Chapter 2

Stability analysis for stochastic neural networks with time-delays

2.1 Introduction

In the past two decades, NNs have received increasing interest owing to their applications in a variety of areas, such as signal processing, pattern recognition, static image processing, associative memory and combinatorial optimization [34]. Stability is one of the main properties of NNs, which is a crucial feature in the design of NNs. In practice, time-delays are often encountered in various engineering, biological and economic systems. Due to the finite speed of information processing, the existence of time-delays frequently causes oscillation, divergence, or even instability of NNs. Recently, many sufficient conditions have been proposed to guarantee the asymptotic or exponential stability for the NNs with various type of time-delays such as constant, time-varying, or distributed, see for example [3, 9, 26, 79, 90, 91, 99], and the references therein.

In addition, the stability criteria for NNs with delays can be classified into two categories, namely, delay-independent criteria and delay-dependent criteria, and the former is more conservative than the latter. Also, there are systems which have non-
zero delays, but they are unstable without delay [28]. Therefore, it is important to perform the stability analysis for the systems with non-zero delays [43, 81, 101], and the non-zero delay can be placed into a given interval. As discussed in [26, 51, 85, 86], distributed delays should be incorporated into the model due to the fact that there may exist a distribution of propagation delays over a period of time in some cases. Recently, some results on stability of SNNs with finite distributed delays have been reported in [51, 85, 86].

Further, in real nervous systems, synaptic transmission is a noisy process brought on by random fluctuations from the release of neurotransmitters and other probabilistic causes, as stated in [6, 15]. Practically, the stochastic phenomenon usually appears in the electrical circuit design of NNs. Moreover, there are many stochastic perturbations that affect the stability of NNs. It has been shown in [63] that a NN could be destabilized by certain stochastic inputs. Therefore, it is of practical importance to study the stochastic effects on the stability property of DNNs, see for example [12, 23, 24, 38, 39, 41, 51, 55, 59, 83, 85, 86, 106],

Based on the above discussions, a class of uncertain SNNs with discrete interval and distributed time-varying delays is considered. The main purpose of this chapter is to study the global robust stability in the mean square for uncertain SNNs with discrete interval and distributed time-varying delays. The parameter uncertainties are assumed to be norm-bounded. By using the Lyapunov-Krasovskii functional technique, global robust stability conditions for the considered uncertain SNNs are given in terms of LMIs, which can be easily calculated by Matlab LMI control toolbox and introducing some free-weighting matrices. Numerical examples are given to illustrate the effectiveness and less conservativeness of the proposed method.
2.2 Robust stability results for uncertain stochastic neural networks with discrete interval and distributed time-varying delays

2.2.1 Problem description and preliminaries

Consider the following HNNs with both discrete and distributed time-varying delays described by

\[ \dot{y}_p(t) = -a_p(y_p(t)) + \sum_{q=1}^{n} w^0_{pq} g_q(y_q(t)) + \sum_{q=1}^{n} w^1_{pq} g_q(y_q(t - \tau(t))) + \sum_{q=1}^{n} e^1_{pq} \int_{t-r(t)}^{t} g_q(y_q(s)) ds + I_p, \quad p = 1, 2, \ldots, n \] (2.1)

or equivalently the vector form

\[ \dot{y}(t) = -Ay(t) + W_0g(y(t)) + W_1g(y(t - \tau(t))) + E_1 \int_{t-r(t)}^{t} g(y(s)) ds + I, \] (2.2)

where \( y(t) = [y_1(t), y_2(t), \ldots, y_n(t)]^T \in \mathbb{R}^n \) denotes the state vector associated with \( n \) neurons. The matrix \( A = \text{diag}\{a_1, a_2, \ldots, a_n\} \) is a diagonal matrix with positive entries \( a_p > 0 \). \( W_0 = (w^0_{pq})_{n \times n}, \ W_1 = (w^1_{pq})_{n \times n} \) and \( E_1 = (e^1_{pq})_{n \times n} \) are connection weights, the discrete delayed connection weights and the distributed delayed connection weights of the \( q \) neuron on the \( p \) neuron respectively. \( I = [I_1, I_2, \ldots, I_n] \) is a constant external input and \( g(x) = [g_1(x_1), \ldots, g_n(x_n)]^T \in \mathbb{R}^n \) is the activation function with \( g(0) = 0 \).

In order to obtain the main results, the following assumptions are always made throughout this section.

\((A1)\) The activation function \( g_p \) is bounded, continuously differentiable with \( g_p(0) = 0 \) and satisfy the Lipschitz condition

\[ |g_p(x_1) - g_p(x_2)| \leq l_p|x_1 - x_2|, \quad \forall \ x_1, x_2 \in \mathbb{R}, \quad p = 1, \ldots, n. \]
Then, by (A1),
\[ |g_p(x_1)| \leq l_p |x_1|, \quad \forall \ x_1 \in \mathbb{R}, \quad p = 1, \ldots, n. \]

(A2) The time-varying delays \( \tau(t) \) and \( r(t) \) satisfy
\[ 0 \leq h_1 \leq \tau(t) \leq h_2, \quad \dot{\tau}(t) \leq \mu < 1, \quad 0 \leq r(t) \leq \bar{r} \]
where \( h_1, h_2, \mu \) and \( \bar{r} \) are constants.

Assume \( y^* = (y_1^*, y_2^*, \ldots, y_n^*)^T \) is an equilibrium point of equation (2.2), one can derive from (2.2) that the transformation \( x(t) = y(t) - y^* \) transforms system (2.2) into the following system:
\[ \dot{x}(t) = -Ax(t) + W_0f(x(t)) + W_1f(x(t - \tau(t))) + E_1 \int_{t-r(t)}^t f(x(s))ds, \quad (2.3) \]
where \( x(t) \) is the state vector of the transformation system, \( f_q(x(t)) = g_q(y_q(t) + y_q^*) - g_q(y_q^*) \) with \( f_q(x(0)) = 0 \) for \( q = 1, 2, \ldots, n \). Then, from (2.3) it follows that
\[ f^T(x)f(x) \leq x^T(t)L^TLx(t). \quad (2.4) \]

In reality, it is often the case that the connection weights of the neurons include uncertainties and the network is distributed by environmental noises that affect the stability of the equilibrium. In this section, as in [51], the HNNs with parameter uncertainties and stochastic perturbations can be described as follows:
\[
\begin{align*}
\dot{x}(t) &= \left[ -A(t)x(t) + W_0(t)f(x(t)) + W_1(t)f(x(t - \tau(t))) + E_1 \int_{t-r(t)}^t f(x(s))ds \right]dt \\
&\quad + \left[ C(t)x(t) + D(t)x(t - \tau(t)) + W_2(t)f(x(t)) + W_3(t)f(x(t - \tau(t))) \right] \\
&\quad + E_2 \int_{t-r(t)}^t f(x(s))ds dw(t), \\
x(t) &= \phi(t), \quad \forall \ t \in [-2\bar{h}, 0], \quad \bar{h} = \max\{h_2, \bar{r}\}, \quad (2.5) \\
\end{align*}
\]
where \( w(t) \) denotes a one-dimensional Brownian motion satisfying \( \mathbb{E}\{dw(t)\} = 0 \) and \( \mathbb{E}\{dw(t)^2\} = dt \). The matrices \( A(t) = A + \Delta A(t) \), \( W_0(t) = W_0 + \Delta W_0(t) \),

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\[ W_1(t) = W_1 + \Delta W_1(t), \quad C(t) = C + \Delta C(t), \quad D(t) = D + \Delta D(t), \quad W_2(t) = W_2 + \Delta W_2(t) \]
and \[ W_3(t) = W_3 + \Delta W_3(t), \] where \( C, D, W_2 \) and \( E_2 \) are connection weight matrices with appropriate dimensions. In this system, the parameter uncertainties are assumed to be of the form:

\[
\begin{bmatrix}
\Delta A(t) & \Delta W_0(t) & \Delta W_1(t) & \Delta C(t) & \Delta D(t) & \Delta W_2(t) & \Delta W_3(t)
\end{bmatrix} = HF(t)[T_1 \ T_2 \ T_3 \ T_4 \ T_5 \ T_6 \ T_7],
\]

where \( H \) and \( T_{c_1} (c = 1, 2, \ldots, 7) \) are real known constant matrices of appropriate dimensions. The unknown matrix \( F(t) \) may be time-varying and satisfies

\[
F^T(t)F(t) \leq I.
\]

It is assumed that all the elements of \( F(t) \) are Lebesque measurable. The matrices \( \Delta A(t), \Delta W_0(t), \Delta W_1(t), \Delta C(t), \Delta D(t), \Delta W_2(t), \Delta W_3(t) \) are said to be admissible if both (2.7) and (2.8) hold. \( \phi(t) \in C([-2\tilde{h}, 0]; \mathbb{R}^n) \) is the initial function. \( f(x) = [f_1(x_1), f_2(x_2), \ldots, f_n(x_n)]^T \in \mathbb{R}^n \) is the activation function with \( f(0) = 0 \).

**Remark 2.2.1** In this section, the interval time-varying delay satisfying assumption (A2) is considered for establishing the stability results different from the previous works. This work will merge the established work of [108] when \( \mu = 0 \), that is \( h_1 = h_2 \) in which case \( \tau(t) \) denotes a constant delay. Further, for \( h_1 = 0 \) it implies that \( 0 \leq \tau(t) \leq h_2 \) which was investigated in [41].

### 2.2.2 Robust stability criteria

In this section, new delay-dependent stability criteria is derived for delayed SNNs using the Lyapunov functional method combined with LMI approach.
Defining two new state variables for the SNNs (2.5),

\[
y(t) = -A(t)x(t) + W_0(t)f(x(t)) + W_1(t)f(x(t - \tau(t))) + E_1 \int_{t-\tau(t)}^{t} f(x(s))ds, \quad (2.9)
\]

\[
g(t) = C(t)x(t) + D(t)x(t - \tau(t)) + W_2(t)f(x(t)) + W_3(t)f(x(t - \tau(t))) + E_2 \int_{t-\tau(t)}^{t} f(x(s))ds.
\]

(2.10)

Then, the SNNs (2.5) can be represented as

\[
dx(t) = y(t)dt + g(t)dw(t).
\]

(2.11)

Moreover, the following equality holds,

\[
x(t) - x(t - \tau(t)) = \int_{t-\tau(t)}^{t} dx(s) = \int_{t-\tau(t)}^{t} y(s)ds + \int_{t-\tau(t)}^{t} g(s)dw(s).
\]

(2.12)

**Theorem 2.2.2** Consider the system (2.5) satisfying assumptions (A1) and (A2). Set \( \bar{k} = \max_{1 \leq \rho \leq n} k_\rho \). The equilibrium point of SNNs (2.5) is robustly exponentially stable in the mean square if there exist symmetric positive-definite matrices \( P, Q, R_1, R_2, R_3, Z, Z_1, Z_2 \), for any matrices \( M_k, N_k, S_k, U_k \) and \( V_k \) \((k = 1, 2, \ldots, 9)\), diagonal matrix \( K > 0 \), positive scalars \( b_1, b_2, \epsilon_1 \) and \( \epsilon_2 \) such that the following LMI holds:

\[
\begin{bmatrix}
-\frac{1}{\bar{k}_2}R_1 & -M & -N & -S & UE_1 + VE_2 & UH & VH \\
* & 0 & 0 & 0 & 0 & 0 & 0 \\
* & * & -\frac{1}{\bar{k}_2-b_1}R_2 & 0 & 0 & 0 & 0 \\
* & * & * & -\frac{1}{\bar{k}_2-b_1}R_3 & 0 & 0 & 0 \\
* & * & * & * & -\frac{1}{\bar{k}_2-b_1}Y & 0 & 0 \\
* & * & * & * & * & -\epsilon_1 I & 0 \\
* & * & * & * & * & * & -\epsilon_2 I
\end{bmatrix} \leq 0, \quad (2.13)
\]

where

\[
II_{11} = (\varphi_{y})_{9 \times 9} + \epsilon_1 T_1^T T_1 + \epsilon_2 T_2^T T_2
\]
with

\[ \varphi_{11} = N_1 + N_1^T - U_1A - A^T U_1 + V_1 C + C^T V_1^T + Q + Z_1 + Z_2 + (h_2 - h_1) Z + b_1 L^T L, \]

\[ \varphi_{12} = N_2^T - U_2^T A^T + C^T V_2^T - N_1 - M_1 + S_1, \quad \varphi_{13} = N_3^T + M_1 - U_3^T A^T + C^T V_3^T, \]

\[ \varphi_{14} = N_4^T - S_1 - U_4^T A^T + C^T V_4^T, \quad \varphi_{15} = P + N_5^T - U_5^T A^T + C^T V_5^T - U_1, \]

\[ \varphi_{16} = N_6^T - U_6^T A^T + C^T V_6^T - V_1, \quad \varphi_{17} = N_7^T - U_7^T A^T + C^T V_7^T, \quad \varphi_{18} = N_8^T - U_8^T A^T + C^T V_8^T - V_1, \]

\[ + S_2 + S_2^T + V_2 D + D^T V_2^T + b_2 L^T L, \quad \varphi_{23} = -N_3^T - M_3 + S_3^T + D^T V_3^T, \]

\[ \varphi_{24} = -N_4^T - M_4 + S_4^T + D^T V_4^T - S_2, \quad \varphi_{25} = -N_5^T - M_5 + S_5^T + D^T V_5^T - U_2, \]

\[ \varphi_{26} = -N_6^T - M_6 + S_6^T + D^T V_6^T - V_2, \quad \varphi_{27} = -N_7^T - M_7 + S_7^T + D^T V_7^T + U_2 W_0 + V_2 W_2, \]

\[ \varphi_{28} = -N_8^T - M_8 + S_8^T + D^T V_8^T + U_2 W_1 + V_2 W_3, \quad \varphi_{29} = -N_9^T - M_9 + S_9^T + D^T V_9^T, \]

\[ \varphi_{30} = -Z_1 + M_3^T - S_3, \quad \varphi_{31} = M_4^T - U_3, \quad \varphi_{32} = M_5^T - V_3, \quad \varphi_{33} = M_7^T + U_3 - U_5, \]

\[ + U_3 W_0 + V_3 W_2, \quad \varphi_{34} = M_8^T + U_3 W_1 + V_3 W_3, \quad \varphi_{35} = M_9^T, \quad \varphi_{36} = -Z_2 - S_4^T - S_4, \]

\[ \varphi_{37} = -S_5^T - U_4, \quad \varphi_{38} = -S_6^T - V_4, \quad \varphi_{39} = -S_7^T + U_4 W_0 + V_4 W_2, \quad \varphi_{40} = -S_8^T + U_4 W_1 + V_4 W_3,
\]

\[ + V_4 W_3, \quad \varphi_{41} = -S_5^T, \quad \varphi_{42} = h_2 R_1 + (h_2 - h_1) R_2 + (h_2 - h_1) R_3 - U_5^T - U_5, \]

\[ \varphi_{43} = -V_5^T - U_6^T, \quad \varphi_{44} = U_5 W_0 + V_5 W_2 - U_7^T + K, \quad \varphi_{45} = U_5 W_1 + V_5 W_3 - U_8^T, \]

\[ \varphi_{46} = -V_6^T, \quad \varphi_{47} = -V_6^T + \tilde{K} I, \quad \varphi_{48} = -V_7^T + U_6 W_0 + V_6 W_2, \quad \varphi_{49} = -V_8^T + U_6 W_1 + V_6 W_3,
\]

\[ + V_6 W_3, \quad \varphi_{50} = -V_9^T, \quad \varphi_{51} = -V_9^T + \tilde{K} I + \tilde{Y} + U_7 W_0 + W_0^T U_7^T + V_7 W_2 + W_2^T V_7^T, \]

\[ \varphi_{52} = U_7 W_1 + U_8 W_0 + V_7 W_3 + V_8 W_2, \quad \varphi_{53} = U_9 W_0 + V_9 W_2, \quad \varphi_{54} = -b_1 I + U_8 W_1 + V_8 W_3 + W_3^T V_8^T, \quad \varphi_{55} = U_9 W_1 + V_9 W_3,
\]

\[ \varphi_{56} = \left( h_2 - h_1 \right)^{-1} Z, \quad \varphi_{57} = \left[ -T_1 \ 0 \ 0 \ 0 \ 0 \ T_2 \ T_3 \ 0 \right], \]
\( \hat{t}_2 = [T_4 \ T_5 \ 0 \ 0 \ 0 \ T_6 \ T_7 \ 0], \ M = [M_1^T \ M_2^T \ M_3^T \ M_4^T \ M_5^T \ M_6^T \ M_7^T \ M_8^T \ M_9^T]^T, \)
\( N = [N_1^T \ N_2^T \ N_3^T \ N_4^T \ N_5^T \ N_6^T \ N_7^T \ N_8^T \ N_9^T]^T, \ S = [S_1^T \ S_2^T \ S_3^T \ S_4^T \ S_5^T \ S_6^T \ S_7^T \ S_8^T \ S_9^T]^T, \)
\( U = [U_1^T \ U_2^T \ U_3^T \ U_4^T \ U_5^T \ U_6^T \ U_7^T \ U_8^T \ U_9^T]^T, \ V = [V_1^T \ V_2^T \ V_3^T \ V_4^T \ V_5^T \ V_6^T \ V_7^T \ V_8^T \ V_9^T]^T. \)

**Proof:** Consider the Lyapunov-Krasovskii functional as follows:

\[
V(x_t, t) = x^T(t)Px(t) + 2\sum_{p=1}^{\eta} k_p \int_0^{x_i} f_i(s)ds + \int_{t-\tau(t)}^t x^T(s)Qx(s)ds + \int_{t-h_1}^t x^T(s)Z_1x(s)ds \\
+ \int_{t-h_2}^t x^T(s)Z_2x(s)ds + \int_{-h_2}^{t-h_1} \int_{t+\theta}^t x^T(s)Zx(s)dsd\theta \\
+ \int_{-\theta}^{\theta} \int_{t-\theta}^t f^T(x(s))Y f(x(s))dsd\theta + \int_{-h_2}^{t-h_1} \int_{t+\theta}^t y^T(s)R_1y(s)dsd\theta \\
+ \int_{-h_2}^{t-h_1} \int_{t+\theta}^t y^T(s)R_2y(s)dsd\theta + \int_{-h_2}^{t-h_1} \int_{t+\theta}^t y^T(s)R_3y(s)dsd\theta, \quad (2.14)
\]

where \( x_t = \{x(t+\theta) : -2\bar{h} \leq \theta \leq 0 \}. \) Then, it can be obtained by Itô's formula that

\[
dV(x_t, t) = \mathcal{L}V(x_t, t)dt + 2x^T(t)P_g(t)dw(t),
\]

where

\[
\mathcal{L}V(x_t, t) = 2x^T(t)P_y(t) + g^T(t)P_g(t) + 2f^T(t)x(t)Ky(t) + g^T(t)KIIg(t) + x^T(t)Qx(t) \\
- (1 - \mu)x^T(t - \tau(t))Qx(t - \tau(t)) + x^T(t)Z_1x(t) - x^T(t - h_1)Z_1x(t - h_1) \\
+ x^T(t)Z_2x(t) - x^T(t - h_2)Z_2x(t - h_2) + (h_2 - h_1)x^T(t)Zx(t) \\
- \int_{t-h_2}^{t-h_1} x^T(s)Zx(s)ds + \int_{t-h_2}^{t-h_1} f^T(s)Y f(x(s))ds \\
+ \int_{t-h_2}^{t-h_1} y^T(s)R_1y(s)ds + (h_2 - h_1)y^T(t)R_2y(t) \\
- \int_{t-h_2}^{t-h_1} y^T(s)R_2y(s)ds + (h_2 - h_1)y^T(t)R_3y(t) - \int_{t-h_2}^{t-h_1} y^T(s)R_3y(s)ds. \quad (2.15)
\]
Then by Lemma 1.16.11 and using $0 \leq h_1 \leq \tau(t) \leq h_2$ and $0 < r(t) \leq \bar{r}$, we have

\[ - \int_{t-r(t)}^{t} y^T(s) R_1 y(s) ds \leq -\frac{1}{h_2} \left[ \int_{t-r(t)}^{t} y(s) ds \right]^T R_1 \left[ \int_{t-r(t)}^{t} y(s) ds \right], \quad (2.16) \]

\[ - \int_{t-r(t)}^{t-h_1} y^T(s) R_2 y(s) ds \leq -\frac{1}{h_2 - h_1} \left[ \int_{t-r(t)}^{t-h_1} y(s) ds \right]^T R_2 \left[ \int_{t-r(t)}^{t-h_1} y(s) ds \right], \quad (2.17) \]

\[ - \int_{t-r(t)}^{t-h_2} y^T(s) R_3 y(s) ds \leq -\frac{1}{h_2 - h_1} \left[ \int_{t-r(t)}^{t-h_2} y(s) ds \right]^T R_3 \left[ \int_{t-r(t)}^{t-h_2} y(s) ds \right], \quad (2.18) \]

\[ - \int_{t-r(t)}^{t-h_1} x^T(s) Z x(s) ds \leq -\frac{1}{h_2 - h_1} \left[ \int_{t-r(t)}^{t-h_1} x(s) ds \right]^T Z \left[ \int_{t-r(t)}^{t-h_1} x(s) ds \right], \quad (2.19) \]

\[ - \int_{t-r(t)}^{t} f^T(x(s)) Y f(x(s)) ds \leq -\frac{1}{\bar{r}} \left[ \int_{t-r(t)}^{t} f^T(x(s)) ds \right]^T Y \left[ \int_{t-r(t)}^{t} f(x(s)) ds \right]. \quad (2.20) \]

Now, we define the new vector $e(t)$ as

\[ e^T(t) = \begin{bmatrix} x^T(t) & x^T(t - \tau(t)) & x^T(t - h_1) & x^T(t - h_2) & y^T(t) & g^T(t) & f^T(x(t)) \end{bmatrix} \left( \int_{t-r(t)}^{t-h_1} x(s) ds \right)^T. \]

From (2.9), (2.10) and (2.12), the following zero equations are true for any matrices $N, M, S, U$ and $V$ with appropriate dimensions

\[ \eta_1(t) = 2e^T(t) N \left[ x(t) - x(t - \tau(t)) - \int_{t-r(t)}^{t} y(s) ds - \int_{t-r(t)}^{t} g(s) dw(s) \right], \quad (2.21) \]

\[ \eta_2(t) = 2e^T(t) M \left[ x(t - h_1) - x(t - \tau(t)) - \int_{t-r(t)}^{t-h_1} y(s) ds - \int_{t-r(t)}^{t-h_1} g(s) dw(s) \right], \quad (2.22) \]

\[ \eta_3(t) = 2e^T(t) S \left[ x(t - \tau(t)) - x(t - h_2) - \int_{t-r(t)}^{t-h_2} y(s) ds - \int_{t-r(t)}^{t-h_2} g(s) dw(s) \right], \quad (2.23) \]

\[ \eta_4(t) = 2e^T(t) U \left[ - A(t) x(t) + W_0(t) f(x(t)) + W_1(t) f(x(t - \tau(t))) + E_1 \int_{t-r(t)}^{t} f(x(s)) ds - y(t) \right], \quad (2.24) \]

\[ \eta_5(t) = 2e^T(t) V \left[ C(t) x(t) + D(t) x(t - \tau(t)) + W_2(t) f(x(t)) + W_3(t) f(x(t - \tau(t))) + E_2 \int_{t-r(t)}^{t} f(x(s)) ds - g(t) \right]. \quad (2.25) \]

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It is obvious from (2.4) that
\[ -b_1 f^T(x(t))f(x(t)) + b_1 x^T(t)L^T L x(t) \geq 0, \quad (2.26) \]
\[ -b_2 f^T(x(t-\tau(t)))f(x(t-\tau(t))) + b_2 x^T(t-\tau(t))L^T L x(t-\tau(t)) \geq 0, \quad (2.27) \]
where \( L = \text{diag}\{l_1, l_2, \cdots, l_n\} > 0 \) is a positive diagonal matrix. Substituting (2.16)-(2.20) into (2.15) and adding \( \eta_1(t), \eta_2(t), \eta_3(t), \eta_4(t), \eta_5(t) \), (2.26) and (2.27) to the right side of (2.15) we get
\[ dV(x_t,t) \leq \zeta^T(t)\Psi\zeta(t) + \xi(dw(t)), \]
where
\[
\Psi = \begin{bmatrix}
\hat{\Omega} & -N & -M & -S & UE_1 + VE_2 \\
* & -\frac{1}{h_2} R_1 & 0 & 0 & 0 \\
* & * & -\frac{1}{h_2-h_3} R_2 & 0 & 0 \\
* & * & * & -\frac{1}{h_2-h_3} R_3 & 0 \\
* & * & * & * & -\frac{1}{\bar{p}} Y 
\end{bmatrix}
\]
with
\[ \hat{\Omega} = (\varphi_M)_{9\times9} + \Delta\Omega_1 + \Delta\Omega_2, \]
\[ \Delta\Omega_1 = UF H F(t)\bar{T}_1 + T_1^T F^T(t)H^T U^T, \]
\[ \Delta\Omega_2 = VH F(t)\bar{T}_2 + T_2^T F^T(t)H^T V^T, \]
\[ \zeta^T(t) = \left[ e^T(t) \int_{\tau(t)}^{t} y^T(s)ds \int_{\tau(t)}^{t-h_1} y^T(s)ds \int_{\tau(t)}^{t-h_2} y^T(s)ds \int_{\tau(t)}^{t} f^T(x(s))ds \right], \]
\[ \xi(dw(t)) = -2e^T(t)N \int_{\tau(t)}^{t} g(s)dw(s) - 2e^T(t)M \int_{\tau(t)}^{t-h_1} g(s)dw(s) \]
\[ -2e^T(t)S \int_{\tau(t)}^{t-h_2} g(s)dw(s) + 2x^T(t)Pg(t)dw(t). \]
According to (2.7) and Lemma 1.16.12(i), \( \Delta\Omega_1 \) and \( \Delta\Omega_2 \) satisfy the following inequalities:
\[ \Delta\Omega_1 \leq \epsilon_1^{-1}UHH^T U^T + \epsilon_1 T_1^T T_1, \quad \Delta\Omega_2 \leq \epsilon_2^{-1}VHH^T V^T + \epsilon_2 T_2^T T_2. \]
From applying Lemma 1.16.10 to (2.13), it follows that

$$dV(x_t, t) \leq -a\left(\|x(t)\|^2 + \|x(t - \tau(t))\|^2 + \|x(t - r(t))\|^2\right)dt + \xi(dw(t)),$$

where $a = \lambda_{\text{min}}\{-\Psi\} > 0$. The remaining part of the proof follows from [51].

Consequently, by the proof of Lyapunov stability theory, the equilibrium point of the SNNs (2.5) is robustly exponentially stochastically stable in the mean square for any $\tau(t), r(t)$ satisfying $0 \leq h_1 \leq \tau(t) \leq h_2$, $\dot{\tau}(t) \leq \mu$ and $0 \leq r(t) \leq \bar{r}$. The proof is completed.

**Remark 2.2.3** When the derivative of $\tau(t)$ is unknown, and the delay $\tau(t)$ satisfies $0 \leq h_1 \leq \tau(t) \leq h_2$, by setting $Q = 0$ in (2.14), the system (2.5) is delay/interval dependent and rate-independent robustly exponentially stable in the mean square for delays $0 \leq h_1 \leq \tau(t) \leq h_2$ and $0 \leq r(t) \leq \bar{r}$.

In the following, the robust stability for the following uncertain SNNs with time-varying delays is discussed

$$dx(t) = \left[-Ax(t) + W_0(t) f(x(t)) + W_1(t) f(x(t - \tau(t)))\right]dt + \left[C(t)x(t) + D(t)x(t - \tau(t)) + W_2(t) f(x(t)) + W_3(t) f(x(t - \tau(t)))\right]dw(t), \quad (2.28)$$

where the time-delay $\tau(t)$ satisfies $0 \leq h_1 \leq \tau(t) \leq h_2$, $\dot{\tau}(t) \leq \mu$. Then, the following results can be obtained.

**Theorem 2.2.4** Consider the system (2.28) satisfying assumptions (A1) and (A2). The equilibrium point of SNNs (2.28) is robustly exponentially stable in the mean square if there exist symmetric positive-definite matrices $P$, $Q$, $R_1$, $R_2$, $R_3$, $Z$, $Z_1$, $Z_2$, any matrices $M_k, N_k, S_k, U_k$ and $V_k$ ($k = 1, 2, \ldots, 9$), diagonal matrix $K > 0$, positive scalars $b_1$, $b_2$, $\epsilon_1$ and $\epsilon_2$ such that the following LMI holds:

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\[
\begin{bmatrix}
  H_{11} & -M & -N & -S & UH & VH \\
  * & -\frac{1}{h_2} R_1 & 0 & 0 & 0 & 0 \\
  * & * & -\frac{1}{h_2} R_2 & 0 & 0 & 0 \\
  * & * & * & -\frac{1}{h_2} R_3 & 0 & 0 \\
  * & * & * & * & -\epsilon_1 I & 0 \\
  * & * & * & * & * & -\epsilon_2 I \\
\end{bmatrix} < 0,
\]

where

\[H_{11} = (\varphi_{ij})_{9 \times 9} + \epsilon_1 \bar{T}_1^T \bar{T}_1 + \epsilon_2 \bar{T}_2^T \bar{T}_2\]

with

\[
\begin{align*}
\varphi_{ij} &= \varphi_{ij}, (i, j = 1, 2, \cdots, 9, ((i, j) \neq (7, 7))), \\
\bar{\varphi}_{ij} &= -b_1 I + U_7 W_0 + W_0^T U_7^T + V_7 W_2 + W_2^T V_7^T.
\end{align*}
\]

**Proof:** Consider the Lyapunov-Krasovskii functional as follows:

\[
V(x_t, t) = x^T(t) P x(t) + 2 \sum_{p=1}^{n} k_p \int_0^{x_p} f_p(s) ds + \int_{t-\tau(t)}^{t} x^T(s) Q x(s) ds + \int_{t-h_1}^{t} x^T(s) Z_1 x(s) ds \\
+ \int_{t-h_2}^{t} x^T(s) Z_2 x(s) ds + \int_{-h_1}^{t} \int_{t-\theta}^{t} x^T(s) Z x(s) ds d\theta + \int_{-h_2}^{0} \int_{t-\theta}^{t} y^T(s) R_1 y(s) ds d\theta \\
+ \int_{-h_2}^{t} \int_{t+\theta}^{t} y^T(s) R_2 y(s) ds d\theta + \int_{-h_1}^{t} \int_{t+\theta}^{t} y^T(s) R_3 y(s) ds d\theta.
\]

The remaining part of the proof immediately follows from Theorem 2.2.2. This completes the proof.

### 2.2.3 Numerical examples

In this section, two examples are given to show the effectiveness of established theories.
Example 2.2.1 Consider the uncertain SNNs

\[
dx(t) = \left[ -A(t)x(t) + W_0(t)f(x(t)) + W_1(t)f(x(t - \tau(t))) + E_1 \int_{t-r(t)}^{t} f(x(s))ds \right] dt \\
+ C(t)x(t) + D(t)x(t - \tau(t)) + W_2(t)f(x(t)) + W_3(t)f(x(t - \tau(t))) \\
+E_2 \int_{t-r(t)}^{t} f(x(s))ds \right] dw(t),
\]

where

\[
A = \begin{bmatrix} 4 & 0 \\ 0 & 6 \end{bmatrix}, \quad W_0 = \begin{bmatrix} 0.2 & 0.1 \\ 0.3 & 0.6 \end{bmatrix}, \quad W_1 = \begin{bmatrix} 0.4 & 0.2 \\ 0.1 & 0.7 \end{bmatrix}, \quad W_2 = \begin{bmatrix} 0.2 & 0.6 \\ 0.5 & 0.1 \end{bmatrix}, \\
W_3 = \begin{bmatrix} 0.3 & 0.6 \\ 0.2 & -0.1 \end{bmatrix}, \quad C = \begin{bmatrix} 0.3 & 0 \\ 0 & -0.1 \end{bmatrix}, \quad D = \begin{bmatrix} 0.5 & -0.1 \\ -0.5 & 0 \end{bmatrix}, \quad H = \begin{bmatrix} 0.1 \\ 0.1 \end{bmatrix}, \\
E_1 = E_2 = I, \quad L = 0.2I, \quad T_1 = T_2 = T_3 = T_4 = T_5 = T_6 = T_7 = \begin{bmatrix} 1 & 1 \end{bmatrix}.
\]

It was reported in [51] that the above system is robustly exponentially stable in the mean square when \(0 < \tau(t) \leq 2.15\), \(0 < r(t) \leq 2.15\). However, by Theorem 2.2.2 and using Matlab LMI control toolbox, for \(\mu = 0\), \(h_1 = 0\) it is found that the equilibrium solution of uncertain SNNs (2.28) is robustly exponentially stable in the mean square for any \(\tau(t)\) and \(r(t)\) satisfying \(0 < \tau(t) \leq h_2 = 19.3247\), \(0 < r(t) \leq 19.3247\). This shows that the established results is finer than the previous results since the stability region is valid upto the upper bound 19.3247 instead of 2.15 in [51]. Figure 2.1 shows the state trajectory of the above SNNs with initial condition \([2 \quad -2]^{T}\).

Example 2.2.2 Consider the uncertain SNNs

\[
dx(t) = \left[ -A(t)x(t) + W_0(t)f(x(t)) + W_1(t)f(x(t - \tau(t))) \right] dt + \left[ C(t)x(t) \\
+ D(t)x(t - \tau(t)) + W_2(t)f(x(t)) + W_3(t)f(x(t - \tau(t))) \right] dw(t), \tag{2.29}
\]
where

\[
A = \begin{bmatrix} 4 & 0 \\ 0 & 5 \end{bmatrix}, \quad W_0 = \begin{bmatrix} 0.4 & -0.7 \\ 0.1 & 0 \end{bmatrix}, \quad W_1 = \begin{bmatrix} -0.2 & 0.6 \\ 0.5 & -0.1 \end{bmatrix}, \quad W_2 = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.1 \end{bmatrix},
\]

\[
W_3 = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.1 \end{bmatrix}, \quad C = \begin{bmatrix} 0.5 & 0 \\ 0 & 0.5 \end{bmatrix}, \quad D = \begin{bmatrix} 0 & -0.5 \\ -0.5 & 0 \end{bmatrix}, \quad H = \begin{bmatrix} 0.1 \\ -0.1 \end{bmatrix},
\]

\[
L = 0.5I, \quad T_1 = \begin{bmatrix} 0.2 & 0.3 \end{bmatrix}, \quad T_2 = \begin{bmatrix} 0.2 & -0.3 \end{bmatrix}, \quad T_3 = \begin{bmatrix} -0.2 & -0.3 \end{bmatrix},
\]

\[
T_4 = T_5 = T_6 = T_7 = \begin{bmatrix} 0.1 & 0.1 \end{bmatrix}.
\]

For \( \mu \geq 1 \), \( Q \) will no longer be helpful to improve the stability condition since \(- (1 - \mu)Q\) is nonnegative definite. Therefore, by setting \( Q = 0 \), an easy delay/interval dependent rate independent criterion is derived for unknown \( \mu \). For the above system, applying Theorem 2 in [12] and Theorem 2 in [51], it is found that the equilibrium solution of SNNs (2.29) is robustly exponentially stable in the mean square for any delay \( \tau(t) \) satisfying \( 0 < \tau(t) \leq 0.5730 \) and \( 0 < \tau(t) \leq 0.7056 \) respectively. However, by using Theorem 2.2.4, we can conclude that if \( 0 < \tau(t) \leq 3.8795 \), system (2.29) is robustly exponentially stable in the mean square sense and finer than the previous works based on the upper bound. Figure 2.2 shows the state trajectory of the above SNNs with initial condition \([2 \quad -2]^T\).

![State trajectory](image)

**Figure 2.1:** The state trajectories of Example 2.2.1 with initial condition \([2 \quad -2]^T\).
2.3 **LMI approach for robust stability analysis for stochastic neural networks with time-varying delay**

2.3.1 **Problem description and preliminaries**

Consider the following HNNs with parameter uncertainties and stochastic perturbations as follows:

\[
\begin{align*}
\dot{x}(t) &= \left[ -Ax(t) + W_0(t)f(x(t)) + W_1(t)f(x(t - \tau(t))) \right] dt + \left[ C(t)x(t) \\
&+ D(t)x(t - \tau(t)) + W_2(t)f(x(t)) + W_3(t)f(x(t - \tau(t))) \right] dw(t),
\end{align*}
\]

\[x(t) = \phi(t), \forall \ t \in [-\eta, 0].\]  

(2.30)

The activation function \( f_q(.) \ q = 1, 2, ..., n \), is continuous and bounded that satisfies the following inequality:

\[0 \leq \frac{f_q(s_1) - f_q(s_2)}{s_1 - s_2} \leq l_q, \quad \forall \ s_1, s_2 \in \mathbb{R}, \ s_1 \neq s_2,\]
where $L = \text{diag}\{l_1, l_2, \ldots, l_n\} > 0$ is a positive diagonal matrix. The time-varying delays $\tau(t)$ satisfy

$$0 \leq \tau(t) \leq \eta, \quad \dot{\tau}(t) \leq \mu,$$

where $\eta$ and $\mu$ are constants.

### 2.3.2 Robust stability criteria

In this section, the following stability criteria is established by using the Lyapunov method in terms of LMIs for the SNNs (2.30). Now defining two new state variables for the SNNs (2.30),

$$y(t) = \left[ -A(t)x(t) + W_0(t)f(x(t)) + W_1(t)f(x(t - \tau(t))) \right],$$

and

$$g(t) = \left[ C(t)x(t) + D(t)x(t - \tau(t)) + W_2(t)f(x(t)) + W_3(t)f(x(t - \tau(t))) \right],$$

the system (2.30) becomes

$$dx(t) = y(t)dt + g(t)dw(t).$$

Moreover, the following equality holds,

$$x(t) - x(t - \tau(t)) = \int_{t-\tau(t)}^{t} dx(s) = \int_{t-\tau(t)}^{t} y(s)ds + \int_{t-\tau(t)}^{t} g(s)dw(s).$$

The following theorem gives the mean square asymptotic stability results for SNNs (2.30) without uncertainty.

**Theorem 2.3.1** For given scalars $\eta \ (0 < \eta), \ \alpha \ (0 < \alpha < 1)$ and $\mu$, the equilibrium point of SNNs (2.30) is globally asymptotically stable in the mean square if there exist symmetric positive-definite matrices $P$, $R_\kappa \ (\kappa = 1, 2, 3)$, $Q_\nu \ (\nu = 1, 2, 3, 4)$, diagonal matrices $K_1 > 0$ and $K_2 > 0$ such that the following LMIs hold

$$R_1 + (1 - \mu)R_3 > 0, \quad R_2 + (1 - \mu)R_3 > 0,$$
\[
\Pi_1 = \begin{bmatrix}
\Pi_{11} & \Pi_{12} & 0 & 0 & \Pi_{15} & PW_1 & -A^T U_1 & C^T P \\
* & \Pi_{22} & \Pi_{23} & 0 & 0 & K_2 L & 0 & D^T P \\
* & * & \Pi_{33} & \Pi_{34} & 0 & 0 & 0 & 0 \\
* & * & * & -Q_3 - \frac{1}{(1-\alpha)\eta} R_2 & 0 & 0 & 0 & 0 \\
* & * & * & * & Q_4 - 2K_1 & 0 & W_0^T U_1 & W_0^T P \\
* & * & * & * & * & -(1-\mu)Q_4 - 2K_2 & W_1^T U_1 & W_1^T P \\
* & * & * & * & * & * & -U_1 & 0 \\
* & * & * & * & * & * & * & -P \\
\end{bmatrix} < 0, 
\]

(2.36)

\[
\Pi_2 = \begin{bmatrix}
\Pi_{11} & 0 & \Pi_{12}^{(1)} & 0 & \Pi_{15} & PW_0 & -A^T U_2 & C^T P \\
* & \Pi_{22}^{(1)} & \Pi_{23}^{(1)} & \Pi_{24}^{(1)} & 0 & K_2 L & 0 & D^T P \\
* & * & \Pi_{33}^{(1)} & \Pi_{34}^{(1)} & 0 & 0 & 0 & 0 \\
* & * & * & -Q_3 - \frac{1}{(1-\alpha)\eta} R_2 & 0 & 0 & 0 & 0 \\
* & * & * & * & Q_4 - 2K_1 & 0 & W_0^T U_2 & W_0^T P \\
* & * & * & * & * & -(1-\mu)Q_4 - 2K_2 & W_1^T U_2 & W_1^T P \\
* & * & * & * & * & * & -U_2 & 0 \\
* & * & * & * & * & * & * & -P \\
\end{bmatrix} < 0, 
\]

(2.37)

where

\[
\Pi_{11} = Q_1 + Q_3 - PA - AP^T - \frac{1}{\alpha \eta} (R_1 + (1 - \mu) R_3), \\
\Pi_{12} = \frac{1}{\alpha \eta} (R_1 + (1 - \mu) R_3), \\
\Pi_{15} = PW_0 + K_1 L, \\
\Pi_{22} = -(1 - \mu) Q_3 - \frac{1}{\alpha \eta} (R_1 + (1 - \mu) R_3) - \frac{1}{\alpha \eta} R_1, \\
\Pi_{23} = \frac{1}{\alpha \eta} R_1, \\
\Pi_{33} = -Q_1 + Q_2 - \frac{1}{\alpha \eta} R_1 - \frac{1}{(1 - \alpha) \eta} R_2, \\
\Pi_{34} = \frac{1}{(1 - \alpha) \eta} R_2, \\
\Pi_{13}^{(1)} = \frac{1}{\alpha \eta} (R_1 + (1 - \mu) R_3), \\
\Pi_{22}^{(1)} = -(1 - \mu) Q_3 - \frac{1}{(1 - \alpha) \eta} (R_2 + (1 - \mu) R_3), \\
\Pi_{23}^{(1)} = \frac{1}{(1 - \alpha) \eta} R_2, \\
\Pi_{33}^{(1)} = -Q_1 + Q_2 - \frac{1}{\alpha \eta} (R_1 + (1 - \mu) R_3) - \frac{1}{(1 - \alpha) \eta} (R_2 + (1 - \mu) R_3), \\
U_1 = (\alpha \eta) R_1 + (1 - \alpha) \eta R_2 + \alpha \eta R_3, \\
U_2 = (\alpha \eta) R_1 + (1 - \alpha) \eta R_2 + \eta R_3.
\]

**Proof:** Consider the Lyapunov-Krasovskii functional

\[
V(x_t, t) = V_1(x_t, t) + V_2(x_t, t) + V_3(x_t, t),
\]

(2.38)
where
\[
V_1(x_t, t) = x^T(t)Px(t),
\]
\[
V_2(x_t, t) = \int_{t-\alpha \eta}^{t} x^T(s)Q_1x(s)ds + \int_{t-\eta}^{t-\alpha \eta} x^T(s)Q_2x(s)ds
+ \int_{t-\tau(t)}^{t} x^T(s)Q_3x(s)ds + \int_{t-\tau(t)}^{t} f^T(x(s))Q_4f(x(s))ds,
\]
\[
V_3(x_t, t) = \int_{-\alpha \eta}^{0} \int_{\theta(t)}^{t} y^T(s)R_1y(s)dsd\theta + \int_{-\eta}^{-\alpha \eta} \int_{t-\theta}^{t} y^T(s)R_2y(s)dsd\theta
+ \int_{-\tau(t)}^{0} \int_{t-\theta}^{t} y^T(s)R_3y(s)dsd\theta.
\]

Then, it can be obtained by Itô’s formula that
\[
dV(x_t, t) = \mathcal{L}V(x_t, t)dt + 2x^T(t)Pg(t)dw(t), \tag{2.39}
\]

where
\[
\mathcal{L}V_1(x_t, t) = 2x^T(t)Py(t) + g^T(t)Pg(t), \tag{2.40}
\]
\[
\mathcal{L}V_2(x_t, t) \leq x^T(t)Q_1x(t) - x^T(t-\alpha \eta)Q_1x(t-\alpha \eta) + x^T(t-\alpha \eta)Q_2x(t-\alpha \eta)
- x^T(t-\eta)Q_2x(t-\eta) + x^T(t)Q_3x(t) - (1-\mu)x^T(t-\tau(t))Q_3x(t-\tau(t))
+ f^T(x(t))Q_4f(x(t)) - (1-\mu)f^T(x(t-\tau(t))Q_4f(x(t-\tau(t))), \tag{2.41}
\]
\[
\mathcal{L}V_3(x_t, t) \leq y^T(t)((\alpha \eta)R_1 + (1-\alpha)\eta R_2 + \tau(t)R_3)y(t)
- \int_{t-\alpha \eta}^{t} y^T(s)R_1y(s)ds - \int_{t-\eta}^{t-\alpha \eta} y^T(s)R_2y(s)ds
- (1-\mu)\int_{t-\tau(t)}^{t} y^T(s)R_3y(s)ds. \tag{2.42}
\]

For any $K_1 = \text{diag}\{k_{11}, k_{21}, \ldots, k_{n1}\} > 0$ and $K_2 = \text{diag}\{k_{12}, k_{22}, \ldots, k_{n2}\} > 0$, it is clear that,
\[
0 \leq -2 \sum_{q=1}^{n} k_{q1}f_q(x_q(t))[f_q(x_q(t) - l_qx_q(t))] - 2 \sum_{q=1}^{n} k_{q2}f_q(x_q(t-\tau(t)))
\times [f_q(x_q(t-\tau(t)) - l_qx_q(t-\tau(t))]
\]

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\[ = 2x^T(t)K_1Lf(x(t)) - 2f^T(x(t))K_1f(x(t)) + 2x^T(t - \tau(t))K_2Lf(x(t - \tau(t))) - 2f^T(x(t - \tau(t)))K_2f(x(t - \tau(t))). \] (2.43)

Now, to estimate the upper bound of the last three terms in the inequality (2.42), the following two cases are considered.

Case 1: If \( 0 \leq \tau(t) \leq \alpha \eta \), then

\[- \int_{t-\eta}^{t} y^T(s)R_1y(s)ds - \int_{t-\eta}^{t-\alpha \tau} y^T(s)R_2y(s)ds
- (1 - \mu) \int_{t-\tau(t)}^{t} y^T(s)R_3y(s)ds \] (2.44)

\[= - \int_{t-\tau(t)}^{t} y^T(s)(R_1 + (1 - \mu)R_3)y(s)ds - \int_{t-\eta}^{t-\alpha \tau} y^T(s)R_1y(s)ds
- \int_{t-\eta}^{t-\alpha \tau} y^T(s)R_2y(s)ds. \] (2.45)

From [48], it follows that

\[-1 = -(\alpha \eta)^{-1}\tau(t) - \left(1 - (\alpha \eta)^{-1}\tau(t)\right). \] (2.46)

Since \( R_1 + (1 - \mu)R_3 > 0 \), by using (2.34), (2.46) and Lemma 1.16.11, it can be estimated that the upper bound of the integral term \( \int_{t-\tau(t)}^{t} y^T(s)(R_1 + (1 - \mu)R_3)y(s)ds \) as follows

\[- \int_{t-\tau(t)}^{t} y^T(s)(R_1 + (1 - \mu)R_3)y(s)ds = -(\alpha \eta)^{-1}\tau(t) \int_{t-\tau(t)}^{t} y^T(s)(R_1 + (1 - \mu)R_3)y(s)ds
- (\alpha \eta)^{-1}\left(1 - (\alpha \eta)^{-1}\tau(t)\right)(\alpha \eta) \int_{t-\tau(t)}^{t} y^T(s)(R_1 + (1 - \mu)R_3)y(s)ds
\leq -(\alpha \eta)^{-1}\tau(t) \int_{t-\tau(t)}^{t} y^T(s)(R_1 + (1 - \mu)R_3)y(s)ds
- (\alpha \eta)^{-1}\left(1 - (\alpha \eta)^{-1}\tau(t)\right)\tau(t) \int_{t-\tau(t)}^{t} y^T(s)(R_1 + (1 - \mu)R_3)y(s)ds, \]

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\[
\leq -(\alpha \eta)^{-1} \left[ \int_{t-\tau(t)}^{t} y(s)ds \right]^T (R_1 + (1 - \mu)R_3) \left[ \int_{t-\tau(t)}^{t} y(s)ds \right] \\
- (\alpha \eta)^{-1} (1 - (\alpha \eta)^{-1}\tau(t)) \left[ \int_{t-\tau(t)}^{t} y(s)ds \right]^T \\
\times (R_1 + (1 - \mu)R_3) \left[ \int_{t-\tau(t)}^{t} y(s)ds \right] \\
\leq -(\alpha \eta)^{-1} \left[ \int_{t-\tau(t)}^{t} y(s)ds \right]^T (R_1 + (1 - \mu)R_3) \left[ \int_{t-\tau(t)}^{t} y(s)ds \right] \\
= -(\alpha \eta)^{-1} \left[ x(t) - x(t-\tau(t)) - \int_{(t-\tau(t))}^{t} g(s)d\omega(s) \right]^T \\
\times (R_1 + (1 - \mu)R_3) \left[ x(t) - x(t-\tau(t)) - \int_{(t-\tau(t))}^{t} g(s)d\omega(s) \right] \\
= -(\alpha \eta)^{-1} \left[ (x(t) - x(t-\tau(t)))^T (R_1 + (1 - \mu)R_3) [x(t) - x(t-\tau(t))] - 2[x(t) - x(t-\tau(t))]^T (R_1 + (1 - \mu)R_3) \\
\times \int_{(t-\tau(t))}^{t} g(s)d\omega(s) + \left[ \int_{(t-\tau(t))}^{t} g(s)d\omega(s) \right]^T \\
\times (R_1 + (1 - \mu)R_3) \left[ \int_{(t-\tau(t))}^{t} g(s)d\omega(s) \right] \right]. \quad (2.47)
\]

By using similar method introduced above, an upper bound of the term

\[- \int_{t-\alpha \eta}^{t-\tau(t)} y^T(s)R_1 y(s)ds \quad \text{and} \quad \int_{t-\alpha \eta}^{t-\eta} y^T(s)R_2 y(s)ds \] can be estimated as follows

\[- \int_{t-\alpha \eta}^{t-\tau(t)} y^T(s)R_1 y(s)ds \leq -(\alpha \eta)^{-1} \left[ (x(t-\tau(t)) - x(t-\alpha \eta))^T R_1 [x(t-\tau(t)) - x(t-\alpha \eta)] \\
- 2[x(t-\tau(t)) - x(t-\alpha \eta)]^T R_1 \int_{t-\alpha \eta}^{t-\tau(t)} g(s)d\omega(s) \\
+ \left[ \int_{t-\alpha \eta}^{t-\tau(t)} g(s)d\omega(s) \right]^T R_1 \left[ \int_{t-\alpha \eta}^{t-\tau(t)} g(s)d\omega(s) \right] \right], \quad (2.48)\]
\[- \int_{t-\eta}^{t-\alpha\eta} y^T(s) R_2 y(s) \, ds \leq - (\eta - \alpha\eta)^{-1} \left[ (x(t - \alpha\eta) - x(t - \eta))^T R_2 [x(t - \alpha\eta) - x(t - \eta)] \right.
\]
\[\left. - 2 [x(t - \alpha\eta) - x(t - \eta)]^T R_2 \int_{t-\eta}^{t-\alpha\eta} g(s) \, d\omega(s) \right]
\[\left. + \left[ \int_{t-\eta}^{t-\alpha\eta} g(s) d\omega(s) \right]^T R_2 \left[ \int_{t-\eta}^{t-\alpha\eta} g(s) d\omega(s) \right] \right]. \quad (2.49)\]

Substituting (2.40)-(2.49) into (2.39), one observes that

\[\mathcal{L} V(x_t, t) \leq \xi^T(t) P_1 \xi(t) + 2 (\delta(t) d\omega(t)), \quad (2.50)\]

\[(\delta(t) d\omega(t)) = (\alpha\eta)^{-1} [x(t - \alpha\eta) - x(t - \eta)]^T \left( R_1 + (1 - \mu) R_3 \right) \int_{(t - \tau(t))}^{t} g(s) d\omega(s)\]
\[+ (\alpha\eta)^{-1} [x(t - \tau(t)) - x(t - \alpha\eta)]^T R_1 \int_{t-\alpha\eta}^{t-\tau(t)} g(s) d\omega(s)\]
\[+ (\eta - \alpha\eta)^{-1} [x(t - \eta) - x(t - \alpha\eta)]^T R_2 \int_{t-\eta}^{t-\alpha\eta} g(s) d\omega(s).\]

Taking the mathematical expectation on both sides of (2.50), there exists a positive scalar $\lambda_1 > 0$ such that

\[\mathbb{E}[\mathcal{L} V(x_t, t)] \leq \mathbb{E}(\xi^T(t) P_1 \xi(t)) \leq - \lambda_1 \mathbb{E} \|x(t)\|^2. \quad (2.51)\]

Case 2: If $\alpha\eta \leq \tau(t) \leq \eta$, then

\[- \int_{t-\alpha\eta}^{t} y^T(s) R_1 y(s) \, ds - \int_{t-\eta}^{t-\alpha\eta} y^T(s) R_2 y(s) \, ds\]
\[= -(1 - \mu) \int_{t-\tau(t)}^{t} y^T(s) R_3 y(s) \, ds \quad (2.52)\]

\[= - \int_{t-\alpha\eta}^{t} y^T(s) (R_1 + (1 - \mu) R_3) y(s) \, ds - \int_{t-\tau(t)}^{t-\alpha\eta} y^T(s) (R_2 + (1 - \mu) R_3) y(s) \, ds\]
\[- \int_{t-\eta}^{t-\tau(t)} y^T(s) R_2 y(s) \, ds. \quad (2.53)\]

Noticing that $R_1 + (1 - \mu) R_3 > 0$, $R_2 + (1 - \mu) R_3 > 0$, an upper bound of the integral
term $-\int_{t-\alpha t}^t y^T(s)(R_1 + (1 - \mu)R_3)y(s)ds$ can be obtained as

$$-\int_{t-\alpha t}^t y^T(s)(R_1 + (1 - \mu)R_3)y(s)ds \leq -(\alpha t)^{-1} \left( [x(t) - x(t - \alpha t)]^T (R_1 + (1 - \mu)R_3) [x(t) - x(t - \alpha t)] - 2[x(t) - x(t - \alpha t)]^T \int_{t-\alpha t}^t g(s)d\omega(s) \right. $$

$$+ \left. \left[ \int_{t-\alpha t}^t g(s)d\omega(s) \right]^T (R_1 + (1 - \mu)R_3) \left[ \int_{t-\alpha t}^t g(s)d\omega(s) \right] \right). \tag{2.54}$$

By using

$$-1 = -(\eta - \alpha t)^{-1}(\tau(t) - \alpha t) \left( 1 - (\eta - \alpha t)^{-1}(\tau(t) - \alpha t) \right), \tag{2.55}$$

$$-1 = -(\eta - \alpha t)^{-1}(\eta - \tau(t)) \left( 1 - (\eta - \alpha t)^{-1}(\eta - \tau(t)) \right) \tag{2.56}$$

and Lemma 11.11, an upper bound of the terms $-\int_{t-\alpha t}^t y^T(s)(R_2 + (1 - \mu)R_3)y(s)ds$, $-\int_{t-\alpha t}^t y^T(s)R_2y(s)ds$ are obtained as follows:

$$-\int_{t-\alpha t}^t y^T(s)(R_2 + (1 - \mu)R_3)y(s)ds \leq -(\eta - \alpha t)^{-1} \left( [x(t - \alpha t) - x(t - \tau(t))]^T \right.$$

$$\times (R_2 + (1 - \mu)R_3) [x(t - \alpha t) - x(t - \tau(t))] - 2[x(t - \alpha t) - x(t - \tau(t))]^T$$

$$\times (R_2 + (1 - \mu)R_3) \int_{t-\tau(t)}^{t-\alpha t} g(s)d\omega(s) + \left[ \int_{t-\tau(t)}^{t-\alpha t} g(s)d\omega(s) \right]^T$$

$$\times (R_2 + (1 - \mu)R_3) \left[ \int_{t-\tau(t)}^{t-\alpha t} g(s)d\omega(s) \right] \right), \tag{2.55}$$

$$-\int_{t-\alpha t}^t y^T(s)R_2y(s)ds \leq -(\eta - \alpha t)^{-1} \left( [x(t - \tau(t)) - x(t - \eta)]^T R_2 [x(t - \tau(t)) - x(t - \eta)] \right.$$

$$- 2[x(t - \tau(t)) - x(t - \eta)]^T R_2 \int_{t-\tau(t)}^{t-\eta} g(s)d\omega(s)$$

$$+ \left[ \int_{t-\tau(t)}^{t-\eta} g(s)d\omega(s) \right]^T R_2 \left[ \int_{t-\tau(t)}^{t-\eta} g(s)d\omega(s) \right] \right). \tag{2.56}$$

Substituting (2.40)-(2.43) and (2.54)-(2.56) into (2.39), it follows that

$$\mathcal{L}V(x_1, t) \leq \xi^T(t) \Pi_2 \xi(t) + (\zeta(t)d\omega(t)), \tag{2.57}$$

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where

\[
(\zeta(t)dw(t)) = 2(\alpha\eta)^{-1}[x(t) - x(t - \alpha\eta)]^T(R_1 + (1 - \mu)R_3) \int_{t-\alpha\eta}^{t} g(s)d\omega(s) \\
+ 2(\eta - \alpha\eta)^{-1}[x(t - \alpha\eta) - x(t - \tau(t))]^T(R_2 + (1 - \mu)R_3) \int_{t-\tau(t)}^{t-\alpha\eta} g(s)d\omega(s) \\
+ 2(\eta - \alpha\eta)^{-1}[x(t - \tau(t)) - x(t - \eta)]^TR_2 \int_{t-\eta}^{t-\tau(t)} g(s)d\omega(s).
\]

Taking the mathematical expectation on both sides of (2.57), there exists a positive scalar $\lambda_2 > 0$ satisfying

\[
\mathbb{E}[LV(x_t,t)] \leq \mathbb{E}(\xi^T(t)\Pi_2\xi(t)) \leq -\lambda_2\mathbb{E}\|x(t)\|^2. \tag{2.58}
\]

$\Pi_1$ and $\Pi_2$ are defined in Theorem 2.3.1 with

\[
\xi^T(t) = \begin{bmatrix}
x^T(t) & x^T(t - \tau(t)) & x^T(t - \alpha\eta) & x^T(t - \eta) & f^T(x(t)) & f^T(x(t - \tau(t)))
\end{bmatrix}.
\]

Thus if $\Pi_i < 0, (i = 1, 2)$ then the stochastic system (2.30) is globally asymptotically stable in the mean square. The proof is completed.

**Remark 2.3.2** In the proof of Theorem 2.3.1, the interval $[t - \eta, t]$ is divided into two subintervals $[t - \eta, t - \alpha\eta]$ and $[t - \alpha\eta, t]$, the information of delayed state $x(t - \alpha\eta)$ can be taken into account.

**Remark 2.3.3** In this section, the LMIs (2.35)-(2.37) obtained for case 1 and case 2 by considering integral terms such as $-\int_{t-\alpha\eta}^{t} y^T(s)R_1y(s)ds$ $-\int_{t-\eta}^{t-\alpha\eta} y^T(s)R_2y(s)ds$ and $-(1 - \mu)\int_{t-\tau(t)}^{t} y^T(s)R_3y(s)ds$ are different from those of [51, 92, 100] and this may lead to less conservatism.

Now, the robust stability analysis for SNNs (2.30) is discussed in the following theorem.

**Theorem 2.3.4** For given scalars $\eta > 0$, $\alpha$ ($0 < \alpha < 1$) and $\mu$, the equilibrium point of SNNs (2.30) is globally robustly asymptotically stable in the mean square if there exist symmetric positive-definite matrices $P$, $R_\kappa$ ($\kappa = 1, 2, 3$), $Q_\iota$ ($\iota = 1, 2, 3, 4$), diagonal matrices $K_1 > 0$, $K_2 > 0$ and scalars $\epsilon_a > 0, (a = 1, 2)$ such that the
following LMIs hold

\[
R_1 + (1 - \mu)R_3 > 0, \quad R_2 + (1 - \mu)R_3 > 0, \quad (2.59)
\]

\[
\begin{bmatrix}
\Xi_1 & \hat{P}H & \Gamma_1U_1 & 0 & \epsilon_1\Gamma_2 & \Gamma_3P & 0 & \epsilon_2\Gamma_4 \\
* & -\epsilon_1I & 0 & 0 & 0 & 0 & 0 & 0 \\
* & * & -U_1 & U_1H & 0 & 0 & 0 & 0 \\
* & * & * & -\epsilon_1I & 0 & 0 & 0 & 0 \\
* & * & * & * & -\epsilon_1I & 0 & 0 & 0 \\
* & * & * & * & * & -\epsilon_1I & 0 & 0 \\
* & * & * & * & * & * & -\epsilon_1I & 0 \\
* & * & * & * & * & * & * & -\epsilon_1I
\end{bmatrix} < 0, \quad (2.60)
\]

\[
\begin{bmatrix}
\Xi_2 & \hat{P}H & \Gamma_1U_2 & 0 & \epsilon_1\Gamma_2 & \Gamma_3P & 0 & \epsilon_2\Gamma_4 \\
* & -\epsilon_1I & 0 & 0 & 0 & 0 & 0 & 0 \\
* & * & -U_2 & U_2H & 0 & 0 & 0 & 0 \\
* & * & * & -\epsilon_1I & 0 & 0 & 0 & 0 \\
* & * & * & * & -\epsilon_1I & 0 & 0 & 0 \\
* & * & * & * & * & -\epsilon_1I & 0 & 0 \\
* & * & * & * & * & * & -\epsilon_1I & 0 \\
* & * & * & * & * & * & * & -\epsilon_1I
\end{bmatrix} < 0, \quad (2.61)
\]

where

\[
\Xi_1 = \begin{bmatrix}
\hat{\Pi}_{11} & \hat{\Pi}_{12} & 0 & 0 & \hat{\Pi}_{15} & PW_0 \\
\hat{\Pi}_{22} & \hat{\Pi}_{23} & 0 & 0 & 0 & 0 \\
\hat{\Pi}_{33} & \hat{\Pi}_{34} & 0 & 0 & 0 & 0 \\
* & * & * & -Q_2 - \frac{1}{(1-\alpha)\eta}R_2 & 0 & 0 \\
* & * & * & Q_4 - 2K_1 + \epsilon_1T_2^T T_2 & 0 & 0 \\
* & * & * & * & -(1-\mu)Q_4 - 2D_2 + \epsilon_1T_3^T T_3 & 0 \\
\end{bmatrix},
\]

\[
\Xi_2 = \begin{bmatrix}
\hat{\Pi}_{11} & \hat{\Pi}_{12} & 0 & 0 & \hat{\Pi}_{15} & PC \\
\hat{\Pi}_{22} & \hat{\Pi}_{23} & \hat{\Pi}_{24} & 0 & 0 & 0 \\
\hat{\Pi}_{33} & \hat{\Pi}_{34} & \hat{\Pi}_{35} & 0 & 0 & 0 \\
* & * & * & -Q_2 - \frac{1}{(1-\alpha)\eta}R_2 & 0 & 0 \\
* & * & * & Q_4 - 2D_1 + \epsilon_1T_2^T T_2 & 0 & 0 \\
* & * & * & * & -(1-\mu)Q_4 - 2K_2 + \epsilon_1T_3^T T_3 & 0 \\
\end{bmatrix},
\]

\[
\hat{P} = \begin{bmatrix} P & 0 & 0 & 0 & 0 \end{bmatrix}^T, \quad \Gamma_1 = [-A^T \ 0 \ 0 \ 0 \ W_0^T \ W_1^T]^T, \quad \Gamma_2 = [-T_1 \ 0 \ 0 \ 0 \ T_2 \ T_3]^T, \\
\Gamma_3 = [C^T \ D^T \ 0 \ 0 \ W_2^T \ W_3^T]^T, \quad \Gamma_4 = [T_4 \ T_5 \ 0 \ 0 \ T_6 \ T_7]^T, \quad \tilde{H} = \Pi_{11} + \epsilon_1T_1^T T_1.
\]

**Proof:** Replacing \(A, W_0, W_1, C, D, W_2\) and \(W_3\) in (2.35) to (2.37) with \(A + HF(t)T_1, \ W_0 + HF(t)T_2, \ W_1 + HF(t)T_3, \ C + HF(t)T_4, \ D + HF(t)T_5, \ W_2 + HF(t)T_6, \ W_3 + HF(t)T_7\) and using Lemma 1.16.10 and 1.16.12, the LMIs (2.59) to (2.61) are obtained. Therefore the proof is completed.
2.3.3 Numerical examples

In this section, numerical examples are given to show the effectiveness of established theories.

**Example 2.3.1** Consider the SNNs (2.30) with following matrices

\[
A = \begin{bmatrix} 4 & 0 \\ 0 & 5 \end{bmatrix}, \quad W_0 = \begin{bmatrix} 0.4 & -0.7 \\ 0.1 & 0 \end{bmatrix}, \quad W_1 = \begin{bmatrix} -0.2 & 0.6 \\ 0.5 & -0.1 \end{bmatrix}, \quad C = \begin{bmatrix} 0.5 & 0 \\ 0 & 0.5 \end{bmatrix},
\]

\[
D = \begin{bmatrix} 0 & -0.5 \\ -0.5 & 0 \end{bmatrix}, \quad W_2 = \begin{bmatrix} 0.1 & 0 \\ 0.1 & 0 \end{bmatrix}, \quad W_3 = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.1 \end{bmatrix}, \quad H = \begin{bmatrix} 0.1 \\ -0.1 \end{bmatrix},
\]

\[
L = 0.5I, \quad T_1 = [0.2 \quad 0.3], \quad T_2 = [0.2 \quad -0.3], \quad T_3 = [-0.2 \quad -0.3],
\]

\[
T_4 = T_5 = T_6 = T_7 = [0.1 \quad 0.1], \quad f(x(t)) = 0.5tanh(x(t)).
\]

For this system, when the differential of \( \tau(t) \) is unknown, applying Theorem 2 in [12] and [51], it is found that the equilibrium point of SNNs (2.30) is robustly exponentially stable in the mean square for any delay \( \tau(t) \) satisfying \( 0 < \tau(t) \leq 0.5730 \) and \( 0 < \tau(t) \leq 0.7056 \). However, using Theorem 2.3.4 and taking \( R_3 = Q_3 = Q_4 = 0 \), the allowable upper bound is obtained as \( \eta = 1.3346 \) for \( \alpha = 0.5 \).

Specially, when \( \Delta C(t) = \Delta D(t) = W_2 = \Delta W_2(t) = W_3 = \Delta W_3(t) = 0 \), the system is same as the one in [41]. It is shown in [41] that the uncertain SNN is globally stable in the mean square for the maximum allowed time-delay being 0.4109. In [12, 51, 92, 100], the maximum allowable upper bounds are obtained as 0.6740, 0.7795, 0.8269 and 0.6194, respectively. However, using Theorem 2.3.4 and taking \( R_3 = Q_3 = Q_4 = 0 \), it is found that allowable upper bound as \( \eta = 1.5187 \) for \( \alpha = 0.5 \). This implies that the system (2.30) is robustly asymptotically stable in the mean square. Therefore, for this example, the results given in this section are less conservative than those in [12, 41, 51, 92] and [100]. The response of the state trajectories for the SNNs (2.30) which converges to zero asymptotically in the mean square are given in Figure 2.3.
Example 2.3.2 Consider the following uncertain SNNs

\[ dx(t) = \left[ -(A + \Delta A(t)) x(t) + (W_1 + \Delta W_1(t)) f(x(t - \tau(t))) \right] dt \\
+ \left[ \Delta C(t) x(t) + \Delta D(t) x(t - \tau(t)) \right] dw(t), \]  

(2.62)

where

\[ A = \begin{bmatrix} 2.2 & 0 & 0 \\ 0 & 2.4 & 0 \\ 0 & 0 & 2.6 \end{bmatrix}, \quad W_1 = \begin{bmatrix} 0.3 & -1.8 & 0.5 \\ -1.1 & 1.6 & 1.1 \\ 0.6 & 0.4 & -0.3 \end{bmatrix}, \quad H = \begin{bmatrix} 0.1 & 0 & 0 \\ 0 & 0.5 & 0 \\ 0 & 0 & 0.3 \end{bmatrix}, \]

\[ T_1 = 0.6I, \quad T_3 = T_4 = T_5 = 0.2I. \]

It was reported in [89] that the system (2.62) with the above matrices is robustly asymptotically stable in the mean square when \( 0 < h \leq 0.8 \). However, using Theorem 2.3.4, taking \( R_3 = Q_3 = Q_4 = 0 \) and \( f(x(t)) = 0.3 \tanh(x(t)) \), the allowable upper bound is obtained as \( \eta = 2.1084 \) when \( \alpha = 0.5 \) and \( L = 0.3I \). Thus, the equilibrium point of uncertain SNNs (2.62) is globally robustly asymptotically stable in the mean square. In the case of \( W_1 = \begin{bmatrix} 0.5 & 0.6 & 0.9 \\ 1.7 & 1.9 & 1.8 \\ 1.3 & 1.5 & 1.9 \end{bmatrix} \), it was reported in [106] that the delay-dependent stability conditions in [89] were not feasible and delay-dependent stability conditions in [106] were satisfied for \( h = 0.8 \). However, using Theorem 2.3.4, taking \( R_3 = Q_3 = Q_4 = 0 \) and \( f(x(t)) = 0.2 \tanh(x(t)) \), the allowable upper bound is obtained as \( \eta = 1.9857 \) when \( \alpha = 0.5 \) and \( L = 0.2I \). This implies that the system (2.62) is globally robustly asymptotically stable in the mean square. The response of the state trajectories for the SNNs (2.62) which converges to zero asymptotically in the mean square are given in Figure 2.4.

Examples 2.3.3 Consider the following SNNs [104]

\[ dx(t) = \left[ -(A + \Delta A(t)) x(t) + (W_0 + \Delta W_0(t)) f(x(t)) + (W_1 + \Delta W_1(t)) f(x(t - \tau(t))) \right] dt \\
+ \sigma(t, x(t), x(t - \tau(t))) dw(t), \]  

(2.63)

where

\[ A = \begin{bmatrix} 1.5 & 0 & 0 \\ 0 & 0.5 & 0 \\ 0 & 0 & 2.3 \end{bmatrix}, \quad W_0 = \begin{bmatrix} 0.3 & -0.19 & 0.3 \\ -0.15 & 0.2 & 0.36 \\ -0.17 & 0.29 & -0.3 \end{bmatrix}, \quad W_1 = \begin{bmatrix} 0.19 & -0.13 & 0.2 \\ 0.26 & 0.09 & 0.1 \\ 0.02 & -0.15 & 0.07 \end{bmatrix}, \]

\[ C = D = 0.1I, \quad H = 0.1I, \quad T_1 = T_2 = T_3 = I, \quad f(x(t)) = 0.5 \tanh(x(t)). \]
When $K_p = \text{diag}\{1,1,1\}$, $K_m = \text{diag}\{-0.5,-0.5,-0.5\}$, it is proved in [104] and [48] that the system is feasible with maximum allowable upper bound $\eta = 2.2471$ and for any $\eta > 0$ respectively. For the case of $K_p = \text{diag}\{1.2,0.5,1.3\}$, $K_m = \text{diag}\{0,0,0\}$, in [48, 104], it is found that the obtained allowable upper bounds are 19.9261, 9.6876, if $\mu = 0.85$ and 4.6364, 2.3879 if $\mu$ is unknown respectively. However, this results, for the case of $L = 0.5I$, when $\mu = 0.85$ and unknown $\mu$ hold for any $\eta > 0, (0 < \alpha < 1)$ and 5.0161($\alpha = 0.2$), which shows that Theorem 2.3.4 improves the feasible region of stability criterion. The response of the state trajectories for the SNNs (2.63) which converges to zero asymptotically in the mean square are given in Figure 2.5.

**Examples 2.3.4** Consider the following uncertain SNNs with time-varying delays:

$$dx(t) = \left[-Ax(t) + W_0f(x(t)) + W_1f(x(t - \tau(t))\right]dt$$

$$+ \left[\Delta C(t)x(t) + \Delta D(t)x(t - \tau(t))\right]dw(t),$$

(2.64)

where

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad W_0 = \begin{bmatrix} -1 & 2 \\ 1 & -2 \end{bmatrix}, \quad W_1 = \begin{bmatrix} -2 & 4 \\ 2 & -4 \end{bmatrix}, \quad H = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

$$C = D = W_2 = W_3 = 0, \quad T_1 = T_2 = T_3 = T_5 = T_7 = 0,$$

$$T_4 = 0.01I, \quad T_5 = 0.02I, \quad f(x(t)) = 0.5\tanh(x(t)).$$

When $L = 0.5I$, the obtained delay upper bounds by Theorem 2.3.4 are listed in Table 2.1. It is clear that these results significantly improved than those results obtained in [12, 48, 92, 100]. This implies that the system (2.64) is globally robustly asymptotically stable in the mean square. The response of the state trajectories for the SNNs (2.64) which converges to zero asymptotically in the mean square are given in Figure 2.6.
Table 2.1: Maximum allowable upper bound of $\eta$ with different $\mu$

<table>
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<tr>
<th></th>
<th>$\mu = 0.5$</th>
<th>$\mu = 1.1$</th>
</tr>
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<td>$\eta$</td>
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<td></td>
</tr>
<tr>
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<td>0.195</td>
</tr>
<tr>
<td>[92]</td>
<td>0.273</td>
<td></td>
</tr>
<tr>
<td>[48]</td>
<td>0.284</td>
<td></td>
</tr>
</tbody>
</table>

Theorem 2.3.4 $0.3851$ ($\alpha = 0.4$) $0.2584$ ($\alpha = 0.4$)

Figure 2.3: State trajectories for Example 2.3.1 with $\eta = 1.3346$ and initial condition $[-0.25 \ 0.5]^T$. 
Figure 2.4: State trajectories of Example 2.3.2 with $\eta = 2.1084$ and initial condition $[-5 \ 3 \ 9]^T$.

Figure 2.5: State trajectories of Example 2.3.3 with $\eta = 5.0161$ and initial condition $[-0.7 \ 0.6 \ -0.9]^T$. 
Figure 2.6: State trajectories of Example 2.3.4 with $\eta = 0.3851$ and initial condition $[-0.7 \ 0.4]^T$. 