmatrix} 0.0747 & 0.0830 \\ 0.0830 & 0.1901 \end{bmatrix}, \quad R_4^2 = \begin{bmatrix} 0.0747 & 0.0830 \\ 0.0830 & 0.1901 \end{bmatrix}, \quad R_5^2 = \begin{bmatrix} 0.7864 & -0.2023 \\ -0.2023 & 0.6647 \end{bmatrix}, \]

\[ Q_1 = \begin{bmatrix} 0.0317 & 0.0385 \\ 0.0385 & 0.1072 \end{bmatrix}, \quad Q_2 = \begin{bmatrix} 0.0931 & 0.2024 \\ 0.2024 & 2.1464 \end{bmatrix}, \quad Q_3 = \begin{bmatrix} 0.0088 & 0.0152 \\ 0.0152 & 0.0810 \end{bmatrix}, \]

\[ Q_4 = \begin{bmatrix} 0.0231 & 0.0393 \\ 0.0393 & 0.2096 \end{bmatrix}, \quad W_1 = \begin{bmatrix} 10.6507 & 0 \\ 0 & 9.6028 \end{bmatrix}, \quad W_2 = \begin{bmatrix} 4.4426 & 0 \\ 0 & 8.8177 \end{bmatrix}. \]

**Remark 3.2.8** For the above Example 3.2.1 the feasible solution is obtained with the maximum allowable upper bound \( h_2 \) by using algorithm in Remark 3.2.2. This implies the maximum allowable upper bound \( h_2 \) of delayed system and the corresponding feasible solution matrices are given in the above Example 3.2.1.

**Example 3.2.3** Consider the neutral SNNs (3.1) and (3.5) with following matrices

\[ A = \begin{bmatrix} 6 & 0 \\ 0 & 9 \end{bmatrix}, \quad W_0 = \begin{bmatrix} -4.1 & 0.8 \\ -0.2 & 0 \end{bmatrix}, \quad W_1 = \begin{bmatrix} -0.3 & 0.5 \\ -0.6 & -0.3 \end{bmatrix}, \quad C = \begin{bmatrix} -0.6 & 0 \\ 0 & -0.5 \end{bmatrix}, \]

\[ D = \begin{bmatrix} 0 & -0.4 \\ -0.6 & 0 \end{bmatrix}, \quad W_2 = \begin{bmatrix} 0.2 & 0 \\ 0 & 0.2 \end{bmatrix}, \quad W_3 = \begin{bmatrix} 0.2 & 0 \\ 0 & 0.2 \end{bmatrix}, \quad H = \begin{bmatrix} 0.2 \\ -0.2 \end{bmatrix}, \]

\[ \bar{D} = \begin{bmatrix} -0.3 & 0 \\ -0.2 & 0.4 \end{bmatrix}, \quad L = 0.5I, \quad T_1 = [0.1 \quad 0.3], \quad T_2 = [0.1 \quad -0.2], \]

\[ T_3 = [-0.4 \quad -0.3], \quad T_4 = T_5 = T_6 = T_7 = [0.2 \quad 0.2]. \]

Using Matlab LMI control toolbox and by Theorem 3.2.5, the given LMI is feasible for any finite constant allowable upper bound \( h_2 \) when \( h_1 = 100 \), \( \mu = 0 \) and \( J = 0.5 \). Now, let \( J = 0.5 \), \( \mu = 3 \), \( h_1 = 0.4 \) and \( h_2 = 0.7 \) by applying Theorem 3.2.5, it is found that the NN (3.1) is globally robustly asymptotically stable in the mean square,
the solutions of the LMI (3.22) are given as follows:

\[
P = \begin{bmatrix} 2.9652 & -0.8814 \\ -0.8814 & 2.1978 \end{bmatrix}, \quad R_1 = \begin{bmatrix} 0.0020 & -0.0000 \\ -0.0000 & 0.0021 \end{bmatrix},
\]

\[
R_2 = 10^{-3} \times \begin{bmatrix} 0.5240 & -0.2426 \\ -0.2426 & 0.6426 \end{bmatrix}, \quad R_3 = \begin{bmatrix} 0.5719 & -0.0993 \\ -0.0993 & 0.5300 \end{bmatrix},
\]

\[
R_4 = \begin{bmatrix} 0.5518 & -0.0774 \\ -0.0774 & 0.4206 \end{bmatrix}, \quad R_5 = \begin{bmatrix} 0.1667 & 0.0133 \\ 0.0133 & 0.0760 \end{bmatrix}, \quad Q_1 = \begin{bmatrix} 0.7778 & -0.3865 \\ -0.3865 & 0.6030 \end{bmatrix},
\]

\[
Q_2 = \begin{bmatrix} 2.5504 & -0.1899 \\ -0.1899 & 1.6996 \end{bmatrix}, \quad Q_3 = \begin{bmatrix} 0.4002 & -0.0567 \\ -0.0567 & 0.3040 \end{bmatrix},
\]

\[
Q_4 = \begin{bmatrix} 0.8868 & -0.1211 \\ -0.1211 & 0.6816 \end{bmatrix}, \quad W_1 = \begin{bmatrix} 14.2899 & 0 \\ 0 & 17.4851 \end{bmatrix}, \quad W_2 = \begin{bmatrix} 2.6357 & 0 \\ 0 & 0.2070 \end{bmatrix},
\]

\[
\epsilon_1 = 2.2747, \quad \epsilon_2 = 2.7481.
\]

**Example 3.2.4** Consider the following uncertain neutral SNNs

\[
d\left[ x(t) - \bar{D}x(t - \tau(t)) \right] = \left[ -A(t)x(t) + W_0(t)f(x(t)) + W_1(t)f(x(t - \tau(t))) \right]dt
\]

\[+ \left[ C(t)x(t) + D(t)x(t - \tau(t)) \right]dw(t),
\]

where

\[
A = \begin{bmatrix} 1.2 & 0 \\ 0 & 1.15 \end{bmatrix}, \quad W_0 = \begin{bmatrix} -0.1 & 0.4 \\ 0.2 & -0.5 \end{bmatrix}, \quad W_1 = \begin{bmatrix} 0.1 & -1 \\ -1.4 & 0.4 \end{bmatrix}, \quad C = \begin{bmatrix} 0.23 & 0.1 \end{bmatrix},
\]

\[
D = \begin{bmatrix} 0.1 & -0.2 \\ 0.2 & 0.3 \end{bmatrix}, \quad \bar{D} = \begin{bmatrix} 0.35 & 0 \\ 0.2 & 0.6 \end{bmatrix}, \quad H = \begin{bmatrix} 0.1 \\ 0.2 \end{bmatrix}, \quad T_1 = [0.1 \ 0.2],
\]

\[
T_2 = [0.5 \ 0.1], \quad T_3 = [-0.2 \ 0.2], \quad T_4 = [-0.1 \ 0.2], \quad T_5 = [0.3 \ 0.1], \quad L = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.
\]

Applying Theorem 3.2.5, we conclude that the given LMI is feasible for any finite constant allowable upper bound \( h_2 \) when \( J = 0, \ \mu = 0, \ \eta_1 = 0. \) From this, let \( J = 0, \ \mu = 3, \ \eta_1 = 0 \) and \( h_2 = 1.5 \) by applying Theorem 3.2.5, it is found that the NN (3.1) is globally robustly asymptotically stable in the mean square, the solutions
of the LMI (3.22) are given by

\[
P = \begin{bmatrix}
15.1117 & 1.9965 \\
1.9965 & 10.3481
\end{bmatrix}, \quad R_1 = \begin{bmatrix}
0.2923 & -0.0391 \\
-0.0391 & 0.2463
\end{bmatrix}, \quad R_2 = \begin{bmatrix}
0.0079 & 0.0006 \\
0.0006 & 0.0152
\end{bmatrix},
\]

\[
R_3 = \begin{bmatrix}
1.6126 & -0.0475 \\
-0.0475 & 1.3502
\end{bmatrix}, \quad R_4 = \begin{bmatrix}
1.5603 & -0.0308 \\
-0.0308 & 1.3368
\end{bmatrix}, \quad R_5 = \begin{bmatrix}
0.4415 & -0.1125 \\
-0.1125 & 0.3824
\end{bmatrix}
\]

\[
Q_1 = \begin{bmatrix}
8.6933 & 0.8893 \\
0.8893 & 2.7625
\end{bmatrix}, \quad Q_2 = \begin{bmatrix}
14.2354 & 1.0346 \\
1.0346 & 7.6599
\end{bmatrix}, \quad Q_3 = \begin{bmatrix}
0.5981 & -0.0132 \\
-0.0132 & 0.5022
\end{bmatrix},
\]

\[
Q_4 = \begin{bmatrix}
0.5981 & -0.0132 \\
-0.0132 & 0.5022
\end{bmatrix}, \quad W_1 = \begin{bmatrix}
4.9958 & 0 \\
0 & 5.7025
\end{bmatrix}, \quad W_2 = \begin{bmatrix}
9.5668 & 0 \\
0 & 16.2389
\end{bmatrix},
\]

\[
\epsilon_1 = 8.5697, \quad \epsilon_2 = 13.9520.
\]

This implies that proposed stability criteria gives a less conservative result that obtained by methods discussed [80].
3.3 Delay-interval dependent robust stability criteria for stochastic neural networks with linear fractional uncertainties

3.3.1 Problem description and preliminaries

Consider the following HNNs with parameter uncertainties and stochastic perturbations as follows:

\[
dx(t) = \left[-A(t)x(t) + W_0(t)f(x(t)) + W_1(t)f(x(t - \tau(t))) + E_1(t) \int_{t-r(t)}^t f(x(s)) ds\right] dt \\
+ \left[C(t)x(t) + D(t)x(t - \tau(t)) + W_2(t)f(x(t)) + W_3(t)f(x(t - \tau(t)))
+ E_2(t) \int_{t-r(t)}^t f(x(s)) ds\right] dw(t), \quad (3.23)
\]

\[
x(t) = \phi(t), \forall t \in \{-2\bar{h}, 0\}, \quad \bar{h} = \max\{h_2, \bar{r}\}, \quad \bar{r} = \max\{r(t)\}.
\]

To establish the stability results assume that the following hold.

(A3) The activation function \( f_q \) is bounded, continuously differentiable with \( f(0) = 0 \) and satisfy the Lipschitz condition

\[
|f_q(x_1) - f_q(x_2)| \leq l_q|x_1 - x_2|, \quad \forall x_1, x_2 \in \mathbb{R}, \quad q = 1, \ldots, n.
\]

where \( L = \text{diag}(l_1, l_2, \ldots, l_n) > 0 \) is a positive diagonal matrix. Then by (A3), it follows that

\[
|f_q(x)| \leq l_q|x|, \quad \forall x \in \mathbb{R}, \quad q = 1, \ldots, n.
\]

(A4) The time-varying delays \( \tau(t) \) and \( r(t) \) are satisfy

\[
0 \leq h_1 \leq \tau(t) \leq h_2, \quad \dot{\tau}(t) \leq \mu < 1, \quad 0 \leq r(t) \leq \bar{r},
\]

where \( h_1, h_2, \mu \) and \( \bar{r} \) are constants.
3.3.2 Global robust stability results

In this section, delay-dependent stability condition is derived for the uncertain NNs (3.23) with time-varying delays. Now defining two new state variables for the stochastic NNs (3.23),

\[ y(t) = \left[ -A(t)x(t) + W_0(t)f(x(t)) + W_1(t)f(x(t - \tau(t))) + E_1(t) \int_{t-\tau(t)}^{t} f(x(s))ds \right], \]

and

\[ g(t) = \left[ C(t)x(t) + D(t)x(t - \tau(t)) + W_3(t)f(x(t)) \right. \]
\[ + W_3(t)f(x(t - \tau(t))) + E_2(t) \int_{t-\tau(t)}^{t} f(x(s))ds \],

then system (3.23) becomes

\[ dx(t) = y(t)dt + g(t)dw(t). \]

Moreover, the following equality holds,

\[ x(t) - x(t - \tau(t)) = \int_{t-\tau(t)}^{t} dx(s) = \int_{t-\tau(t)}^{t} y(s)ds + \int_{t-\tau(t)}^{t} g(s)dw(s). \]  \( (3.24) \)

**Theorem 3.3.1** Consider NNs (3.23) satisfying assumptions (A3) and (A4). The equilibrium point of SNNS (3.23) is globally asymptotically stable in the mean square if there exist symmetric matrices \( P_1 > 0, \) \( R_\ell > 0, \) \( (\ell = 1, 2, 3), \) \( Q_j > 0 \) \( (j = 1, 2, 3, 4), \) diagonal matrix \( K > 0, \) \( K_1 > 0, \) \( K_2 > 0 \) and for any matrices \( P_i, (i = 2, ..., 46) \) such that the LMI holds

\[
\Pi_1 = \begin{bmatrix}
\hat{Q} & -N & -M & -S \\
* & -\frac{1}{k_2}R_2 & 0 & 0 \\
* & * & -\frac{1}{k_2-h_1}R_3 & 0 \\
* & * & * & -\frac{1}{k_2-h_1}(R_2 + R_3)
\end{bmatrix} < 0
\]  \( (3.25) \)
where 

\[
\ddot{\Omega} = \begin{bmatrix}
\Omega_{11} & \Omega_{12} & \Omega_{13} & \Omega_{14} & \Omega_{15} & \Omega_{16} & \Omega_{17} & \Omega_{18} & \Omega_{19} \\
\ddot{\Omega}_{21} & \ddot{\Omega}_{22} & \ddot{\Omega}_{23} & \ddot{\Omega}_{24} & \ddot{\Omega}_{25} & \ddot{\Omega}_{26} & \ddot{\Omega}_{27} & \ddot{\Omega}_{28} & \ddot{\Omega}_{29} \\
\ddot{\Omega}_{31} & \ddot{\Omega}_{32} & \ddot{\Omega}_{33} & \ddot{\Omega}_{34} & \ddot{\Omega}_{35} & \ddot{\Omega}_{36} & \ddot{\Omega}_{37} & \ddot{\Omega}_{38} & \ddot{\Omega}_{39} \\
\ddot{\Omega}_{41} & \ddot{\Omega}_{42} & \ddot{\Omega}_{43} & \ddot{\Omega}_{44} & \ddot{\Omega}_{45} & \ddot{\Omega}_{46} & \ddot{\Omega}_{47} & \ddot{\Omega}_{48} & \ddot{\Omega}_{49} \\
\ddot{\Omega}_{51} & \ddot{\Omega}_{52} & \ddot{\Omega}_{53} & \ddot{\Omega}_{54} & \ddot{\Omega}_{55} & \ddot{\Omega}_{56} & \ddot{\Omega}_{57} & \ddot{\Omega}_{58} & \ddot{\Omega}_{59} \\
\ddot{\Omega}_{61} & \ddot{\Omega}_{62} & \ddot{\Omega}_{63} & \ddot{\Omega}_{64} & \ddot{\Omega}_{65} & \ddot{\Omega}_{66} & \ddot{\Omega}_{67} & \ddot{\Omega}_{68} & \ddot{\Omega}_{69} \\
\ddot{\Omega}_{71} & \ddot{\Omega}_{72} & \ddot{\Omega}_{73} & \ddot{\Omega}_{74} & \ddot{\Omega}_{75} & \ddot{\Omega}_{76} & \ddot{\Omega}_{77} & \ddot{\Omega}_{78} & \ddot{\Omega}_{79} \\
\ddot{\Omega}_{81} & \ddot{\Omega}_{82} & \ddot{\Omega}_{83} & \ddot{\Omega}_{84} & \ddot{\Omega}_{85} & \ddot{\Omega}_{86} & \ddot{\Omega}_{87} & \ddot{\Omega}_{88} & \ddot{\Omega}_{89} \\
\ddot{\Omega}_{91} & \ddot{\Omega}_{92} & \ddot{\Omega}_{93} & \ddot{\Omega}_{94} & \ddot{\Omega}_{95} & \ddot{\Omega}_{96} & \ddot{\Omega}_{97} & \ddot{\Omega}_{98} & \ddot{\Omega}_{99}
\end{bmatrix} \quad (3.26)
\]

with

\[
\Omega_{11} = Q_1 + Q_2 + Q_3 + P_2 + P_2^T - P_{29}A - A^TP_{29}^T + P_{38}C + C^TP_{38}^T, \quad \Omega_{12} = -P_2 + P_3^T \\
- P_{11} + P_{20} - A^TP_{30}^T + C^TP_{39}^T + P_{38}D, \quad \Omega_{13} = P_4^T + P_{11} - A^TP_{31}^T + C^TP_{40}^T, \\
\Omega_{14} = P_5^T - P_{20} - A^TP_{32}^T + C^TP_{41}^T, \quad \Omega_{15} = P_1 + P_6^T - P_{29} - A^TP_{33}^T + C^TP_{42}^T, \\
\Omega_{16} = P_7^T - A^TP_{34}^T + P_{38} + C^TP_{43}^T, \quad \Omega_{17} = P_8^T - P_{29} - W_0 + A^TP_{35}^T + P_{38}W_2 + C^TP_{44}^T + \text{LK}_1, \\
\Omega_{18} = P_9^T - P_{29}W_1 - A^TP_{36}^T + P_{38}W_3 + C^TP_{45}^T, \quad \Omega_{19} = P_1^T - P_3 - P_{12} - P_{10}^T, \\
- A^TP_{37}^T + P_{38}E_2 + C^TP_{46}^T, \quad \Omega_{22} = -(1 - \mu)Q_1 - P_3^T - P_3 - P_{12} - P_{10}^T, \\
+ P_{21} + P_{22}^T + P_{39}D + D^TP_{39}^T, \quad \Omega_{23} = -P_4^T + P_{21} - P_{22}^T + P_{22}^T + D^TP_{40}^T, \\
\Omega_{24} = -P_5^T - P_{14}^T + P_{21}^T + D^TP_{41}^T, \quad \Omega_{25} = -P_6^T - P_{30}^T + P_{30}^T + D^TP_{42}^T, \\
\Omega_{26} = -P_7^T - P_{16}^T - P_{39}^T + D^TP_{43}^T, \quad \Omega_{27} = -P_8^T + P_7^T + P_26^T + P_{30}W_0 + P_{30}W_2, \\
+ C^TP_{44}^T, \quad \Omega_{28} = -P_9^T - P_{18}^T + P_{27}^T + P_{30}W_1 + P_{39}W_3 + D^TP_{45} + \text{LK}_2, \\
\Omega_{29} = -P_{10}^T - P_{19}^T + P_{28}^T + P_{36}E_1 + P_{39}E_2 + D^TP_{46}^T, \quad \Omega_{33} = Q_2 + P_{13} + P_{13}^T, \\
\Omega_{34} = P_{14}^T - P_{22}, \quad \Omega_{35} = P_{15}^T - P_{31}, \quad \Omega_{36} = P_{16}^T - P_{40}, \quad \Omega_{37} = P_{17}^T + P_{31}W_0 + P_{40}W_2, \\
\Omega_{38} = P_{18}^T + P_{31}W_1 + P_{40}W_3, \quad \Omega_{39} = P_{19}^T + P_{31}E_1 + P_{40}E_2, \quad \Omega_{44} = Q_3 - P_{23} - P_{23}^T, \\
\Omega_{45} = -P_{24}^T - P_{32}, \quad \Omega_{46} = -P_{25}^T - P_{41}, \quad \Omega_{47} = -P_{26}^T + P_{39}W_0 + P_{41}W_2, \quad \Omega_{48} = -P_{27}^T + P_{32}W_1 + P_{41}W_3, \quad \Omega_{49} = -P_{28}^T + P_{32}E_1 + P_{41}E_2, \quad \Omega_{55} = h_2R_2 + (h_2 - h_1)R_3, \\
- P_{33} - P_{33}^T, \quad \Omega_{56} = -P_{34}^T - P_{42}, \quad \Omega_{57} = K^T + P_{38}W_0 - P_{35}^T + P_{42}W_2, \\
\end{align*}
\[
\Omega_{68} = P_{39}W_1 - P_{36}^T + P_{42}W_3, \quad \Omega_{59} = P_{33}E_1 - P_{37}^T + P_{42}E_2, \quad \Omega_{66} = P_1 + K - P_{43} - P_{43}^T, \\
\Omega_{67} = P_{34}W_0 + P_{43}W_2 - P_{44}^T, \quad \Omega_{68} = P_{34}W_1 + P_{43}W_3 - P_{45}^T, \quad \Omega_{69} = P_{34}E_1 + P_{43}E_2 - P_{46}^T, \\
\Omega_{77} = -2K_1 + \tilde{r}R_1 + Q_4 + P_{35}W_0 + W_0^TP_3^T + P_{44}W_2 + W_2^TP_4^T, \\
\Omega_{78} = P_{35}W_1 + W_0^TP_{36}^T + P_{44}W_3 + W_2^TP_{45}^T, \quad \Omega_{79} = P_{35}E_1 + W_0^TP_{37}^T + P_{44}E_2 + W_2^TP_{46}^T, \\
\Omega_{88} = -(1 - \mu)Q_4 + P_{36}W_1 + W_1^TP_{36}^T + P_{46}W_3 + W_3^TP_{45}^T - 2K_2, \quad \Omega_{89} = P_{36}E_1 + W_1^TP_{37}^T \\
+ P_{45}E_2 + W_3^TP_{46}^T, \quad \Omega_{99} = P_{37}E_1 + E_1^TP_{37}^T + P_{46}E_2 + E_2^TP_{46}^T - \frac{1}{R_1}\tilde{r},
\]

\[N = \begin{bmatrix} P_2 & P_3 & P_4 & P_5 & P_6 & P_7 & P_8 & P_9 & P_{10} \end{bmatrix}^T,
\]

\[M = \begin{bmatrix} P_{11} & P_{12} & P_{13} & P_{14} & P_{15} & P_{16} & P_{17} & P_{18} & P_{19} \end{bmatrix}^T,
\]

\[S = \begin{bmatrix} P_{20} & P_{21} & P_{22} & P_{23} & P_{24} & P_{25} & P_{26} & P_{27} & P_{28} \end{bmatrix}^T,
\]

\[U = \begin{bmatrix} P_{29} & P_{30} & P_{31} & P_{32} & P_{33} & P_{34} & P_{35} & P_{36} & P_{37} \end{bmatrix}^T,
\]

\[V = \begin{bmatrix} P_{38} & P_{39} & P_{40} & P_{41} & P_{42} & P_{43} & P_{44} & P_{45} & P_{46} \end{bmatrix}^T.
\]

**Proof:** Consider the Lyapunov-Krasovskii functional

\[V(x_t, t) = V_1(x_t, t) + V_2(x_t, t) + V_3(x_t, t) + V_4(x_t, t) + V_5(x_t, t) \quad (3.27)
\]

where

\[V_1(x_t, t) = \xi^T(t)EP\xi(t),
\]

\[V_2(x_t, t) = 2\sum_{p=1}^{n} k_p \int_{0}^{T} f_q(x(s))ds,
\]

\[V_3(x_t, t) = \int_{-\tau}^{0} \int_{t+\theta}^{t} f(x(s))^T R_1 f(x(s)) d\sigma d\theta,
\]

\[V_4(x_t, t) = \int_{-h_2}^{0} \int_{t+\theta}^{t} y^T(s) R_2 y(s) d\sigma d\theta + \int_{-h_2}^{0} \int_{t+\theta}^{t} y^T(s) R_3 y(s) d\sigma d\theta.
\]

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\[ V_5(x_t, t) = \int_{t-r(t)}^{t} x^T(s)Q_1 x(s) ds + \int_{t-h_2}^{t} x^T(s)Q_2 x(s) ds \\
+ \int_{t-h_2}^{t} x^T(s)Q_3 x(s) ds + \int_{t-r(t)}^{t} f^T(x(s))Q_4 f(x(s)) ds \]

with

\[ E^T = \begin{bmatrix}
I & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}, \]

\[ P = \begin{bmatrix}
P_1 & P_2 & P_3 & P_4 & P_5 & P_6 & P_7 & P_8 & P_9 & P_{10} \\
P_{11} & P_{12} & P_{13} & P_{14} & P_{15} & P_{16} & P_{17} & P_{18} & P_{19} \\
P_{20} & P_{21} & P_{22} & P_{23} & P_{24} & P_{25} & P_{26} & P_{27} & P_{28} \\
P_{29} & P_{30} & P_{31} & P_{32} & P_{33} & P_{34} & P_{35} & P_{36} & P_{37} \\
P_{38} & P_{39} & P_{40} & P_{41} & P_{42} & P_{43} & P_{44} & P_{45} & P_{46}
\end{bmatrix}, \]

\[ \xi^T(t) = \begin{bmatrix}
x^T(t) & x^T(t-h_2) & x^T(t-h_1) & x^T(t-h_2) & y^T(t) \\
g^T(t) & f^T(x(t)) & f^T(x(t-h_2)) & f^T(x(t-h_1)) & f^T(x(s))ds
\end{bmatrix} \]

and

\[ EP = P^T E^T \geq 0. \]

It is noted that \( \xi^T(t)EP\xi(t) \) is actually \( x^T(t)P_1 x(t) \). Then, it can be obtained by Itô’s formula that

\[ dV(x_t, t) = \mathcal{L}V(x_t, t)dt + 2x^T(t)P_1 g(t) dw(t) \tag{3.28} \]

where

\[ \mathcal{L}V_1(x_t, t) = 2x^T(t)P_1 y(t) + g^T(t)P_1 g(t). \]
On the other hand, from (3.24), the following equations are true

\[ \eta_1(t) = [x(t) - x(t - \tau(t))] - \int_{t-\tau(t)}^{t} y(s)ds - \int_{t-\tau(t)}^{t} g(s)dw(s) = 0, \]

\[ \eta_2(t) = [x(t - h_1) - x(t - \tau(t))] - \int_{t-h_1}^{t-\tau(t)} y(s)ds - \int_{t-h_1}^{t-\tau(t)} g(s)dw(s) = 0, \]

\[ \eta_3(t) = [x(t - \tau(t)) - x(t - h_2) - \int_{t-h_2}^{t-\tau(t)} y(s)ds - \int_{t-h_2}^{t-\tau(t)} g(s)dw(s) = 0, \]

\[ \eta_4(t) = [-A(t)x(t) + B(t)f(x(t)) + C(t)f(x(t - \tau(t))) + \int_{t-\tau(t)}^{t} f(x(s))ds - y(t) = 0, \]

\[ \eta_5(t) = [A_1(t)x(t) + B_1(t)f(x(t)) + C_1(t)f(x(t - \tau(t))) + \int_{t-\tau(t)}^{t} f(x(s))ds - g(t) = 0. \]

Therefore,

\[ \mathcal{L}V_1(x, t) = 2x^T(t)P_1y(t) + g^T(t)P_1g(t) + 2\xi^T(t)P_\xi, \]

\[ \mathcal{L}V_2(x, t) = 2f^T(x(t))Ky(t) + g^T(t)Kg(t), \]

\[ \mathcal{L}V_3(x, t) = \bar{r}f^T(x(t))R_1f(x(t)) - \int_{t-\tau(t)}^{t} f^T(x(s))R_1f(x(s))ds, \]

\[ \mathcal{L}V_4(x, t) = h_2y^T(t)R_2y(t) - \int_{t-h_2}^{t} y^T(s)R_2y(s)ds + (h_2 - h_1)y^T(t)R_3y(t) \]

\[ - \int_{t-h_2}^{t-h_1} y^T(s)R_3y(s)ds, \]

\[ \mathcal{L}V_5(x, t) = x^T(t)Q_1x(t) - (1 - \mu)x^T(t - \tau(t))Q_1x(t - \tau(t)) + x^T(t)Q_2x(t) \]

\[ - x^T(t - h_1)Q_2x(t - h_1) + x^T(t)Q_3x(t) - x^T(t - h_2)Q_3x(t - h_2) \]

\[ + f^T(x(t))Q_4f(x(t)) - (1 - \mu)f^T(x(t - \tau(t)))Q_4f(x(t - \tau(t))). \]
It is obvious that

\[ 2x^T(t)LK_1f(x(t)) - 2f^T(x(t))K_1f(x(t)) \geq 0, \quad (3.34) \]

\[ 2x^T(t-\tau(t))LK_2f(x(t-\tau(t))) - 2f^T(x(t-\tau(t)))K_2f(x(t-\tau(t))) \geq 0. \quad (3.35) \]

Then by Lemma 1.16.11 and using \(0 \leq h_1 \leq \tau(t) \leq h_2\) and \(0 < r(t) \leq \tilde{r}\), we have

\[ -\int_{t-r(t)}^{t} f^T x(s) R_1 f x(s) ds \leq -\frac{1}{\tilde{r}} \left[ \int_{t-r(t)}^{t} f(x(s)) ds \right]^T R_1 \left[ \int_{t-r(t)}^{t} f(x(s)) ds \right], \quad (3.36) \]

\[ -\int_{t-r(t)}^{t} y^T(s) R_2 y(s) ds \leq -\frac{1}{h_2} \left[ \int_{t-r(t)}^{t} y(s) ds \right]^T R_2 \left[ \int_{t-r(t)}^{t} y(s) ds \right], \quad (3.37) \]

\[ -\int_{t-h_2}^{t-\tau(t)} y^T(s)(R_2 + R_3)y(s) ds \leq -\frac{1}{h_2 - h_1} \left[ \int_{t-h_2}^{t-\tau(t)} y(s) ds \right]^T (R_2 + R_3) \left[ \int_{t-h_2}^{t-\tau(t)} y(s) ds \right], \quad (3.38) \]

\[ -\int_{t-r(t)}^{t-h_1} y^T(s) R_3 y(s) ds \leq -\frac{1}{h_2 - h_1} \left[ \int_{t-r(t)}^{t-h_1} y(s) ds \right]^T R_3 \left[ \int_{t-r(t)}^{t-h_1} y(s) ds \right]. \quad (3.39) \]

Substituting (3.29)-(3.39) into (3.28), it follows that

\[ dV(x_t, t) \leq \psi^T(t) P_1 \psi(t) dt + \zeta(dw(t)), \]

where \( P_1 \) is defined in Theorem 3.3.1 with

\[ \psi^T(t) = \left[ \xi^T(t) \int_{t-\tau(t)}^{t} y^T(s) ds \int_{t-r(t)}^{t-h_1} y^T(s) ds \int_{t-h_2}^{t-r(t)} y^T(s) ds \right]. \quad (3.40) \]

\[ \zeta(dw(t)) = -2\xi^T(t) N \int_{t-\tau(t)}^{t} g(s) dw(s) - 2\xi^T(t) M \int_{t-r(t)}^{t-h_1} g(s) dw(s) \]

\[ -2\xi^T(t) S \int_{t-h_2}^{t-r(t)} g(s) dw(s) + 2\xi^T(t) P_1 g(t) dw(t) \]

According to \( P_1 < 0 \) and there exists a scalar \( \alpha > 0 \) such that

\[ P_1 + \text{diag}\{\alpha I_n, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0\} < 0. \]

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Hence we have
\[ E \left[ \frac{dV(x_t, t)}{dt} \right] = E(\psi^T(t) \Pi_1 \psi(t)) \leq -\alpha E|x(t)|^2. \]

Thus if \( \Pi_1 < 0 \), then the stochastic system (3.23) is asymptotically stable in the mean square. The proof is completed.

**Theorem 3.3.2** Consider NNs (3.23) satisfying assumptions (A3) and (A4). The equilibrium point of SNNs (3.23) is linear fractional norm-bounded uncertainties (3.5) is globally asymptotically stable in the mean square if there exist scalars \( \epsilon_1 > 0, \epsilon_2 > 0 \), some symmetric matrices \( P_1 > 0, R_\ell > 0, (\ell = 1, 2, 3), Q_j > 0, (j = 1, 2, 3, 4) \), diagonal matrices \( K > 0, K_1 > 0, K_2 > 0 \) and for any matrices \( P_\nu, (\nu = 2, ..., 46) \) such that the LMI holds
\[
\begin{bmatrix}
\hat{Q} & -N & -M & -S & S_1 & \epsilon_1 N_1^T & S_2 & \epsilon_2 N_2^T \\
* & -\frac{1}{h_2} R_2 & 0 & 0 & 0 & 0 & 0 \\
* & * & -\frac{1}{h_2-h_1} R_3 & 0 & 0 & 0 & 0 \\
* & * & * & -\frac{1}{h_2-h_1} (R_2 + R_3) & 0 & 0 & 0 \\
* & * & * & * & -\epsilon_1 I & \epsilon_1 J^T & 0 & 0 \\
* & * & * & * & * & -\epsilon_1 I & 0 & 0 \\
* & * & * & * & * & * & -\epsilon_2 I & \epsilon_2 J^T \\
* & * & * & * & * & * & * & -\epsilon_2 I
\end{bmatrix} < 0 \quad (3.41)
\]

where \( \hat{Q} \) is defined in (3.26).

**Proof:** Assume that inequality (3.41) holds. From (3.41), it can be easily obtained that
\[
\Psi = \begin{bmatrix}
\Pi_1 & S_1 & \epsilon_1 N_1^T & S_2 & \epsilon_2 N_2^T \\
* & -\epsilon_1 I & \epsilon_1 J^T & 0 & 0 \\
* & * & -\epsilon_1 I & 0 & 0 \\
* & * & * & -\epsilon_2 I & \epsilon_2 J^T \\
* & * & * & * & -\epsilon_2 I
\end{bmatrix} < 0,
\]

where
\[
S_1 = \begin{bmatrix}
HP_{29} & HP_{30} & HP_{31} & HP_{32} & HP_{33} & HP_{34} & HP_{35} & HP_{36} & HP_{37} & 0 & 0 & 0
\end{bmatrix}^T,
\]
\[
N_1 = \begin{bmatrix}
-E_1 & 0 & 0 & 0 & 0 & 0 & E_2 & E_3 & E_4 & 0 & 0 & 0
\end{bmatrix},
\]
\[
S_2 = \begin{bmatrix}
HP_{38} & HP_{39} & HP_{40} & HP_{41} & HP_{42} & HP_{43} & HP_{44} & HP_{45} & HP_{46} & 0 & 0 & 0
\end{bmatrix}^T,
\]
\[
N_2 = \begin{bmatrix}
E_5 & E_6 & 0 & 0 & 0 & 0 & E_7 & E_8 & E_9 & 0 & 0 & 0
\end{bmatrix}
\]

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and $\Pi_1$ in defined in (3.25). Thus, $\Psi = \Pi_1 + S_1A(t)N_1 + N_1^T A(t)S_1^T + S_2A(t)N_2 + N_2^T A(t)S_2^T < 0$ holds according to Lemma 1.16.13. It can be verified that $\Psi$ is exactly the same $\Pi_1$ of (3.25) when $A$, $B$, $C$, $D$, $A_0$, $A_1$, $B_1$, $C_1$ and $D_1$ are replaced by $A + HA(t)T_1$, $B + HA(t)T_2$, $C + HA(t)T_3$, $D + HA(t)T_4$, $A_0 + HA(t)T_5$, $A_1 + HA(t)T_6$, $B_1 + HA(t)T_7$, $C_1 + HA(t)T_8$ and $D_1 + HA(t)T_9$ respectively.

**Remark 3.3.3** It is clear to see that if we set $J = 0$, the linear fractional norm-bounded uncertainties reduced to the routine norm-bounded uncertainties. Therefore, one can easily derive a corresponding results for routine norm-bounded uncertainties from Theorem 3.3.2.

Now, the robust stability analysis is discussed for the following uncertain SNNs with time-varying delays through the next theorem.

$$
\frac{dx(t)}{dt} = \left[ -A(t)x(t) + B(t)f(x(t)) + C(t)f(x(t - \tau(t))) \right] dt
+ \left[ A_0(t)x(t) + A_1(t)x(t - \tau(t)) + B_1(t)f(x(t)) + C_1(t)f(x(t - \tau(t))) \right] dw(t)
$$

(3.42)

where the time-delay $\tau(t)$ satisfies $0 \leq \tau_1 \leq \tau(t) \leq \tau_2$, $\dot{\tau}(t) \leq \mu$. Then, we have the following result.

**Theorem 3.3.4** Consider SNNs (3.42) satisfying assumptions (A3) and (A4). The equilibrium point of SNNs (3.42) is linear fractional norm-bounded uncertainties (3.5) is globally asymptotically stable in the mean square if there exist scalars $\varepsilon_1 > 0$, $\varepsilon_2 > 0$, some symmetric matrices $P_1 > 0$, $R_0 > 0$ ($\varphi = 2, 3$), $Q_j > 0$ ($j = 1, 2, 3, 4$) diagonal matrices $K > 0$, $K_1 > 0$, $K_2 > 0$ and for any matrices $P_i$ ($\iota = 2, ..., 41$) such that the LMI holds

\[
\begin{bmatrix}
\Omega & -\bar{N} & -\bar{M} & -\bar{S} & S_1 & \varepsilon_1 N_1^T & \bar{S}_2 & \varepsilon_2 N_2^T \\
* & -\frac{1}{h_2} R_2 & 0 & 0 & 0 & 0 & 0 & 0 \\
* & * & -\frac{1}{h_2 - h_1} R_3 & 0 & 0 & 0 & 0 & 0 \\
* & * & * & -\frac{1}{h_2 - h_1} (R_2 + R_3) & 0 & 0 & 0 & 0 \\
* & * & * & * & -\varepsilon_1 I & \varepsilon_1 J^T & 0 & 0 \\
* & * & * & * & * & -\varepsilon_1 I & 0 & 0 \\
* & * & * & * & * & * & -\varepsilon_2 I & \varepsilon_2 J^T \\
* & * & * & * & * & * & * & -\varepsilon_2 I \\
\end{bmatrix} < 0
\] (3.43)
where \( \tilde{\Omega} = (\tilde{\Omega})_{a \times b} (a, b = 1, 2, \ldots, 8) \) with

\[
\begin{align*}
\tilde{\Omega}_{11} &= Q_1 + Q_2 + Q_3 + P_2 + P_2^T - P_{26} A - A^T P_{26}^T + P_{34} C + C^T P_{34}^T, \\
&\quad - P_{10} + P_{18} - A^T P_{27}^T + C^T P_{35}^T + P_{34} D, \\
\tilde{\Omega}_{12} &= P_4^T + P_{10} - A^T P_{28}^T + C^T P_{36}^T, \\
\tilde{\Omega}_{14} &= P_5^T - P_{18} - A^T P_{29}^T + C^T P_{37}^T, \\
\tilde{\Omega}_{15} &= P_7 + P_6^T - P_{26} - A^T P_{30}^T + C^T P_{38}^T, \\
\tilde{\Omega}_{16} &= P_7^T - A^T P_{31}^T - P_{34} + C^T P_{39}^T, \\
&\quad + L K_1, \\
\tilde{\Omega}_{18} &= P_0^T + P_{26} W_1 - A^T P_{33}^T + P_{34} W_3 + A_0^T P_{41}^T, \\
\tilde{\Omega}_{22} &= -(1 - \mu) Q_1 - P_3^T, \\
- P_3 - P_{11} - P_9 + P_{19} + P_{10}^T + P_{35} D + D^T P_{35}^T, \\
\tilde{\Omega}_{24} &= - P_5^T - P_{13} - P_9 + P_{21}^T + D^T P_{37}^T, \\
\tilde{\Omega}_{25} &= - P_6^T - P_{14} - P_{27}, \\
\tilde{\Omega}_{26} &= - P_7^T - P_{15} + P_{35} + P_{23} + D^T P_{39}^T, \\
\tilde{\Omega}_{27} &= - P_8^T - P_{16} + P_{24}, \\
\tilde{\Omega}_{28} &= - P_9^T - P_{17} + P_{25} + P_{27} W_1 + P_{35} W_3 + A_1^T P_{41}^T, \\
&\quad + L K_2, \\
\tilde{\Omega}_{33} &= - Q_2 + P_{12} + P_{13}^T, \\
\tilde{\Omega}_{34} &= P_{10}^T + P_{13} - P_{20}, \\
\tilde{\Omega}_{35} &= P_{14}^T - P_{28}, \\
\tilde{\Omega}_{36} &= P_{15}^T - P_{26}, \\
\tilde{\Omega}_{37} &= P_{16}^T + P_{28} W_0 + P_{36} W_2, \\
\tilde{\Omega}_{38} &= P_{17}^T + P_{28} W_1 + P_{36} W_3, \\
\tilde{\Omega}_{44} &= - Q_3 - P_{21} - P_{21}^T, \\
\tilde{\Omega}_{45} &= - P_{22} - P_{29}, \\
\tilde{\Omega}_{46} &= - P_{23}^T - P_{37}, \\
\tilde{\Omega}_{47} &= - P_{24}^T + P_{29} W_0 + P_{37} W_2, \\
&\quad + P_{37} W_2, \\
\tilde{\Omega}_{48} &= - P_{25}^T + P_{29} W_1 + P_{47} W_3, \\
\tilde{\Omega}_{55} &= h_2 R_2 + (h_2 - h_1) R_3 - P_{30} - P_{30}^T, \\
\tilde{\Omega}_{56} &= - P_{31}^T - P_{38}, \\
\tilde{\Omega}_{57} &= K^T + P_{30} W_0 - P_{32}^T + P_{39} W_2, \\
\tilde{\Omega}_{58} &= P_{30} W_1 - P_{33}^T + P_{38} W_3, \\
\tilde{\Omega}_{66} &= P_1 + K - P_{39} - P_{39}^T, \\
\tilde{\Omega}_{67} &= P_{41} W_0 + P_{39} W_2 - P_{40}^T, \\
\tilde{\Omega}_{68} &= P_{31} W_1 + P_{39} W_3 - P_{41}^T, \\
\tilde{\Omega}_{77} &= -2 K_1 + Q_4 + P_{32} W_0 + W_0^T P_{32}^T + P_{40} W_2 + W_2^T P_{40}^T, \\
\tilde{\Omega}_{78} &= P_{32} W_1 + W_1^T P_{33} + P_{41} W_3 + W_3^T P_{41}^T - L K_2,
\end{align*}
\]
\[
\tilde{\Omega}_{88} = - (1 - \mu) Q_4 + P_{33} W_1 + W_1^T P_{33} + P_{41} W_3 + W_3^T P_{41} - L K_2,
\]
\[
\tilde{N} = \begin{bmatrix} P_2 & P_3 & P_4 & P_5 & P_6 & P_7 & P_8 & P_9 \end{bmatrix}^T,
\]
\[
\tilde{M} = \begin{bmatrix} P_{10} & P_{11} & P_{12} & P_{13} & P_{14} & P_{15} & P_{16} & P_{17} \end{bmatrix}^T,
\]

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\[
\tilde{S} = [P_{18} \quad P_{19} \quad P_{20} \quad P_{21} \quad P_{22} \quad P_{23} \quad P_{24} \quad P_{25}]^T,
\]
\[
\tilde{U} = [P_{26} \quad P_{27} \quad P_{28} \quad P_{29} \quad P_{30} \quad P_{31} \quad P_{32} \quad P_{33}]^T,
\]
\[
\tilde{V} = [P_{34} \quad P_{35} \quad P_{36} \quad P_{37} \quad P_{38} \quad P_{39} \quad P_{40} \quad P_{41}]^T,
\]
\[
\tilde{S}_1 = [HP_{26} \quad HP_{27} \quad HP_{28} \quad HP_{29} \quad HP_{30} \quad HP_{31} \quad HP_{32} \quad HP_{33}]^T,
\]
\[
\tilde{N}_1 = [-E_1 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad E_2 \quad E_3],
\]
\[
\tilde{S}_2 = [HP_{34} \quad HP_{35} \quad HP_{36} \quad HP_{37} \quad HP_{38} \quad HP_{39} \quad HP_{40} \quad HP_{41}]^T,
\]
\[
\tilde{N}_2 = [E_4 \quad E_5 \quad 0 \quad 0 \quad 0 \quad 0 \quad E_6 \quad E_7].
\]

**Proof:** Consider the Lyapunov-Krasovskii functional

\[
V(x_t, t) = V_1(x_t, t) + V_2(x_t, t) + V_3(x_t, t) + V_4(x_t, t)
\]

where

\[
V_1(x_t, t) = \xi^T(t)EP \xi(t),
\]
\[
V_2(x_t, t) = 2 \sum_{p=1}^{n} k_p \int_{0}^{x_p} f_q(x(s))ds,
\]
\[
V_3(x_t, t) = \int_{t-h_4}^{t} x^T(s)Q_1 x(s)ds + \int_{t-h_1}^{t} x^T(s)Q_2 x(s)ds
\]
\[
+ \int_{t-h_2}^{t} x^T(s)Q_3 x(s)ds + \int_{t-h_4}^{t} f^T(x(s))Q_4 f(x(s))ds,
\]
\[
V_4(x_t, t) = \int_{-h_2}^{0} \int_{t+\theta}^{t} y^T(s)R_2 y(s)dsd\theta + \int_{-h_4}^{0} \int_{t+\theta}^{t} y^T(s)R_3 y(s)dsd\theta,
\]

with

\[
E^T = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix},
\]

\[
P = \begin{bmatrix}
P_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
P_2 & P_3 & P_4 & P_5 & P_6 & P_7 & P_8 & P_9 \\
P_{10} & P_{11} & P_{12} & P_{13} & P_{14} & P_{15} & P_{16} & P_{17} \\
P_{18} & P_{19} & P_{20} & P_{21} & P_{22} & P_{23} & P_{24} & P_{25} \\
P_{26} & P_{27} & P_{28} & P_{29} & P_{30} & P_{31} & P_{32} & P_{33} \\
P_{34} & P_{35} & P_{36} & P_{37} & P_{38} & P_{39} & P_{40} & P_{41}
\end{bmatrix}
\]

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and
\[
\xi^T(t) = \begin{bmatrix} x^T(t) & x^T(t - \tau(t)) & x^T(t - h_1) & x^T(t - h_2) & y^T(t) \\
g^T(t) & f^T(x(t)) & f^T(x(t - \tau(t))) \end{bmatrix}.
\]

The remaining part of the proof follows immediately from Theorem 3.3.2. This completes the proof.

### 3.3.3 Numerical examples

In this section, three examples are given to show the effectiveness of established theories.

**Example 3.3.1** Consider the system (3.23) with following matrices

\[
A = \begin{bmatrix} 4 & 0 \\ 0 & 6 \end{bmatrix}, \quad W_0 = \begin{bmatrix} 0.2 & -4 \\ 0.1 & 0.3 \end{bmatrix}, \quad W_1 = \begin{bmatrix} 0.4 & 0.2 \\ 0.1 & 0.7 \end{bmatrix}, \quad C = \begin{bmatrix} 0.3 & 0 \\ 0 & 0.3 \end{bmatrix}, \\
D = \begin{bmatrix} 0.5 & -0.1 \\ -0.5 & 0 \end{bmatrix}, \quad W_2 = \begin{bmatrix} 0.2 & 0.6 \\ 0.5 & 0.1 \end{bmatrix}, \quad W_3 = \begin{bmatrix} 0.3 & 0.6 \\ 0.2 & -0.1 \end{bmatrix}, \\
E_1 = E_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad L = 0.2I.
\]

Using Matlab LMI control toolbox to solve the LMI (3.25) (without uncertainties), we obtain the upper bound of the time-varying delays is \( h_2 = 7.2546 \) when \( \mu = 0 \). This shows that the approach developed in this section is effective and less conservative than some existing results.

**Example 3.3.2** Consider the system (3.23) with following matrices

\[
A = \begin{bmatrix} 4 & 0 \\ 0 & 6 \end{bmatrix}, \quad W_0 = \begin{bmatrix} 0.2 & -4 \\ 0.1 & 0.3 \end{bmatrix}, \quad W_1 = \begin{bmatrix} 0.4 & 0.2 \\ 0.1 & 0.7 \end{bmatrix}, \quad C = \begin{bmatrix} 0.3 & 0 \\ 0 & 0.3 \end{bmatrix}, \\
D = \begin{bmatrix} 0.5 & -0.1 \\ -0.5 & 0 \end{bmatrix}, \quad W_2 = \begin{bmatrix} 0.2 & 0.6 \\ 0.5 & 0.1 \end{bmatrix}W_3 = \begin{bmatrix} 0.3 & 0.6 \\ 0.2 & -0.1 \end{bmatrix}, \quad H = \begin{bmatrix} 0.1 \\ 0.1 \end{bmatrix}, \\
E_1 = E_2 = I, \quad L = 0.2I, \quad T_1 = T_2 = T_3 = T_4 = T_5 = T_6 = T_7 = T_8 = T_9 = [1 \ 1].
\]

It was reported in [51] that the above matrices is robustly asymptotically stable in the mean square when \( 0 < \tau(t) \leq 2.15, \ 0 < r(t) \leq 2.15 \). However, by Theorem 3.3.2 and
using Matlab LMI control toolbox, $\mu = 0$, $h_1 = 0$ and $J = 0$ it is found that the equilibrium solution of uncertain SNNs (3.23) is robustly asymptotically stable in the mean square for any $\tau(t)$ and $r(t)$ satisfying $0 < \tau(t) \leq h_2 = 7.2353$, $0 < r(t) \leq 7.2353$. This shows that the established results in this section is finer than the previous results since the stability region is valid upto the upper bound 7.2353 instead of 2.15 in [51].

<table>
<thead>
<tr>
<th>$h_1$</th>
<th>$h_2 = \bar{r}$</th>
<th>$J$</th>
<th>$\mu$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>7.2349</td>
<td>0.1</td>
<td>0</td>
</tr>
<tr>
<td>0</td>
<td>7.2326</td>
<td>0.3</td>
<td>0</td>
</tr>
<tr>
<td>0</td>
<td>7.2271</td>
<td>0.5</td>
<td>0</td>
</tr>
<tr>
<td>0</td>
<td>7.2114</td>
<td>0.7</td>
<td>0</td>
</tr>
<tr>
<td>0</td>
<td>7.1876</td>
<td>0.8</td>
<td>0</td>
</tr>
<tr>
<td>0</td>
<td>7.1121</td>
<td>0.9</td>
<td>0</td>
</tr>
</tbody>
</table>

Table 3.1: Maximum allowable upper bounds for various $J$

It is clear from the Table 3.1 that when $h_1 = 0$ and $\mu$ are fixed, the different values of $h_2 = \bar{r}$ are plotted for corresponding different values of $J$.

<table>
<thead>
<tr>
<th>$h_1$</th>
<th>$h_2 = \bar{r}$</th>
<th>$J$</th>
<th>$\mu$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>7.1245</td>
<td>0</td>
<td>0.1</td>
</tr>
<tr>
<td>0</td>
<td>6.3494</td>
<td>0</td>
<td>0.5</td>
</tr>
<tr>
<td>0</td>
<td>1.8250</td>
<td>0</td>
<td>0.9</td>
</tr>
<tr>
<td>0</td>
<td>7.1163</td>
<td>0.5</td>
<td>0.1</td>
</tr>
<tr>
<td>0</td>
<td>6.3417</td>
<td>0.5</td>
<td>0.5</td>
</tr>
<tr>
<td>0</td>
<td>1.8251</td>
<td>0.5</td>
<td>0.9</td>
</tr>
</tbody>
</table>

Table 3.2: Maximum allowable upper bounds for various $J$ and $\mu$

The Table 3.2 clearly shows that when $h_1$ is fixed, we get different values of $h_2$ corresponding to different values of $J$ and $\mu$.

**Example 3.3.3** Consider the SNNs with linear fractional uncertainties

\[
\begin{align*}
    dx(t) &= \left[ -A(t)x(t) + W_0(t)f(x(t)) + W_1(t)f(x(t - \tau(t))) \right] dt + C(t)x(t) \\
          &+ D(t)x(t - \tau(t)) + W_2(t)f(x(t)) + W_3(t)f(x(t - \tau(t))) \right] dw(t) \\
\end{align*}
\]  

(3.44)

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where

\[
A = \begin{bmatrix} 4 & 0 \\ 0 & 5 \end{bmatrix}, \quad W_0 = \begin{bmatrix} 0.4 & -0.7 \\ 0.1 & 0 \end{bmatrix}, \quad W_1 = \begin{bmatrix} -0.2 & 0.6 \\ 0.5 & -0.1 \end{bmatrix}, \quad C = \begin{bmatrix} 0.5 & 0 \\ 0 & 0.5 \end{bmatrix},
\]

\[
D = \begin{bmatrix} 0 & -0.5 \\ -0.5 & 0 \end{bmatrix}, \quad W_2 = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.1 \end{bmatrix}, \quad W_3 = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.1 \end{bmatrix}, \quad H = \begin{bmatrix} 0.1 \\ -0.1 \end{bmatrix},
\]

\[
L = 0.5I, \quad T_1 = [0.2 \; 0.3], \quad T_2 = [0.2 \; -0.3], \quad T_3 = [-0.2 \; -0.3],
\]

\[
T_4 = T_5 = T_6 = T_7 = [0.1 \; 0.1].
\]

Applying Theorem 2 in [12], Theorem 2 in [51] and Theorem 3.3 in [76], for the above systems it is found that the equilibrium solution of SNNs (3.44) is robustly asymptotically stable in the mean square for delay \( \tau(t) \) satisfying \( 0 < \tau(t) \leq 0.5730 \), \( 0 < \tau(t) \leq 0.7056 \) and \( 0 < \tau(t) \leq 3.8795 \) respectively. However applying Theorem 3.3.4, we can conclude that if \( 0 < \tau(t) < \infty \), system (3.44) is robustly asymptotically stable in the mean square sense and finer then the previous works based on the upper bound. In the Table 3.3, the different values of \( h_2 \) are plotted for corresponding to

<table>
<thead>
<tr>
<th>( h_1 )</th>
<th>( h_2 )</th>
<th>( J )</th>
<th>( \mu )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( 0 )</td>
<td>( &lt; \infty )</td>
<td>0.1</td>
<td>0.5</td>
</tr>
<tr>
<td>( 0 )</td>
<td>( &lt; \infty )</td>
<td>0.3</td>
<td>0.9</td>
</tr>
<tr>
<td>( 0 )</td>
<td>( &lt; \infty )</td>
<td>0.5</td>
<td>0.1</td>
</tr>
<tr>
<td>( 0 )</td>
<td>( &lt; \infty )</td>
<td>0.5</td>
<td>0.9</td>
</tr>
</tbody>
</table>

Table 3.3: Maximum allowable upper bounds for various \( J \)

different values of \( J \) when \( h_1 = 0 \) and \( \mu \) are fixed.

<table>
<thead>
<tr>
<th>( h_1 )</th>
<th>( h_2 )</th>
<th>( J )</th>
<th>( \mu )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( 0 )</td>
<td>( &lt; \infty )</td>
<td>0.1</td>
<td>0.1</td>
</tr>
<tr>
<td>( 0 )</td>
<td>( &lt; \infty )</td>
<td>0.5</td>
<td>0.5</td>
</tr>
<tr>
<td>( 0 )</td>
<td>( &lt; \infty )</td>
<td>0.9</td>
<td>0.9</td>
</tr>
<tr>
<td>( 0 )</td>
<td>( &lt; \infty )</td>
<td>0.5</td>
<td>0.9</td>
</tr>
<tr>
<td>( 0 )</td>
<td>( &lt; \infty )</td>
<td>0.5</td>
<td>0.1</td>
</tr>
</tbody>
</table>

Table 3.4: Maximum allowable upper bounds for various \( J \) and \( \mu \)

The Table 3.4 clearly shows that when \( h_1 \) is fixed, we get different values of \( h_2 \) corresponding to different values of \( J \) and \( \mu \).
3.4 LMI conditions for robust stability analysis of stochastic Hopfield neural networks with interval time-varying delays and linear fractional uncertainties

3.4.1 Problem description and preliminaries

In this section, consider the following uncertain stochastic HNNs with time-varying delays:

$$
\begin{align*}
\dot{x}(t) &= \left[ -A(t)x(t) + W_0(t)f(x(t)) + W_1(t)f(x(t - \tau(t))) \right] dt \\
&\quad + \left[ C(t)x(t) + D(t)x(t - \tau(t)) \right] dw(t) \\
x(t) &= \phi(t), \quad \forall t \in [-h_2, 0],
\end{align*}
$$

where $x(t) \in \mathbb{R}^n$ is the neural state vector, $f(x(t)) = [f_1(x_1(t)), \ldots, f_n(x_n(t))]^T \in \mathbb{R}^n$ is the neuron activation function with initial condition $f(0) = 0$. $w(t)$ is one-dimensional Brownian motion defined on a complete probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathcal{P})$. Further assume that the time-varying delays $\tau(t)$ satisfies

$$
0 \leq h_1 \leq \tau(t) < h_2, \quad \dot{\tau}(t) \leq \mu,
$$

where $h_1, h_2$ and $\mu$ are constants.

3.4.2 Global robust stability results

In this section, global asymptotically stability criteria for system (3.45) is derived using Lyapunov method in terms of LMIs. Now defining two new state variables for the SNNs (3.45),

$$
y(t) = -A(t)x(t) + B(t)f(x(t)) + C(t)f(x(t - \tau(t))),
$$
and
\[ g(t) = D_1(t)x(t) + D_2(t)x(t - \tau(t)), \]
then the SNNs (3.45) can be written as
\[ dx(t) = y(t)dt + g(t)d\omega(t). \]
Moreover, the following equality holds,
\[ x(t) - x(t - \tau(t)) = \int_{t-\tau(t)}^{t} dx(s) = \int_{t-\tau(t)}^{t} y(s)ds + \int_{t-\tau(t)}^{t} g(s)d\omega(s). \]
The following theorem gives the mean square asymptotic stability results for SNNs (3.45) without uncertainty.

**Theorem 3.4.1** For given scalars \( h_2 > h_1 \geq 0 \) and \( \mu \), the equilibrium point of SNNs (3.45) without uncertainty is asymptotically stable in the mean square if there exist symmetric matrices
\[ P = \begin{bmatrix} P_{11} & P_{12} & P_{13} & P_{14} \\ P_{21}^T & P_{22} & P_{23} & P_{24} \\ P_{31}^T & P_{32} & P_{33} & P_{34} \\ P_{41}^T & P_{42}^T & P_{43}^T & P_{44} \end{bmatrix} > 0, \ Q_\delta > 0 \ (\delta = 1, 2, 3), \ Z_h > 0, \ R_h > 0, \] for any appropriately dimensioned matrices \( N_h, M_h, S_h \ (h = 1, 2), \ V_k, U_k \ (k = 1, 2, 3) \), and positive diagonal matrices \( K_1 \) and \( K_2 \) such that the following LMI is feasible
\[
\Psi = \begin{bmatrix} \tilde{\Omega} & \sqrt{h_2+1} \ N & \sqrt{(h_2-h_1)+1} \ M & \sqrt{(h_2-h_1)+1} \ S \\ * & -R_1 & 0 & 0 \\ * & * & -(R_1+R_2) & 0 \\ * & * & * & -R_2 \end{bmatrix} < 0, \quad (3.47)
\]
where \( \tilde{\Omega} = (\Omega_{\ell,\rho})_{11 \times 11} \) with
\[
\begin{align*}
\Omega_{1,1} &= Q_1 + Q_2 + Q_3 + h_2 Z_1 + (h_2 - h_1) Z_2 + P_2 + P_2^T + N_1 + N_1^T, \\
\Omega_{1,2} &= -(1 - \mu) P_2 + (1 - \mu) P_3 - (1 - \mu) P_4 - N_1 + N_2^T + M_1 - S_1 - A^T U_1^T + C^T V_1^T, \\
\Omega_{1,3} &= P_4 + S_1, \quad \Omega_{1,4} = -P_3 - M_1, \quad \Omega_{1,5} = P_1 - A^T U_2^T + C^T V_2^T, \quad \Omega_{1,6} = -A^T U_3^T \\
&+ C^T V_3^T, \quad \Omega_{1,7} = L K_1, \quad \Omega_{1,8} = 0, \quad \Omega_{1,9} = P_5^T, \quad \Omega_{1,10} = P_6, \quad \Omega_{1,11} = P_7, \\
\end{align*}
\]
\[
\begin{align*}
\Omega_{2,2} &= -(1-\mu)Q_1 - N_2 - N_2^T + M_2 + M_2^T - S_2 - S_2^T + V_1D + D^TV_1^T, \quad \Omega_{2,3} = S_2, \\
\Omega_{2,4} &= -M_2, \quad \Omega_{2,5} = D^TV_2^T - U_1, \quad \Omega_{2,6} = D^TV_3^T - V_1, \quad \Omega_{2,7} = U_1W_0, \\
\Omega_{2,8} &= LK_2 + U_1W_1, \quad \Omega_{2,9} = -(1-\mu)P_5^T + (1-\mu)P_6^T - (1-\mu)P_7^T, \quad \Omega_{2,10} = (1-\mu)P_8^T \\
&\quad - (1-\mu)P_9^T - (1-\mu)P_6, \quad \Omega_{2,11} = -(1-\mu)P_{10}^T - (1-\mu)P_7 + (1-\mu)P_9, \\
\Omega_{3,3} &= -Q_2, \quad \Omega_{3,4} = 0, \quad \Omega_{3,5} = 0, \quad \Omega_{3,6} = 0, \quad \Omega_{3,7} = 0, \quad \Omega_{3,8} = 0, \quad \Omega_{3,9} = P_7^T, \\
\Omega_{3,10} &= P_9^T, \quad \Omega_{3,11} = P_{10}^T, \quad \Omega_{4,4} = -Q_3, \quad \Omega_{4,5} = 0, \quad \Omega_{4,6} = 0, \quad \Omega_{4,7} = 0, \quad \Omega_{4,8} = 0, \\
\Omega_{4,9} &= -P_6^T, \quad \Omega_{4,10} = -P_8^T, \quad \Omega_{4,11} = -P_9 \quad \Omega_{5,5} = h_2R_1 + (h_2 - h_1)R_2 - U_2 - U_2^T, \\
\Omega_{5,6} &= -U_3^T - V_2, \quad \Omega_{5,7} = U_2W_0, \quad \Omega_{5,8} = U_2W_1, \quad \Omega_{5,9} = P_2, \quad \Omega_{5,10} = P_3, \quad \Omega_{5,11} = P_4, \\
\Omega_{6,6} &= P_1 + h_2R_1 + (h_2 - h_1)R_2 - V_3 - V_3^T, \quad \Omega_{6,7} = U_3W_0, \quad \Omega_{6,8} = U_3W_1, \quad \Omega_{6,9} = 0, \\
\Omega_{6,10} &= 0, \quad \Omega_{6,11} = 0, \quad \Omega_{7,7} = -2K_1, \quad \Omega_{7,8} = 0, \quad \Omega_{7,9} = 0, \quad \Omega_{7,10} = 0, \quad \Omega_{7,11} = 0, \\
\Omega_{8,8} &= -2K_2, \quad \Omega_{8,9} = 0, \quad \Omega_{8,10} = 0, \quad \Omega_{9,111} = 0, \quad \Omega_{9,9} = \frac{-1}{h_2}Z_1, \quad \Omega_{9,10} = 0, \quad \Omega_{9,11} = 0, \\
\Omega_{10,10} &= \frac{-1}{h_2 - h_1}(Z_1 + Z_2), \quad \Omega_{10,11} = 0, \quad \Omega_{11,11} = \frac{-1}{h_2 - h_1}Z_2, \\
N &= \begin{bmatrix} N_1^T & N_2^T & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}^T, \\
M &= \begin{bmatrix} M_1^T & M_2^T & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}^T, \\
S &= \begin{bmatrix} S_1^T & S_2^T & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}^T, \\
U &= \begin{bmatrix} 0 & U_1^T & 0 & 0 & U_2^T & U_3^T & 0 & 0 & 0 & 0 \end{bmatrix}^T, \\
V &= \begin{bmatrix} 0 & V_1^T & 0 & 0 & V_2^T & V_3^T & 0 & 0 & 0 & 0 \end{bmatrix}^T.
\end{align*}

**Proof:** Consider the Lyapunov-Krasovskii functional

\[
V(x_t,t) = V_1(x_t,t) + V_2(x_t,t) + V_3(x_t,t) + V_4(x_t,t),
\]
where

\[
V_1(x_t, t) = \chi^T(t)P\chi(t),
\]

\[
V_2(x_t, t) = \int_{t-h_2}^t x^T(s)Q_1x(s)ds + \int_{t-h_1}^t x^T(s)Q_2x(s)ds
+ \int_{t-h_2}^t x^T(s)Q_3x(s)ds,
\]

\[
V_3(x_t, t) = \int_{-h_2}^0 \int_{t+\theta}^{t+\theta} x^T(s)Z_1x(s)dsd\theta + \int_{-h_2}^0 \int_{t+\theta}^{t+\theta} x^T(s)Z_2x(s)dsd\theta,
\]

\[
V_4(x_t, t) = \int_{-h_2}^0 \int_{t+\theta}^{t+\theta} y^T(s)R_1y(s)dsd\theta + \int_{-h_2}^0 \int_{t+\theta}^{t+\theta} g^T(s)R_1g(s)dsd\theta
+ \int_{-h_2}^{-h_1} \int_{t+\theta}^{t+\theta} g^T(s)R_2g(s)dsd\theta + \int_{-h_2}^{-h_1} \int_{t+\theta}^{t+\theta} y^T(s)R_2y(s)dsd\theta,
\]

with

\[
\chi(t) = \left[ x^T(t) \begin{pmatrix} \int_{t-\tau(t)}^t x(s)ds \\ \int_{t-h_2}^{t-\tau(t)} x(s)ds \\ \int_{t-h_1}^{t-\tau(t)} x(s)ds \end{pmatrix} \right] T
\]

and \( x_t = \{x(t+\theta): -h_2 \leq \theta \leq 0\} \). Then, it can be obtained by Itô’s formula that

\[
dV(x_t, t) = \mathcal{L}V(x_t, t)dt + 2x^T(t)P_1g(t)d\omega(t), \tag{3.48}
\]

where

\[
\mathcal{L}V_1(x_t, t) \leq 2 \begin{bmatrix} \int_{t-\tau(t)}^t x(s)ds \\ \int_{t-h_2}^{t-\tau(t)} x(s)ds \\ \int_{t-h_1}^{t-\tau(t)} x(s)ds \end{bmatrix}^T \begin{bmatrix} P_1 & P_2 & P_3 \\ P_2^T & P_3 & P_4 \\ P_3^T & P_4 & P_5 \end{bmatrix} \begin{bmatrix} x(t) - (1-\mu)x(t-\tau(t)) \\ y(t) \\ (1-\mu)x(t-\tau(t)) - x(t-h_2) \end{bmatrix}
+ g^T(t)P_1g(t),
\]

\[
\mathcal{L}V_2(x_t, t) \leq x^T(t)Q_1x(t) - (1-\mu)x^T(t-\tau(t))Q_1x(t-\tau(t)) + x^T(t)Q_2x(t)
- x^T(t-h_2)Q_2x(t-h_2) + x^T(t)Q_3x(t) - x^T(t-h_2)Q_3x(t-h_2),
\]

\[
\mathcal{L}V_3(x_t, t) = h_2x^T(t)Z_1x(t) - \int_{t-h_2}^t x^T(s)Z_1x(s)ds + (h_2-h_1)x^T(t)Z_2x(t)
- \int_{t-h_2}^{t-h_1} x^T(s)Z_2x(s)ds,
\]

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\[ \mathcal{L}V_4(x_t, t) = h_2 y^T(t)R_1 y(t) - \int_{t-h_2}^{t} g^T(s)R_1 y(s) ds + h_2 g^T(t)R_1 g(t) \]
\[ - \int_{t-h_2}^{t} g^T(s)R_2 g(s) ds + (h_2 - h_1) y^T(t)R_2 y(t) - \int_{t-h_2}^{t-h_1} y^T(s)R_2 y(s) ds \]
\[ + (h_2 - h_1) g^T(t)R_2 g(t) - \int_{t-h_2}^{t-h_1} g^T(s)R_2 g(s) ds. \]

It is obvious that, there exist positive diagonal matrices \( K_1 \) and \( K_2 \) such that the following inequalities hold

\[ 2x^T(t)LK_1 f(x(t)) - 2f^T(x(t))K_1 f(x(t)) \geq 0, (3.49) \]
\[ 2x^T(t) - \tau(t)LK_2 f(x(t)) - \tau(t)) - 2f^T(x(t) - \tau(t))K_2 f(x(t) - \tau(t)) \geq 0, (3.50) \]

where \( L \) is a matrix with appropriate dimensions. Using Lemma 1.16.11 and \( 0 \leq h_1 \leq \tau(t) < h_2 \), that is,

\[ - \int_{t-\tau(t)}^{t} x^T(s)Z_1 x(s) ds \leq - \frac{1}{h_2} \left[ \int_{t-\tau(t)}^{t} x(s) ds \right]^T Z_1 \left[ \int_{t-\tau(t)}^{t} x(s) ds \right], \quad (3.51) \]
\[ - \int_{t-h_2}^{t-\tau(t)} x^T(s)(Z_1 + Z_2) x(s) ds \leq - \frac{1}{h_2 - h_1} \left[ \int_{t-h_2}^{t-\tau(t)} x(s) ds \right]^T (Z_1 + Z_2) \left[ \int_{t-h_2}^{t-\tau(t)} x(s) ds \right], \]
\[ - \int_{t-\tau(t)}^{t-h_1} x^T(s)Z_2 x(s) ds \leq - \frac{1}{h_2 - h_1} \left[ \int_{t-\tau(t)}^{t-h_1} x(s) ds \right]^T Z_2 \left[ \int_{t-\tau(t)}^{t-h_1} x(s) ds \right]. \quad (3.52) \]

Now, define the new vector as follows

\[
\xi^T(t) = \begin{bmatrix} x^T(t) & x^T(t - \tau(t)) & x^T(t - h_1) & x^T(t - h_2) & y^T(t) & g^T(t) \\
 f^T(x(t)) & f^T(x(t) - \tau(t))) & \int_{t-\tau(t)}^{t} x^T(s) ds & \int_{t-h_2}^{t-\tau(t)} x^T(s) ds & \int_{t-h_2}^{t-h_1} x^T(s) ds \end{bmatrix}.
\]

The following equations hold for any matrices \( N, M, S, U \) and \( V \) with appropriate
\begin{align}
\eta_1(t) &= 2\xi^T(t)N\left[ x(t) - x(t - \tau(t)) - \int_{t-\tau(t)}^{t} y(s)ds + \int_{t-\tau(t)}^{t} g(s)d\omega(s) \right] = 0, \quad (3.54) \\
\eta_2(t) &= 2\xi^T(t)M\left[ x(t - \tau(t)) - x(t - h_2) - \int_{t-h_2}^{t-\tau(t)} y(s)ds - \int_{t-h_2}^{t-\tau(t)} g(s)d\omega(s) \right] = 0, \quad (3.55) \\
\eta_3(t) &= 2\xi^T(t)S\left[ x(t - h_1) - x(t - \tau(t)) - \int_{t-\tau(t)}^{t-h_1} y(s)ds - \int_{t-\tau(t)}^{t-h_1} g(s)d\omega(s) \right] = 0, \quad (3.56) \\
\eta_4(t) &= 2\xi^T(t)U\left[ -Ax(t) + Bf(x(t)) + Cf(x(t - \tau(t))) - y(t) \right] = 0, \quad (3.57) \\
\eta_5(t) &= 2\xi^T(t)V\left[ -D_1x(t) + D_2x(t - \tau(t)) - g(t) \right] = 0. \quad (3.58)
\end{align}

Then, one can observe that
\begin{align}
&-2\xi^T(t)N\left[ \int_{t-\tau(t)}^{t} y(s)ds + \int_{t-\tau(t)}^{t} g(s)d\omega(s) \right] \\
&- \int_{t-\tau(t)}^{t} y^T(s)R_1y(s)ds - \int_{t-\tau(t)}^{t} g^T(s)R_1g(s)ds \\
&\leq - \int_{t-\tau(t)}^{t} \left[ \xi^T(t)N + y(s)R_1 \right]^T R_1^{-1} \left[ \xi(t)N + y(s)R_1 \right]ds + \xi^T(t)NR_1^{-1}N^T\xi(t) \\
&+ h_2\xi^T(t)NR_1^{-1}N^T\xi(t) + \left( \int_{t-\tau(t)}^{t} g(s)d\omega(s) \right)R_1 \left( \int_{t-\tau(t)}^{t} g(s)d\omega(s) \right) \\
&- \int_{t-\tau(t)}^{t} g^T(s)R_1g(s)ds \\
&\leq (h_2 + 1)\xi^T(t)NR_1^{-1}N^T\xi(t) + \left( \int_{t-\tau(t)}^{t} g(s)d\omega(s) \right)R_1 \left( \int_{t-\tau(t)}^{t} g(s)d\omega(s) \right) \\
&- \int_{t-\tau(t)}^{t} g^T(s)R_1g(s)ds \quad (3.59)
\end{align}
\[-2\xi^T(t)M \left[ \int_{t-h_2}^{t-\tau(t)} y(s)ds + \int_{t-h_2}^{t-\tau(t)} g(s)d\omega(s) \right] - \int_{t-h_2}^{t-\tau(t)} y^T(s)(R_1 + R_2)y(s)ds \\
\leq - \int_{t-h_2}^{t-\tau(t)} \left[ \xi(t)M + y(s)(R_1 + R_2) \right]^T(R_1 + R_2)^{-1} \left[ \xi(t)M + y(s)(R_1 + R_2) \right]ds \\
+ \xi^T(t)M(R_1 + R_2)^{-1}M^T \xi(t) + (h_2 - h_1)\xi^T(t)M(R_1 + R_2)^{-1}M^T \xi(t) \\
+ \left( \int_{t-h_2}^{t-\tau(t)} g(s)d\omega(s) \right)(R_1 + R_2) \left( \int_{t-h_2}^{t-\tau(t)} g(s)d\omega(s) \right) - \int_{t-h_2}^{t-\tau(t)} g^T(s)(R_1 + R_2)g(s)ds \\
\leq (h_2 - h_1 + 1)\xi^T(t)M(R_1 + R_2)^{-1}M^T \xi(t) + \left( \int_{t-h_2}^{t-\tau(t)} g(s)d\omega(s) \right)(R_1 + R_2) \\
\times \left( \int_{t-h_2}^{t-\tau(t)} g(s)d\omega(s) \right) - \int_{t-h_2}^{t-\tau(t)} g^T(s)(R_1 + R_2)g(s)ds, \tag{3.60}
\]

\[-2\xi^T(t)S \left[ \int_{t-\tau(t)}^{t-h_1} y(s)ds + \int_{t-\tau(t)}^{t-h_1} g(s)d\omega(s) \right] \\
- \int_{t-\tau(t)}^{t-h_1} y^T(s)R_2y(s)ds - \int_{t-\tau(t)}^{t-h_1} g^T(s)R_2g(s)ds \\
\leq - \int_{t-\tau(t)}^{t-h_1} \left[ \xi(t)S + y(s)R_2 \right]^T R_2^{-1} \left[ \xi(t)S + y(s)R_2 \right]ds + \xi^T(t)SR_2^{-1}S^T \xi(t) \\
+ (h_2 - h_1)\xi^T(t)SR_2^{-1}S^T \xi(t) + \left( \int_{t-\tau(t)}^{t-h_1} g(s)d\omega(s) \right)R_2 \left( \int_{t-\tau(t)}^{t-h_1} g(s)d\omega(s) \right) \\
- \int_{t-\tau(t)}^{t-h_1} g^T(s)R_2g(s)ds \\
\leq (h_2 - h_1 + 1)\xi^T(t)SR_2^{-1}S^T \xi(t) + \left( \int_{t-\tau(t)}^{t-h_1} g(s)d\omega(s) \right)R_2 \left( \int_{t-\tau(t)}^{t-h_1} g(s)d\omega(s) \right) \\
- \int_{t-\tau(t)}^{t-h_1} g^T(s)R_2g(s)ds. \tag{3.61}
\]

Substituting (3.49)-(3.61) into (3.48), it follows that
\[
dV(x_t, t) \leq \xi^T(t)\Psi \xi(t)dt + 2x^T(t)P_1g(t)d\omega(t),
\]
where
\[
\Psi = \begin{bmatrix}
\hat{\Omega} & \sqrt{h_2 + 1} N & \sqrt{(h_2 - h_1) + 1} M & \sqrt{(h_2 - h_1) + 1} S \\
-\hat{R}_1 & 0 & 0 & 0 \\
*(R_1 + R_2) & 0 & -R_2 & 0 \\
0 & 0 & 0 & 0 \\
\end{bmatrix} < 0.
\]

From (3.47) there exists a scalar \( \alpha > 0 \) such that
\[
\Psi + diag\{\alpha I_n, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0\} < 0.
\]

Hence, we have
\[
E\left[\frac{dV(x_t, t)}{dt}\right] \leq E(\xi(t)^T\Psi\xi(t)) \leq -\alpha E|x(t)|^2.
\]

Thus if \( \Psi < 0 \), then the SNNs (3.45) is asymptotically stable in the mean square. The proof is completed.

**Remarks 3.4.2** In this section, by constructing a new matrix \( P \) in the Lyapunov-Krasovskii functional, less conservative results have been obtained than those results discussed in [41, 48, 92, 98, 104]. Moreover, only few free-weighting matrices are considered to remove the restrictive condition \( \mu < 1 \) and also to avoid the computational complexity [12].

Next, we derive the following robust stability conditions for the system (3.45).

**Theorem 3.4.3** For given scalars \( h_2 > h_1 > 0 \) and \( \mu \), the equilibrium point of SNNs (3.45) subject to linear fractional norm-bounded uncertainty (3.5) is robustly asymptotically stable in the mean square if there exist symmetric matrices \( P > 0, Q_\delta > 0, (\delta = 1, 2, 3), Z_h > 0, R_h > 0 \), any appropriately dimensioned matrices \( N_h, M_h, S_h, (h = 1, 2), V_k, U_k, (k = 1, 2, 3) \), positive diagonal matrices \( K_1, K_2 \) and scalars \( \epsilon_1 > 0, \epsilon_2 > 0 \), such that the following LMI is feasible
\[
\begin{bmatrix}
\Psi & \hat{U}H & \epsilon_1 \Pi_1^T & \hat{V}H & \epsilon_2 \Pi_2^T \\
* & -\epsilon_1 I & \epsilon_1 J_1^T & 0 & 0 \\
* & * & -\epsilon_1 I & 0 & 0 \\
* & * & * & -\epsilon_2 I & \epsilon_2 J_2^T \\
* & * & * & * & -\epsilon_2 I \\
\end{bmatrix} < 0,
\]

where \( \Psi \) is defined in (3.47).

**Proof:** Assume that the inequality (3.62) holds. From (3.62), it can be easily obtained
that
\[
\Pi = \begin{bmatrix}
\Psi & \hat{U} H & \epsilon_1 \Pi_1^T & \hat{V} H & \epsilon_2 \Pi_2^T \\
* & -\epsilon_1 I & \epsilon_1 J^T & 0 & 0 \\
* & * & -\epsilon_1 I & 0 & 0 \\
* & * & * & -\epsilon_2 I & \epsilon_2 J^T \\
* & * & * & * & -\epsilon_2 I
\end{bmatrix} < 0,
\]

\[
\Pi_1 = \begin{bmatrix}
- T_1^T & 0 & 0 & 0 & 0 & T_2^T & T_3^T & 0 & 0 & 0 & 0 & 0
\end{bmatrix},
\]

\[
\Pi_2 = \begin{bmatrix}
T_4^T & T_5^T & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}.
\]

\[
\hat{U} = \begin{bmatrix}
0 & U_1^T & 0 & 0 & U_2^T & U_3^T & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}^T,
\]

\[
\hat{V} = \begin{bmatrix}
0 & V_1^T & 0 & 0 & V_2^T & V_3^T & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}^T.
\]

Thus, \( \Pi = \Psi + \hat{U} H A(t) \Pi_1 + \Pi_1^T A(t) H^T \hat{U}^T + \hat{V} H A(t) \Pi_2 + \Pi_2^T A(t) H^T \hat{V}^T < 0 \) holds according to Lemma 1.16.13. It can be verified that \( \Pi \) is exactly the same as \( \Psi \) of (3.47) when \( A, \ W_0, \ W_1, \ C, \ \text{and} \ D \) are replaced by \( A + H A(t) T_1, \ W_0 + H A(t) T_2, \ W_1 + H A(t) T_3, \ C + H A(t) T_4 \) and \( D + H A(t) T_5 \) respectively.

**Remarks 3.4.4** Theorem 3.4.3 provides a delay-dependent stability condition for uncertain SNNs (3.45) with interval time-varying delay. When time-delay is invariant, that is, \( h_1 = h_2 = h \), the stability result to be considered for the following SNNs

\[
dx(t) = \left[ -A(t)x(t) + W_0(t)f(x(t)) + W_1(t)f(x(t-h)) \right] dt
\]

\[
+ \left[ C(t)x(t) + D(t)x(t-h) \right] dw(t), \quad (3.63)
\]

\[
x(t) = \phi(t), \quad \forall \ t \in [-h, 0].
\]

For system (3.63), we have the following result.

**Corollary 3.4.5** For given scalar \( h > 0 \), the equilibrium point of SNNs (3.63) subject to linear fractional norm-bounded uncertainty (3.5) is robustly asymptotically stable in the mean square if there exist symmetric matrices \( Q_3 > 0, \ Z_1 > 0, \ R_1 > 0 \), \( P = \begin{bmatrix} P_1 & P_2^T \\ P_2 & P_3 \end{bmatrix} > 0 \), any appropriately dimensioned matrices \( N_h \), (h =
1, 2), \( V_k, U_k \), \((k = 1, 2, 3)\), positive diagonal matrices \( K_1, K_2 \) and positive scalars \( \epsilon_1, \epsilon_2 \) such that the following LMI is feasible

\[
\mathcal{E} = \begin{bmatrix}
\sqrt{h+1} N & \bar{U} H & \epsilon_1 I^T_1 & \bar{V} H & \epsilon_2 I^T_2 \\
* & -R_1 & 0 & 0 & 0 \\
* & * & -\epsilon_1 I & \epsilon_1 J^T & 0 & 0 \\
* & * & * & -\epsilon_1 I & 0 & 0 \\
* & * & * & * & -\epsilon_2 I & \epsilon_2 J^T \\
* & * & * & * & * & -\epsilon_2 I 
\end{bmatrix} < 0, \quad (3.64)
\]

where

\[ \mathcal{E} = (\mathcal{E}_{\epsilon, \nu})_{7 \times 7} \text{ with} \]

\[ \mathcal{E}_{1,1} = Q_3 + hZ_1 + P_2 + P_2^T + N_1 + N_1^T, \quad \mathcal{E}_{1,2} = -P_2 - N_1 + N_2^T - A^T U_1^T + C^T V_1^T, \]

\[ \mathcal{E}_{1,3} = -P_1 - A^T U_2^T + C^T V_2^T, \quad \mathcal{E}_{1,4} = -A^T U_3^T + C^T V_3^T, \quad \mathcal{E}_{1,5} = L K_1, \quad \mathcal{E}_{1,6} = 0, \]

\[ \mathcal{E}_{1,7} = P_3^T, \quad \mathcal{E}_{2,2} = -Q_3 - N_2 - N_2^T + V_1 C + C^T V_1^T, \quad \mathcal{E}_{2,3} = D^T V_2^T - U_1, \]

\[ \mathcal{E}_{2,4} = D^T V_3^T - V_1, \quad \mathcal{E}_{2,5} = U_1 W_0, \quad \mathcal{E}_{2,6} = U_1 W_1 + L K_2, \quad \mathcal{E}_{2,7} = -P_3^T, \]

\[ \mathcal{E}_{3,3} = h R_1 - U_2 - U_2^T, \quad \mathcal{E}_{3,4} = U_3^T - V_2, \quad \mathcal{E}_{3,5} = U_2 W_0, \quad \mathcal{E}_{3,6} = U_2 W_1, \quad \mathcal{E}_{3,7} = 0, \]

\[ \mathcal{E}_{4,4} = P_1 + h R_1 - V_3 - V_3^T, \quad \mathcal{E}_{4,5} = U_3 W_0, \quad \mathcal{E}_{4,6} = U_3 W_1, \quad \mathcal{E}_{4,7} = P_2, \quad \mathcal{E}_{5,5} = -2 K_1, \]

\[ \mathcal{E}_{5,6} = \mathcal{E}_{5,7} = 0, \quad \mathcal{E}_{6,6} = -2 K_2, \quad \mathcal{E}_{6,7} = 0, \quad \mathcal{E}_{7,7} = -\frac{1}{h} Z_1. \]

\[ \bar{N} = \begin{bmatrix} N_1^T & N_2^T & 0 & 0 & 0 & 0 & 0 \end{bmatrix}^T, \quad \bar{U} = \begin{bmatrix} 0 & U_1^T & U_2^T & U_3^T & 0 & 0 & 0 \end{bmatrix}^T, \]

\[ \bar{V} = \begin{bmatrix} 0 & V_1^T & V_2^T & V_3^T & 0 & 0 & 0 \end{bmatrix}^T, \quad \Gamma_1 = \begin{bmatrix} -T_1^T & 0 & 0 & 0 & T_2^T & T_3^T & 0 \end{bmatrix}^T, \]

\[ \Gamma_2 = \begin{bmatrix} T_4^T & T_5^T & 0 & 0 & 0 & 0 & 0 \end{bmatrix}^T. \]

**Proof:** Consider the Lyapunov-Krasovskii functional

\[ V(x_t, t) = V_1(x_t, t) + V_2(x_t, t) + V_3(x_t, t) + V_4(x_t, t), \]

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where
\[ V_1(x, t) = \zeta^T(t)P\zeta(t), \quad V_2(x, t) = \int_{t-h}^{t} x^T(s)Q_3x(s)ds, \]
\[ V_3(x, t) = \int_{-h}^{0} \int_{t+\theta}^{t} x^T(s)Z_1x(s)d\theta \]
\[ V_4(x, t) = \int_{-h}^{0} \int_{t+\theta}^{t} y^T(s)R_1y(s)d\theta + \int_{-h}^{0} \int_{t+\theta}^{t} g^T(s)R_1g(s)d\theta \]
with
\[ \zeta(t) = \left[ x^T(t) \left( \int_{t-h}^{t} x(s)ds \right)^T \right]. \]

Now, define the new vector as follows
\[ \eta^T(t) = \left[ x^T(t) \quad x^T(t-h) \quad y^T(t) \quad g^T(t) \quad f^T(x(t)) \quad f^T(x(t-h)) \quad \int_{t-h}^{t} x^T(s)ds \right]. \]

The following equations hold for any matrices \( \bar{N}, \bar{U} \) and \( \bar{V} \) with appropriate dimensions,
\[ 2\eta^T(t)\bar{N}\left[ x(t) - x(t-h) - \int_{t-h}^{t} y(s)ds - \int_{t-h}^{t} g(s)d\omega(s) \right] = 0, \]
\[ 2\eta^T(t)\bar{U}\left[ -A(t)x(t) + W_0(t)f(x(t)) + W_1(t)f(x(t-h)) - y(t) \right] = 0, \]
\[ 2\eta^T(t)\bar{V}\left[ C(t)x(t) + D(t)x(t-h) - g(t) \right] = 0. \]

Following the similar arguments as in the proof of Theorem 3.4.3, we can obtain the desired result immediately, hence the detailed proof is omitted.

### 3.4.3 Numerical examples

In this section, three examples are given to show the effectiveness of the established theories.

**Example 3.4.1.** Consider the following uncertain SNNs with time-varying delays
\[ dx(t) = \left[ -A(t)x(t) + W_0(t)f(x(t)) + W_1(t)f(x(t-h)) \right]dt \]
\[ + \left[ Cx(t) + Dx(t-\tau(t)) \right]dw(t), \]
where

\[
A = \begin{bmatrix} 4 & 0 \\ 0 & 5 \end{bmatrix}, \quad W_0 = \begin{bmatrix} 0.4 & -0.7 \\ 0.1 & 0 \end{bmatrix}, \quad W_1 = \begin{bmatrix} -0.2 & 0.6 \\ 0.5 & -0.1 \end{bmatrix},
\]

\[
C = \begin{bmatrix} 0.5 & 0 \\ 0 & 0.5 \end{bmatrix}, \quad D = \begin{bmatrix} 0 & -0.5 \\ -0.5 & 0 \end{bmatrix}, \quad L = \begin{bmatrix} 0.5 & 0 \\ 0 & 0.5 \end{bmatrix},
\]

\[
H = \begin{bmatrix} 0.1 \\ -0.1 \end{bmatrix}, \quad T_1 = [0.2 \ 0.3], \quad T_2 = [0.2 \ -0.3],
\]

\[
T_3 = [-0.2 \ -0.3].
\]

Applying results in [41, 100] to the above system, the achieved maximum allowable upper bounds are 0.4109 and 0.6194 respectively. However, using Theorem 3.4.3 with \( P_R = 0, (R = 2 \ldots 10), Q_1 = 0, h_1 = 0 \) and \( J = 0 \), it is found that the maximum allowable upper bound is \( h_2 = 0.7681 \). It was reported in [104] that for the system (3.45), the achieved maximum allowable delay \( \tau(t) \) is 0.6633 when \( \mu = 0.95 \). However using Theorem 3.4.3 it is found that the maximum allowable upper bound is \( h_2 = 0.7681 \) when \( \mu = 0.95 \), \( h_1 = 0 \) and \( J = 0 \) and also Table 3.5 describes allowable upper bounds \( h_2 \) for different \( h_1 \), \( \mu \), and \( J \). Therefore, for this example, the results are less conservative than those results in [41, 100, 104]. The response of the state trajectories for SNNs are given in Figure 3.1.

<table>
<thead>
<tr>
<th>( J = 0 ), ( h_1 = 0 )</th>
<th>( \mu = 0 )</th>
<th>( \mu = 0.5 )</th>
<th>( \mu = 0.7 )</th>
<th>( \mu = 0.9 )</th>
<th>( \mu = 1 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( J = 0 ), ( h_1 = 0.5 )</td>
<td>0.9661</td>
<td>0.7801</td>
<td>0.7681</td>
<td>0.7681</td>
<td>0.7681</td>
</tr>
<tr>
<td>( J = 0.5 ), ( h_1 = 0 )</td>
<td>0.9593</td>
<td>0.7778</td>
<td>0.7664</td>
<td>0.7663</td>
<td>0.7663</td>
</tr>
<tr>
<td>( J = 0.5 ), ( h_1 = 0.5 )</td>
<td>0.9593</td>
<td>0.7895</td>
<td>0.7866</td>
<td>0.7866</td>
<td>0.7866</td>
</tr>
</tbody>
</table>

**Example 3.4.2** Consider the following SNNs

\[
dx(t) = \left[ -(A + \Delta A(t))x(t) + (W_1 + \Delta W_1(t))f(x(t - h)) \right] dt
\]

\[+ \left[ (C + \Delta C(t))x(t) + (D + \Delta D(t))x(t - h) \right] d\omega(t), \quad (3.65)
\]

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where

\[
A = \begin{bmatrix}
4.1989 & 0 & 0 \\
0 & 0.7160 & 0 \\
0 & 0 & 1.9985
\end{bmatrix}, \quad W_1 = \begin{bmatrix}
-0.1052 & -0.5069 & -0.1121 \\
-0.0257 & -0.2808 & 0.0212 \\
0.1205 & -0.2153 & 0.1315
\end{bmatrix},
\]

\[
C = \begin{bmatrix}
-0.1038 & -0.4879 & -0.1088 \\
-0.0268 & -0.2798 & 0.0245 \\
0.1209 & -0.2098 & 0.1311
\end{bmatrix}, \quad D = \begin{bmatrix}
-0.1064 & -0.5073 & -0.1125 \\
-0.0253 & -0.2811 & 0.0202 \\
0.1197 & -0.2136 & 0.1289
\end{bmatrix},
\]

\[
H = \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}, \quad L = \begin{bmatrix}
0.4129 & 0 & 0 \\
0 & 3.8993 & 0 \\
0 & 0 & 1.0160
\end{bmatrix}, \quad T_1 = T_3 = T_4 = 0.1I, \\
T_5 = 0.2I.
\]

It was reported in [98] that for the system (3.65), the achieved maximum allowable delay is \( h = 1.3802 \) when \( r = 1 \) (without decomposition approach). However, using Corollary 3.4.5 with \( J = 0 \), it is found that the maximum allowable upper bound is \( h = 2.5237 \). This shows that the established results are finer than the previous results since the stability region is valid up to the upper bound 2.5237 instead of 1.3802 in [98].

The response of the state trajectories for SNNs are given in Figure 3.2.

**Example 3.4.3** Consider the following uncertain SNNs with time-varying delays:

\[
dx(t) = \left[ -Ax(t) + W_0f(x(t)) + W_1f(x(t - \tau(t)) \right] dt
\]

\[
+ \left[ \Delta C(t)x(t) + \Delta D(t)x(t - \tau(t)) \right] dw(t),
\]

where

\[
A = \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}, \quad W_0 = \begin{bmatrix}
-1 & 2 \\
-2 & 1 \\
-2 & 4
\end{bmatrix}, \quad W_1 = \begin{bmatrix}
-2 & 4 \\
2 & -4 \\
1 & 0
\end{bmatrix}, \quad H = \begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix},
\]

\[
T_4 = 0.01I, \quad T_5 = 0.02I.
\]

When \( L = 0.3I \), \( h_1 = 0 \) and \( J = 0 \), the obtained delay upper bounds by Theorem 3.4.3 are listed in Table 3.6 with the recent ones in [12, 48, 92, 100]. From Table 3.6, one can see that Theorem 3.4.3 has less number of decision variables than those variables discussed in the papers [12, 48, 92, 100]. Also it provides larger delay bounds for different condition of \( \mu \). The response of the state trajectories for SNNs are given in Figure 3.3.
Table 3.6: Maximum allowable upper bound of $h_2$ with different $\mu$

<table>
<thead>
<tr>
<th>$\mu = 0.5$</th>
<th>$\mu = 1.1$</th>
<th>unknown $\mu$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$h_2$</td>
<td>$h_2$</td>
<td>No. of variables</td>
</tr>
<tr>
<td>[12]</td>
<td>0.264</td>
<td>-</td>
</tr>
<tr>
<td>[100]</td>
<td>0.264</td>
<td>0.195</td>
</tr>
<tr>
<td>[92]</td>
<td>0.273</td>
<td>-</td>
</tr>
<tr>
<td>[48]</td>
<td>0.284</td>
<td>-</td>
</tr>
<tr>
<td>Theorem 3.4.3</td>
<td>0.4018</td>
<td>0.2899</td>
</tr>
</tbody>
</table>

Figure 3.1: State trajectories for Example 3.4.1 with $h_2 = 0.7681$ initial condition $[-0.25 \ 0.5]^T$. 
Figure 3.2: State trajectories of Example 3.4.2 with $h = 2.5237$ and initial condition $[0.1 \ 0.2 \ 0.3]^T$.

Figure 3.3: State trajectories of Example 3.4.3 with $h_2 = 0.4018$ and initial condition $[-0.5 \ 0.25]^T$. 

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