6.1 INTRODUCTION

In framing the traditional inventory model, it was assumed that the payment must be made to the supplier for the items immediately after receiving the consignment. However, a supplier permits the buyer a period of time (say 30 days), to settle the total amount owed to him/her. Usually, interest is not charged for the outstanding amount if it is paid within the permissible delay period. This credit term in financial management is denoted as “net 30”. But if the payment is not paid within the permissible delay period, then interest is charged on the outstanding amount under the previously agreed terms and conditions. Hence, the buyer earns the interest on the accumulated revenue received. The buyer can delay the payment upto the last moment of the permissible period allowed by the supplier. Thus the buyer is very much benefitted by making possible use of the permissible delay period. This reduces the amount of capital invested in stock for the duration of the permissible period. Allowing credit period is a marketing strategy for the supplier to attract new customers who consider it to be a type of cost (or price) reduction. However, the strategy of granting credit terms adds not only an additional cost to the supplier but also an additional dimension of default risk to the supplier.
In reality, to encourage the retailer to buy more, the supplier offers permissible delay period. Thus to be more practical, in this chapter, we have developed an appropriate inventory model for non-instantaneous deteriorating items with permissible delay in payments. However, there is no inventory model for non-instantaneous deteriorating items with trade credit and shortages partially backlogged. In this chapter, we frame a more general model which is a general framework that comprises numerous previous models such as in Ghare and Schrader [49], Goyal [52], Teng [133] and Ouyang et al. [102] as special cases.

The rest of the chapter is organized as follows: In section 6.2, the problem description of this chapter is given followed by assumptions and notations used throughout this chapter. In section 6.3, the mathematical model to minimize the total inventory cost is established. Section 6.4 presents useful theorems to characterize the optimal solutions and provides a simple algorithm. Several numerical examples are provided in section 6.5 to illustrate the theory and the solution procedure. This is followed by managerial implications in section 6.6 and conclusion in section 6.7.

6.2 PROBLEM DESCRIPTION

In this chapter, an appropriate inventory model for non-instantaneous deteriorating items with permissible delay in payments is proposed. Further, in this model, shortages are allowed and partially backlogged. The backlogging rate is variable and dependent on the waiting time for the next replenishment. Some useful theorems have been framed to characterize the optimal solutions. An easy-to-use algorithm designed in
this chapter helps to find the optimal replenishment cycle time and the optimal order quantity under various circumstances.

6.2.1 ASSUMPTIONS

1) $D$ is the annual demand rate for the item. It is constant.
2) Replenishment rate is infinite and lead time is zero.
3) Shortages are allowed and partially backlogged. It is to be noted that, the longer the waiting time is, the smaller the backlogging rate will be. Let $B(t)$ denote this fraction where $t$ is the waiting time up to the next replenishment. $B(t) = \frac{1}{1 + \delta t}$, where $\delta$ is the backlogging parameter such that $0 \leq \delta \leq 1$.
4) The product life (time to deterioration) $t$ has a probability density function $f(t) = \theta e^{-\theta(t-t_d)}$ for $t > t_d$ where $t_d$ is the length of time in which the product has no deterioration (fresh product time) and $\theta$ is a parameter. The cumulative distribution function of $t$ is given by $F(t) = \int_{t_d}^{t} f(x) \, dx = 1 - e^{-\theta(t-t_d)}$ for $t > t_d$, so that the deterioration rate is $r(t) = \frac{f(t)}{1 - F(t)} = \theta$, for $t > t_d$.
5) $t_d$ can be estimated by utilizing the random sample data of the product during past time and statistical maximum likelihood method. For simplicity, we assume that $t_d$ is a given constant and $t_d \leq t_1$ in this chapter.
6) During the trade credit period, $M$, the account is not settled; generated sales revenue is deposited in an interest bearing account. At the end of the period, the retailer pays off all units bought, and starts to pay the capital opportunity cost for the items in stock.

7) The system operates for an infinite planning horizon.

6.2.2 NOTATIONS

Even though the common notations are given in section 1.8, the following notations are used only in this chapter.

$TC(t_1, T)$ the total annual inventory cost

$M$ trade credit period

$h$ holding cost per unit per unit time excluding interest charges

6.3 MODEL FORMULATION

The inventory system evolves as follows: $I_m$ units of items arrive at the inventory system at the beginning of each cycle. During the time interval $[0, t_d]$, the inventory level is decreasing only owing to demand rate. The inventory level is dropping to zero due to demand and deterioration during the time interval $[t_d, t_1]$. Then the shortage interval keeps to the end of the current order cycle. The whole process is repeated.

Based on the above description, the differential equation representing the inventory status is given by,
\[
\frac{dI(t)}{dt} = \begin{cases} 
-D & \text{if } 0 \leq t \leq t_d, \\
-D - \theta I(t) & \text{if } t_d \leq t \leq t_1, \\
\frac{-D}{1 + \delta(T-t)} & \text{if } t_1 \leq t \leq T,
\end{cases} \tag{56}
\]

with the boundary conditions \(I(0) = I_m, I(t_1) = 0\).

The solution of equation (56) is

\[
I(t) = \begin{cases} 
I_1(t) & \text{if } 0 \leq t \leq t_d, \\
I_2(t) & \text{if } t_d \leq t \leq t_1, \\
I_3(t) & \text{if } t_1 \leq t \leq T,
\end{cases} \tag{57}
\]

where

\[
I_1(t) = \frac{D}{\theta} \left[ e^{\theta(t-t_d)} - \theta(t-t_d) - 1 \right],
\]

\[
I_2(t) = \frac{D}{\theta} \left[ e^{\theta(t-t)} - 1 \right],
\]

\[
I_3(t) = -\frac{D}{\delta} \left[ \ln[1 + \delta(T-t_1)] - \ln[1 + \delta(T-t)] \right].
\]

The total annual cost which is a function of \(t_1\) and \(T\) is given by (refer to Appendix D for detailed calculations),

\[
TC(t_1, T) = \begin{cases} 
TC_1(t_1, T) & \text{if } 0 < M \leq t_d, \\
TC_2(t_1, T) & \text{if } t_d < M \leq t_1, \\
TC_3(t_1, T) & \text{if } M > t_1,
\end{cases} \tag{58}
\]

where,

\[
TC_1(t_1, T) = \frac{D}{T} \left[ \frac{A}{D} + \frac{ht_d}{\theta} \left[ e^{\theta(t-t_1)} - 1 \right] + \frac{ht_d^2}{2} + \frac{h + p\theta}{\theta^2} \left[ e^{\theta(t-t_1)} - \theta(t_1 - t_d) - 1 \right] \right]
\]
\[ \frac{s + \delta \pi}{\delta} \left[ (T - t_1) - \frac{1}{\delta} \ln \left[ 1 + \delta (T - t_1) \right] \right] + p I_p \left\{ \frac{(t_1 - M)^2}{\theta} \left[ e^{\theta (t_1 - t_2)} - 1 \right] \right\}, \] (59)

\[ TC_2(t_1, T) = \frac{D}{T} \left\{ \frac{A}{D} + \frac{h t_d}{\theta} \left[ e^{\theta (t_1 - t_2)} - 1 \right] + \frac{h t_d^2}{2} + \frac{h + p \theta}{\theta^2} \left[ e^{\theta (t_1 - t_2)} - \theta (t_1 - t_2) - 1 \right] \right\} + \frac{s + \delta \pi}{\delta} \left[ (T - t_1) - \frac{1}{\delta} \ln \left[ 1 + \delta (T - t_1) \right] \right] + p I_p \left\{ \frac{e^{\theta (t_1 - M)} - \theta (t_1 - M) - 1}{\theta^2} \right\} \] (60)

and

\[ TC_3(t_1, T) = \frac{D}{T} \left\{ \frac{A}{D} + \frac{h t_d}{\theta} \left[ e^{\theta (t_1 - t_2)} - 1 \right] + \frac{h t_d^2}{2} + \frac{h + p \theta}{\theta^2} \left[ e^{\theta (t_1 - t_2)} - \theta (t_1 - t_2) - 1 \right] \right\} + \frac{s + \delta \pi}{\delta} \left[ (T - t_1) - \frac{1}{\delta} \ln \left[ 1 + \delta (T - t_1) \right] \right] - p I_p t_1 (M - t_1 / 2) \] (61)
6.4 THEORETICAL RESULTS

Case 1. \(0 < M \leq t_d\) (refer to Figure 3)

The necessary conditions for the total annual cost in (59) to be minimum are

\[
\frac{\partial TC_1(t_1, T)}{\partial t_1} = 0 \text{ and } \frac{\partial TC_1(t_1, T)}{\partial T} = 0,
\]

which gives

\[
\frac{\partial TC_1}{\partial t_1} = \frac{D}{T} \left\{ e^{\theta(t_1-t_d)} \left( ht_d + \frac{h+p\theta}{\theta} \right) - \frac{h+p\theta}{\theta} + \left( pl_p(t_d-M) + \frac{pl_p}{\theta} \right) e^{\theta(t_1-t_d)} \right. \\

\left. - \frac{pl_p}{\theta} + \left( \frac{s+\delta\pi}{\delta} \right) \left( -1 + \frac{1}{1+\delta(T-t_1)} \right) \right\} = 0
\]

(62)

and

\[
\frac{\partial TC_1}{\partial T} = \frac{D}{T^2} \left\{ \frac{(T-t_1)(\delta t_1-1)}{1+\delta(T-t_1)} + \frac{1}{\delta} \ln \left[ 1+\delta(T-t_1) \right] \right\} \left( \frac{s+\delta\pi}{\delta} \right) \\

\left. - \frac{A}{D} \frac{ht_d}{\theta} \left( e^{\theta(t_1-t_d)} - 1 \right) - \frac{h^2}{2} - \frac{h+p\theta}{\theta^2} \left( e^{\theta(t_1-t_d)} - \theta(t_1-t_d) - 1 \right) \right. \\

\left. - \frac{pl_p}{\theta} \left\{ \frac{(t_d-M)}{\theta} (e^{\theta(t_1-t_d)} - 1) + \frac{(t_d-M)^2}{2} + \frac{1}{\theta^2} \left( e^{\theta(t_1-t_d)} - 1 - \theta(t_1-t_d) \right) \right\} \\

\left. + \frac{p_t I_e M^2}{2} \right\} = 0.
\]

(63)
Figure 3. Graphical representation of inventory system for the Case $0 < M \leq t_d$

For notational convenience, let

$$N = \frac{s + \delta \pi}{\delta}, \quad L = h t_d + W, \quad W = \frac{h + p \theta}{\theta}, \quad U = \frac{p I_p}{\theta}, \quad \text{and} \quad V = p I_p (t_d - M) + U.$$  

then equations (62) and (63) become

$$T = t_i + \frac{(L + V) e^{\delta (1-t_d)} - (U + W)}{\delta \left[(U + W) + N - (L + V) e^{\delta (1-t_d)}\right]}$$  

(64)

and

$$N\left[\frac{(T-t_i)(\delta t_i - 1)}{1 + \delta (T-t_i)} + \frac{1}{\delta} \ln \left[1 + \delta (T-t_i)\right]\right] - \frac{A}{D} - \frac{(L + V)}{\theta} \left[e^{\delta (1-t_d)} - 1\right] - \frac{ht_d^2}{2} - \frac{pI_p (t_d - M)^2}{2} + (W + U) (t_i - t_d) + \frac{p I_p M^2}{2} = 0$$  

(65)

respectively. Substituting equation (64) into equation (65), we have
\[
\frac{\delta t_d^{-1}}{\delta} \left[ (L+V)e^{\alpha(t_{1}-t_d)} - (W+U) \right] - \frac{N}{\delta} \ln \left[ \frac{W+U}{N} + 1 - \frac{(L+V)}{N}e^{\alpha(t_{1}-t_d)} \right] \\
- \frac{A}{D} \frac{(L+V)}{\theta} \left[ e^{\alpha(t_{1}-t_d)} - 1 \right] - \frac{ht_d^2}{2} - \frac{pl_p(t_d-M)^2}{2} \\
+ (W+U)(t_1-t_d) + \frac{p_I e M^2}{2} = 0 . \tag{66}
\]

**Lemma 6.1.** For \(0 < M \leq t_d\), we have

(a) If \(\frac{\delta t_d^{-1}}{\delta} \left[ (L+V) - (W+U) \right] - \frac{N}{\delta} \ln \left[ \frac{W+U}{N} + 1 - \frac{(L+V)}{N} \right] \]
\[- \frac{A}{D} \frac{ht_d^2}{2} - \frac{pl_p(t_d-M)^2}{2} + \frac{p_I e M^2}{2} \leq 0\], then the solution of \((t_1, T)\) say \((t_{11}, T_1)\) which satisfies (64) and (65) not only exists but also is unique.

(b) If \(\frac{\delta t_d^{-1}}{\delta} \left[ (L+V) - (W+U) \right] - \frac{N}{\delta} \ln \left[ \frac{W+U}{N} + 1 - \frac{(L+V)}{N} \right] \]
\[- \frac{A}{D} \frac{ht_d^2}{2} - \frac{pl_p(t_d-M)^2}{2} + \frac{p_I e M^2}{2} > 0\], then the solution of \((t_1, T)\) say \((t_{11}, T_1)\) which satisfies (64) and (65) does not exist.

**Proof of part (a)**

By assumptions, we have \(T > t_1\). Therefore from equation (64),
\[
\frac{(L+V)e^{\alpha(t_{1}-t_d)} - (U+W)}{\delta \left[ (U+W) + N - (L+V)e^{\alpha(t_{1}-t_d)} \right]} > 0 . \tag*{That is, \((L+V)e^{\alpha(t_{1}-t_d)} - (U+W) > 0\).} 
\]

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Thus, the denominator part, $\delta [(U + W) + N - (L + V) e^{\theta (t_1 - t_d)}] > 0$. This is equivalent to, $(U + W) + N > (L + V) e^{\theta (t_1 - t_d)}$. This implies that

$\quad t_1 < t_d + \frac{1}{\theta} \ln \left( \frac{(U + W) + N}{L + V} \right) = t_1^b$.

We let,

$$F_1(x) = \frac{dx}{\delta} \left[ (L + V) e^{\theta (x - t_d)} - (W + U) \right] - \frac{N}{\delta} \ln \left[ \frac{W + U}{N} + 1 - \frac{(L + V) e^{\theta (x - t_d)}}{N} \right]$$

$$- \frac{A}{D} \frac{(L + V)}{\theta} \left[ e^{\theta (x - t_d)} - 1 \right] - \frac{h t_d^2}{2} - \frac{p I_x (t_d - M)^2}{2}$$

$$+ (W + U)(x - t_d) + \frac{p I_x M^2}{2}, \quad x \geq t_d.$$

(67)

Taking the first-order derivative of $F_1(x)$ with respect to $x \in (t_d, t_1^b)$, we have

$$\frac{dF_1(x)}{dx} = \frac{d}{dx} \left[ (L + V) e^{\theta (x - t_d)} - (W + U) \right]$$

$$- \frac{N}{\delta} \left[ \frac{W + U}{N} + 1 - \frac{(L + V) e^{\theta (x - t_d)}}{N} \right] > 0.$$

Therefore, $\frac{dF_1(x)}{dx} > 0$. Thus $F_1(x)$ is a strictly increasing function with respect to $x$ in the interval $[t_d, t_1^b]$. Furthermore, by using assumption we have $F_1(t_d) \leq 0$ and $\lim_{x \to t_1^b} F_1(x) = +\infty$. Therefore, by using the intermediate value theorem, there exists a unique $t_1$ say $t_1^* \in [t_d, t_1^b]$ such that $F_1(t_1^*) = 0$, (i.e.) $t_1^*$ is the unique solution of (66). Once we obtain $t_1^*$, then the value of $T$ (denoted by $T_1^*$) can be found from (64) and is given by,
\[ T_i^* = t_{i1}^* + \frac{(L+V) e^{\alpha (\bar{\gamma}_1 - \bar{\gamma}_u)} - (U + W)}{\delta \left[ (U + W) + N - (L + V) e^{\alpha (\bar{\gamma}_1 - \bar{\gamma}_u)} \right]} \]

**Proof of part (b)**

If \( \frac{\delta t_d - 1}{\delta} \left[ (L+V) - (W+U) \right] - \frac{N}{\delta} \ln \left[ \frac{W+U}{N} + 1 - \frac{(L+V)}{N} \right] - \frac{A}{D} \)

\[-\frac{ht_d^2}{2} - \frac{p I_p (t_d - M)^2}{2} + p I_e M^2 > 0, \]

then from (67) we have \( F_1(t_d) > 0 \). Since \( F_1(x) \) is a strictly increasing function of \( x \in [t_d, t_i^b] \) we have \( F_1(x) > 0 \) for all \( x \in [t_d, t_i^b] \). Thus, we cannot find a value \( t_i \in [t_d, t_i^b] \) such that \( F_1(t_i) = 0 \). This completes the proof.

**Theorem 6.1.** When \( 0 < M \leq t_d \), we have

(a) If \( \frac{\delta t_d - 1}{\delta} \left[ (L+V) - (W+U) \right] - \frac{N}{\delta} \ln \left[ \frac{W+U}{N} + 1 - \frac{(L+V)}{N} \right] - \frac{A}{D} \frac{ht_d^2}{2} \)

\[-\frac{p I_p (t_d - M)^2}{2} + p I_e M^2 \leq 0, \]

then the total annual cost \( TC_1(t_i, T) \) is convex and reaches its global minimum at the point \( (t_{i1}^*, T_1^*) \), where \( (t_{i1}^*, T_1^*) \) is the point which satisfies equations (64) and (65).
(b) If \( \frac{\delta}{\delta} \left[ (L + V) - (W + U) \right] - \frac{N}{\delta} \ln \left[ \frac{W + U}{N} + 1 - \frac{(L + V)}{N} \right] - \frac{A}{D} \frac{ht_d}{2} > 0 \), then the total annual cost \( TC_1(t_1, T) \) has a minimum value at the point \( (t_{11}^*, T_1^*) \), where \( t_{11}^* = t_d \) and

\[
T_1^* = t_d + \frac{ht_d + pl_p(t_d - M)}{\delta \left[ N - ht_d - pl_p(t_d - M) \right]}.
\]

Proof of part (a)

Taking the second derivative of \( TC_1(t_1, T) \) with respect to \( t_1 \) and \( T \), and then finding the values of these functions at point \( (t_{11}^*, T_1^*) \) we obtain

\[
\frac{\partial^2 TC_1(t_1, T)}{\partial t_1^2}\bigg|_{(t_{11}^*, T_1^*)} = \frac{D}{T_1^*} \left\{ \theta(L + V) e^{\theta(t_{11}^* - t_d)} + \frac{s + \delta \pi}{\left[ 1 + \delta(T_1^* - t_{11}^*) \right]^2} \right\} > 0,
\]

\[
\frac{\partial^2 TC_1(t_1, T)}{\partial t_1 \partial T}\bigg|_{(t_{11}^*, T_1^*)} = -\frac{D}{T_1^*} \left\{ \frac{s + \delta \pi}{\left[ 1 + \delta(T_1^* - t_{11}^*) \right]^2} \right\},
\]

\[
\frac{\partial^2 TC_1(t_1, T)}{\partial T^2}\bigg|_{(t_{11}^*, T_1^*)} = \frac{D}{T_1^*} \left\{ \frac{s + \delta \pi}{\left[ 1 + \delta(T_1^* - t_{11}^*) \right]^2} \right\} > 0
\]

and

\[
\frac{\partial^2 TC_1(t_1, T)}{\partial t_1^2}\bigg|_{(t_{11}^*, T_1^*)} \frac{\partial^2 TC_1(t_1, T)}{\partial T^2}\bigg|_{(t_{11}^*, T_1^*)} - \left[ \frac{\partial^2 TC_1(t_1, T)}{\partial t_1 \partial T}\bigg|_{(t_{11}^*, T_1^*)} \right]^2
\]

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\[
\left( \frac{D}{T_1^*} \right)^2 \theta (L + V)e^{\delta(t_{11} - t_d)} \left\{ \frac{s + \delta \pi}{1 + \delta \left( T_1^* - t_{11}^* \right)^2} \right\} > 0.
\]

(69)

From equations (68), (69) and Lemma 6.1, we find that \( t_{11}, T_1^* \) is the global minimum point of \( TC_1(t_1, T) \).

Proof of part (b)

If
\[
\frac{\delta t_d - 1}{\delta} \left[ (L + V) - (W + U) \right] - \frac{N}{\delta} \ln \left[ \frac{W + U}{N} + 1 - \frac{L + V}{N} \right]
- \frac{A}{D} + \frac{ht_d^2}{2} - \frac{pI_p(t_d - M)^2}{2} + \frac{pI_pM^2}{2} > 0,
\]
then we know that \( F_1(x) > 0 \), for all \( x \in [t_d, t_1^b] \). Thus, \( \frac{\partial TC_1(t_1, T)}{\partial T} = \frac{DF_1(t_1)}{T^2} > 0, \forall t_1 \in [t_d, t_1^b] \) which implies \( TC_1(t_1, T) \) is a strictly increasing function of \( T \). Thus \( TC_1(t_1, T) \) has a minimum value when \( T \) is minimum. Therefore, \( TC_1(t_1, T) \) has a minimum value at the point \( (t_{11}^*, T_1^*) \) where \( t_{11}^* = t_d \) and
\[
T_1^* = t_d + \frac{ht_d + pI_p(t_d - M)}{\delta \left[ N - ht_d - pI_p(t_d - M) \right]}.
\]
This completes the proof.

Case 2. \( t_d < M \leq t_1 \) (refer to Figure 4)

The necessary conditions for the total annual cost in equation (60) to be minimum are
\[
\frac{\partial TC_2}{\partial t_1} = 0
\]

(70)
Figure 4. Graphical representation of inventory system for the Case \( t_d < M \leq t_1 \)

and \( \frac{\partial TC_2}{\partial T} = 0 \). \( \text{(71)} \)

Equations (70) and (71) on simplifying and using the notations given in case (1) become

\[
T = t_1 + \frac{Le^{\theta(t_1-t_d)} + Ue^{\theta(t_1-M)}}{\delta \left[ (W+U) + N - Le^{\theta(t_1-t_d)} - Ue^{\theta(t_1-M)} \right]} \quad \text{(72)}
\]

and

\[
N \left[ \frac{(T-t_1)(\delta t_1-1)}{1+\delta(T-t_1)} + \frac{1}{\delta} \ln \left[ 1 + \delta(T-t_1) \right] \right] - \frac{A}{D} = \frac{I}{\theta} \left[ e^{\theta(t_1-t_d)} - 1 \right] \\
- \frac{ht_d^2}{2} + W(t_1-t_d) - U \left[ e^{\theta(t_1-M)} - 1 \right] + U(t_1 - M) + \frac{p_1 I_s M^2}{2} = 0, \quad \text{(73)}
\]

respectively.

Substituting (72) into (73), we have
\[
\left( \frac{\delta t_1 - 1}{\delta} \right) \left[ L e^{\theta(t_1 - \theta)} - U e^{\theta(M - \theta)} - (W + U) \right] \\
- \frac{N}{\delta} \ln \left[ \frac{W + U}{N} + 1 - \frac{L}{N} e^{\theta(t_1 - \theta)} - \frac{U}{N} e^{\theta(M - \theta)} \right] - \frac{A}{D} - \frac{L}{\theta} \left[ e^{\theta(t_1 - \theta)} - 1 \right] \\
- \frac{h \tau^2}{2} + W(t_1 - t_d) - \frac{U}{\theta} \left[ e^{\theta(M - \theta)} - 1 \right] + U(t_1 - M) + \frac{P(t^2)}{2} = 0
\] (74)

Lemma 6.2. For \( t_d < M \leq t_1 \), we have

a) If \( \left( \frac{\delta M - 1}{\delta} \right) \left[ L e^{\theta(M - \theta)} - W \right] - \frac{N}{\delta} \ln \left[ \frac{W}{N} + 1 - \frac{L}{N} e^{\theta(M - \theta)} \right] \\
- \frac{A}{D} - \frac{L}{\theta} \left[ e^{\theta(M - \theta)} - 1 \right] - \frac{h \tau^2}{2} + W(M - t_d) \leq 0 \), then the solution of \( (t_1, T) \)
which satisfies (72) and (73) not only exists but also is unique.

b) If \( \left( \frac{\delta M - 1}{\delta} \right) \left[ L e^{\theta(M - \theta)} - W \right] - \frac{N}{\delta} \ln \left[ \frac{W}{N} + 1 - \frac{L}{N} e^{\theta(M - \theta)} \right] \\
- \frac{A}{D} - \frac{L}{\theta} \left[ e^{\theta(M - \theta)} - 1 \right] - \frac{h \tau^2}{2} + W(M - t_d) > 0 \), then the solution of \( (t_1, T) \)
which satisfies (72) and (73) does not exist.

Proof.

Define, \( F_2(x) = \left( \frac{\delta x - 1}{\delta} \right) \left[ L e^{\theta(x - \theta)} + U e^{\theta(M - x)} - (W + U) \right] \)
\[
- \frac{N}{\delta} \ln \left[ \frac{W + U}{N} + 1 - \frac{L}{N} e^{\theta(x - \theta)} - \frac{U}{N} e^{\theta(M - x)} \right] - \frac{A}{D} - \frac{L}{\theta} \left[ e^{\theta(x - \theta)} - 1 \right] - \frac{h \tau^2}{2} + W(x - t_d) \\
- \frac{U}{\theta} \left[ e^{\theta(M - x)} - 1 \right] + U(x - M) \), for \( x \in [M, \infty) \). (75)
Taking the first order derivative of $F_2(x)$ with respect to $x \in [M, \infty)$, we have

$$\frac{dF_2(x)}{dx} = \theta \left[ L e^{\theta(x-t_d)} + U e^{\theta(x-M)} \right] x + \frac{Le^{\theta(x-t_d)} + U e^{\theta(x-M)} - (W + U)}{\delta \left[ (W + U) + N - L e^{\theta(x-t_d)} - U e^{\theta(x-M)} \right]} > 0.$$  

Further, by using the assumption we have $F_2(M) \leq 0$. It can be shown that $\lim_{x \to \infty} F_2(x) = \infty$. Therefore, by using the intermediate value theorem, there exists a unique $t_{12}^* \in [M, \infty)$ such that $F_2(t_{12}^*) = 0$. That is $t_{12}^*$ is the unique solution of equation (74). Once we obtain the value $t_{12}^*$, then the value of $T_2^*$ (denoted by $T_2^*$) can be found from equation (72), and is given by

$$T_2^* = t_{12}^* + \frac{L e^{\theta(t_{12}^*-t_d)} + U e^{\theta(t_{12}^*-M)} - (W + U)}{\delta \left[ (W + U) + N - L e^{\theta(t_{12}^*-t_d)} - U e^{\theta(t_{12}^*-M)} \right]}.$$

**Proof of part (b)**

If $\left( \frac{\delta M - 1}{\delta} \right) \left[ L e^{\theta(M-t_d)} - W \right] - \frac{N \ln \left[ \frac{W}{N} + 1 - \frac{L}{N} e^{\theta(M-t_d)} \right]}{\delta} - \frac{A}{D} - \frac{L}{\theta} \left[ e^{\theta(M-t_d)} - 1 \right] - \frac{h t_d^2}{2} + W(M - t_d) > 0$, then from equation (75), we have $F_2(M) > 0$. Since $F_2(x)$ is a strictly increasing function of $x \in [M, \infty)$, it implies that $F_2(x) > 0$ for all $x \in [M, \infty)$. Thus, we cannot find a value $t_1 \in [M, \infty)$ such that $F_2(t_1) = 0$. **This completes the proof.**
Lemma 6.3. For $t_d < M \leq t_1$, we have:

(a) If \( \left( \frac{\delta M - 1}{\delta} \right) \left[ L e^{\theta(M-t_d)} - W \right] - \frac{N}{\delta} \ln \left[ \frac{W}{N} + 1 - \frac{L}{N} e^{\theta(M-t_d)} \right] \)

\[- \frac{A}{D} - \frac{L}{\theta} \left[ e^{\theta(M-t_d)} - 1 \right] - \frac{h t_d^2}{2} + W(M - t_d) \leq 0, \] then the total annual inventory cost $TC_2(t_1, T)$ has the global minimum value at the point $(t_{12}^*, T_2^*)$ which satisfies equations (72) and (73).

(b) If \( \left( \frac{\delta M - 1}{\delta} \right) \left[ L e^{\theta(M-t_d)} - W \right] - \frac{N}{\delta} \ln \left[ \frac{W}{N} + 1 - \frac{L}{N} e^{\theta(M-t_d)} \right] \)

\[- \frac{A}{D} - \frac{L}{\theta} \left[ e^{\theta(M-t_d)} - 1 \right] - \frac{h t_d^2}{2} + W(M - t_d) > 0, \] then the total annual cost $TC_2(t_1, T)$ has a minimum value at the point $(t_{12}^*, T_2^*)$, where $(t_{12}^* = M)$ and $T_2^* = M + \frac{L e^{\theta(M-t_d)} - W}{\delta \left[ W + N - L e^{\theta(M-t_d)} \right]}$.

Proof of part (a) Taking the second order derivative of $TC_2(t_1, T)$ with respect to $t_1$ and $T$, and then finding the values of these functions at point $(t_{12}^*, T_2^*)$, we obtain.

\[ \frac{\partial^2 TC_2(t_1, T)}{\partial t_1^2} \bigg|_{t_{12}^*, T_2^*} = \frac{D}{T^*} \theta \left[ L e^{\theta(t_{12}^*-t_d)} + U e^{\theta(t_{12}^*-M)} \right] \frac{s + \delta \pi}{\left[ 1 + \delta(T_2^*-t_{12}^*) \right]^2} > 0, \]

\[ \frac{\partial^2 TC_2(t_1, T)}{\partial t_1 \partial T} \bigg|_{t_{12}^*, T_2^*} = \frac{D}{T^*} \frac{s + \delta \pi}{\left[ 1 + \delta(T_2^*-t_{12}^*) \right]^2}, \]
\[
\frac{\partial^2 TC_2(t_1, T)}{\partial T^2} \bigg|_{(i_{12}^*, i_{2}^*)} = D \left( \frac{s + \delta \pi}{1 + \delta (T_2^* - i_{12}^*)^2} \right) > 0 \quad (76)
\]

and
\[
\frac{\partial^2 TC_2(t_1, T)}{\partial t_1^2} \bigg|_{(i_{12}^*, i_{2}^*)} - \left[ \frac{\partial^2 TC_2(t_1, T)}{\partial t_1 \partial T} \bigg|_{(i_{12}^*, i_{2}^*)} \right]^2 = \left( \frac{D}{L^*} \right)^2 \left[ Le^{\theta(i_{12} - t_d)} + U e^{\theta(i_{12} - M)} \right] \left( \frac{s + \delta \pi}{1 + \delta (T_2^* - i_{12}^*)^2} \right) > 0. \quad (77)
\]

From equations (76), (77) and Lemma 6.2, \((i_{12}^*, T_2^*)\) is the global minimum point of \(TC_2(i_{12}^*, T_2^*)\).

**Proof of part (b)**

If
\[
\left( \frac{\delta M - 1}{\delta} \right) \left[ L e^{\theta(M - t_d)} - W \right] - \frac{N}{\delta} \ln \left[ \frac{W}{N} + 1 \right] = \frac{L}{\theta} \left[ e^{\theta(M - t_d)} - 1 \right] - \frac{ht_d^2}{2} + W(M - t_d) > 0,
\]

then we know that \(F_2(x) > 0\), for all \(x \in [M, \infty)\). Thus,
\[
\frac{\partial TC_2(t_1, T)}{\partial T} = \frac{DF_2(t_1)}{T^2} > 0, \quad \forall t_1 \in [M, \infty). \]

This implies that \(TC_2(t_1, T)\) is a strictly increasing function of \(T\). Thus, \(TC_2(t_1, T)\) has a minimum value at the point \((i_{12}^*, T_2^*)\), where \(i_{12}^* = M\) and \(T_2^* = M + \frac{L e^{\theta(M - t_d)} - W}{\delta \left[ W + N - L e^{\theta(M - t_d)} \right]}\). This completes the proof.
Case 3. $M > t_1$ (refer to Figure 5)

The necessary conditions for the total annual cost in equation (61) to be minimum are

$$\frac{\partial TC_3(t_1, T)}{\partial t_1} = 0$$  \hspace{1cm} (78)

and $$\frac{\partial TC_3(t_1, T)}{\partial T} = 0.$$  \hspace{1cm} (79)

Using the notations given in case 1 equations (78) and (79) become

$$T = t_1 + \frac{Le_0^{(t_1-t_d)} - W + p_1 I_s (t_1 - M)}{\delta \left[ W - L e_0^{(t_1-t_d)} - p_1 I_s (t_1 - M) + N \right]}$$  \hspace{1cm} (80)

Figure 5. Graphical representation of inventory system for the Case $M > t_1$
and 
\[
N \left[ \frac{(T - t_1)(\delta t_2 - 1)}{1 + \delta (T - t_1)} + \frac{1}{\delta} \ln \left[ 1 + \delta (T - t_1) \right] \right] - \frac{A}{D} - \frac{L}{\theta} \left[ e^{\theta (t_1 - t_d)} - 1 \right] \\
- \frac{ht_d^2}{2} + W(t_1 - t_d) + p_1 I_e t_1 (M - t_1 / 2) = 0
\] (81)
respectively.

Substituting equation (80) into equation (81), we have
\[
\left( \frac{\delta t_1 - 1}{\delta} \right) \left[ L e^{\theta (t_1 - t_d)} - W + p_1 I_e (t_1 - M) \right] \\
- \frac{N}{\delta} \ln \left[ \frac{W}{N} - \frac{L}{N} e^{\theta (t_1 - t_d)} + 1 - \frac{p_1 I_e (t_1 - M)}{N} \right] \\
- \frac{A}{D} - \frac{L}{\theta} \left[ e^{\theta (t_1 - t_d)} - 1 \right] - \frac{ht_d^2}{2} + W(t_1 - t_d) + p_1 I_e t_1 (M - t_1 / 2) = 0.
\] (82)

By using a similar method as in the proof of Lemma 6.1 and Lemma 6.2, we can show that the value of \((t_1, T)\) which satisfies (80) and (81) not only exists but also is unique under certain conditions. Let,
\[
\Delta_1 = t_d (L - W) - \frac{1}{\delta} \left[ (L - W) + p_1 I_e (t_d - M) \right] - \frac{N}{\delta} \ln \left[ \frac{W}{N} - \frac{L}{N} + 1 - \frac{p_1 I_e (t_d - M)}{N} \right] \\
- \frac{A}{D} - \frac{ht_d^2}{2} + \frac{p_1 I_e t_d^2}{2}
\]
and
\[
\Delta_2 = \left( \frac{\delta M - 1}{\delta} \right) \left[ L e^{\theta (M - t_d)} - W \right] - \frac{N}{\delta} \ln \left[ \frac{W}{N} - \frac{L}{N} e^{\theta (M - t_d)} + 1 \right] \\
- \frac{A}{D} - \frac{L}{\theta} \left[ e^{\theta (M - t_d)} - 1 \right] - \frac{ht_d^2}{2} + W(M - t_d) + p_1 I_e M^2 / 2.
\]
Lemma 6.4. For $M > t_1$, we have:

a) If $\Delta_1 \leq 0 \leq \Delta_2$, then the total annual inventory cost $TC_3$ has the global minimum value at the point $(t^*_{13}, T^*_3)$, where $(t^*_{13}, T^*_3)$ is the point which satisfies equations (80) and (81) and $t^*_{13} \in [t_d, M]$.

b) If $\Delta_2 < 0$, then the total annual inventory cost $TC_3(t_1, T)$ has a minimum value at the point $(t^*_{13}, T^*_3)$, where $t^*_{13} = M$ and

$$T^*_3 = M + \frac{Le^\theta(M-t_d) - W}{\delta \left[ W - Le^\theta(M-t_d) + N \right]}.$$

c) If $\Delta_1 > 0$, then the total annual cost $TC_3(t_1, T)$ has a minimum value at the point $(t^*_{13}, T^*_3)$, where $t^*_{13} = t_d$ and

$$T^*_3 = t_d + \frac{L-W + p_1 I_e(t_d-M)}{\delta \left[ W-L - p_1 I_e(t_d-M) + N \right]}.$$

Proof.

(a) $\Delta_1 \leq 0 \leq \Delta_2$

Let, $F_3(x) = \left( \frac{\delta x - 1}{\delta} \right) \left[ Le^{\theta(x-t_d)} - W + p_1 I_e(x-M) \right]$

$$- \frac{N}{\delta} \ln \left[ \frac{W - L}{N} e^{\theta(x-t_d)} + 1 - \frac{p_1 I_e(x-M)}{N} \right] - \frac{A}{D} \frac{L}{\theta} \left[ e^{\theta(x-t_d)} - 1 \right]$$

$$- \frac{ht_d^2}{2} + W(x-t_d) + p_1 I_e(x-M-x/2), \text{ for } t_d \leq x \leq M.$$

Since the first order derivative of $F_3(x)$ with respect to $x \in (t_d, M)$ is
\[
\frac{dF_3(x)}{dx} = \left[L \theta e^{\theta(x-t_d)} + p_1 I_e \right] \left[ x + \frac{Le^{\theta(x-t_d)} - W + p_1 I_e (x-M)}{\delta \left(W - Le^{\theta(x-t_d)} - p_1 I_e (x-M) + N\right)} \right] > 0,
\]

\(F_3(x)\) is a strictly increasing function of \(x\) in the interval \([t_d, M]\). Moreover, by assumption \(F_3(t_d) \leq 0\) and \(F_3(M) \geq 0\). That is, \(F_3(t_d) \leq 0 \leq F_3(M)\). Thus, we can find a unique value \(x \in [t_d, M]\) such that \(F_3(x) = 0\) which implies that the solution of \((t_{13}^*, T_3^*)\) which satisfies (80) and (81) not only exists but also is unique.

Furthermore,

\[
\frac{\partial^2 T C_3}{\partial t_1^2} \bigg|_{(t_{13}^*, T_3^*)} = \frac{D}{T_3^*} \left[ L \theta e^{\theta(t_{13}^*-t_d)} + p_1 I_e + \frac{s + \delta \pi}{\left[1 + \delta \left(T_3^* - t_{13}^*\right)\right]^2} \right] > 0,
\]

\[
\frac{\partial^2 T C_3}{\partial t_1 \partial T} \bigg|_{(t_{13}^*, T_3^*)} = -\frac{D}{T_3^*} \left[ \frac{s + \delta \pi}{\left[1 + \delta \left(T_3^* - t_{13}^*\right)\right]^2} \right],
\]

\[
\frac{\partial^2 T C_3}{\partial T^2} \bigg|_{(t_{13}^*, T_3^*)} = \frac{D}{T_3^*} \left[ \frac{s + \delta \pi}{\left[1 + \delta \left(T_3^* - t_{13}^*\right)\right]^2} \right] > 0
\]

and

\[
\frac{\partial^2 T C_3(t_1, T)}{\partial t_1^2} \bigg|_{(t_{13}^*, T_3^*)} \frac{\partial^2 T C_3(t_1, T)}{\partial T^2} \bigg|_{(t_{13}^*, T_3^*)} - \left[ \frac{\partial^2 T C_3(t_1, T)}{\partial t_1 \partial T} \bigg|_{(t_{13}^*, T_3^*)} \right]^2
\]
\[ \left( \frac{D}{T} \right)^2 \left[ L \theta e^{\theta (t_{13} - t_d)} + p_1 I_e \right] \left( \frac{s + \delta \pi}{1 + \delta (T_3^* - t_{i3}^*)^2} \right) > 0. \]

Therefore, \( TC_3(t_{i3}^*, T_3^*) \) has the global minimum value at \( (t_{i3}^*, T_3^*) \).

(b) On the other hand if \( \Delta_2 < 0, F_3(M) < 0 \). Since \( F_3(x) \) is a strictly increasing function of \( x \) in the interval \([t_d, M]\), we can get \( F_3(x) < 0 \) for all \( x \in [t_d, M] \). This implies that \( \frac{\partial TC_3(t_1, T)}{\partial T} = \frac{DF_3(t_1)}{T^2} < 0 \), for all \( t_1 \in [t_d, M] \).

So, \( TC_3(t_1, T) \) is a strictly decreasing function of \( T \) in the interval \([t_d, M]\).

Thus \( TC_3(t_1, T) \) has a minimum value at \( (t_{i3}^*, T_3^*) \) where \( t_{i3}^* = M \) and the corresponding minimum value of \( T_3^* \) is

\[ T_3^* = M + \frac{Le^{\theta (M - t_d)} - W}{\delta \left[ W - Le^{\theta (M - t_d)} + N \right]}, \]

(c) If \( \Delta_1 > 0, F_3(t_d) > 0 \), then we can get \( F_3(x) > 0 \) for all \( x \in [t_d, M] \), which implies \( \frac{\partial TC_3(t_1, T)}{\partial T} = \frac{DF_3(t_1)}{T^2} > 0 \) for all \( t_1 \in [t_d, M] \).

So, \( TC_3(t_1, T) \) is a strictly increasing function of \( T \) in the interval \([t_d, M]\).

Thus \( TC_3(t_1, T) \) has a minimum value at the point \( (t_{i3}^*, T_3^*) \) where \( t_{i3}^* = t_d \) and

\[ T_3^* = t_d + \frac{L - W + p_1 I_e (t_d - M)}{\delta \left[ W - L - p_1 I_e (t_d - M) + N \right]} \]
Theorem 6.2
When $M > t_d$, we have the following results.

(a) If $\Delta_1 \leq 0 \leq \Delta_2$, then $TC(t_1^*, T^*) = TC_3(t_{13}, T_3)$ and $t_1^* = t_{13}$; $T^* = T_3$, where $(t_{13}, T_3)$ is the point which satisfies (80) and (81).

(b) If $\Delta_2 < 0$, then $TC(t_1^*, T^*) = \min\{ TC_2(t_{12}, T_2), TC_3(M, T_3) \}$ where $(t_{12}, T_2)$ is the point which satisfies (72) and (73). Hence $(t_1^*, T^*) = (t_{12}, T_2)$ or $(M, T_3)$ associated with lower cost.

(c) If $\Delta_1 > 0$, then $TC(t_1^*, T^*) = TC_3(t_d, T_3)$. That is, $(t_1^*, T^*) = (t_{13}, T_3) = (t_d, T_3)$.

Proof.
It immediately follows from Lemmas 6.2, 6.3 and 6.4.

6.4.1 ALGORITHM

Step 1 Compare the values of $M$ and $t_d$. If $M \leq t_d$, then go to Step 2.

Otherwise, if $M > t_d$, go to Step 3.

Step 2 Calculate

$$\Delta = \frac{\delta t_d - 1}{\delta} \left[ (L + V) - (W + U) \right] - \frac{N}{\delta} \ln \left[ \frac{W + U}{N} + 1 - \frac{(L + V)}{N} \right]$$

$$- \frac{A}{D} - \frac{ht_d^2}{2} - \frac{pL}{2} (t_d - M)^2 + \frac{p_1 t_4 M^2}{2}$$

(i) If $\Delta \leq 0$, $TC^*(t_1^*, T^*) = TC_1(t_{11}, T_1)$ where $(t_{11}, T_1)$ is the point which satisfies (64) and (65). Go to Step 4.

(ii) If $\Delta > 0$, $TC^*(t_1^*, T^*) = TC_1(t_d, T_1)$. That is, $(t_1^*, T^*) = (t_{11}, T_1) = (t_d, T_1)$. Go to Step 4.
Step 3 Calculate

\[ \Delta_1 = t_d (L-W) - \frac{1}{\delta} \left[ (L-W) + p_1 I_e (t_d - M) \right] \]

\[ - \frac{N}{\delta} \ln \left[ \frac{W}{N} - \frac{L}{N} e^{\theta (M-t_d)} + 1 - \frac{p_1 I_e (t_1 - M)}{N} \right] - \frac{A}{D} \frac{ht_d^2}{2} + \frac{p_1 I_e t_d^2}{2} \]

and

\[ \Delta_2 = \left( \frac{\delta M - 1}{\delta} \right) \left[ I_e e^{\theta (M-t_d)} - W \right] - \frac{N}{\delta} \ln \left[ \frac{W}{N} - \frac{L}{N} e^{\theta (M-t_d)} + 1 \right] \]

\[ - \frac{A}{D} - \frac{L}{\theta} \left[ e^{\theta (M-t_d)} - 1 \right] - \frac{ht_d^2}{2} + W (M - t_d) + p_1 I_e M^2 / 2. \]

(i) If \( \Delta_1 \leq 0 \leq \Delta_2 \), then \( TC(t_1^*, T^*) = TC_3(t_{13}, T_3) \) and \( (t_1^*, T^*) = (t_{13}, T_3) \)

where \( (t_{13}, T_3) \) is the point which satisfies (80) and (81). Go to Step 4.

(ii) If \( \Delta_2 < 0 \), then \( TC(t_1^*, T^*) = \min \{ TC_2(t_{12}, T_2), TC_3(M, T_3) \} \) and

\( (t_1^*, T^*) = (t_{12}, T_2) \) or \((M, T_3)\) associated with lower cost and \( (t_{12}, T_2) \)

is the point which satisfies (72) and (73). Go to Step 4.

(iii) If \( \Delta_1 > 0 \), then \( TC(t_1^*, T^*) = TC_3(t_d, T_3) \) and \( (t_1^*, T^*) = (t_{13}, T_3) = (t_d, T_3) \).

Go to Step 4.

Step 4 Stop.

After obtaining the optimal values of \( t_1 \) and \( T \), the optimal order quantity \( Q \) (denoted by \( Q^* \)) can be obtained from \( Q = I_m + I_b \) where

\[ I_b = -I_3(T) \] and \( I_m = I_1(0) \).
6.5 NUMERICAL EXAMPLES

In order to illustrate the above solution procedure, we consider the following numerical examples.

Example 6.1

Consider an inventory system with the following data: \( A=250; \quad h=15; \quad s=30; \quad \pi=25; \quad p=80; \quad p_1=85; \quad D=1000; \quad t_d = 0.0685; \quad \theta = 0.08 ; \quad \delta = 0.56 ; \quad M=0.1233, \quad I_p = 0.15; \quad I_e =0.12 \) in appropriate units. We see that \( M > t_d \). Therefore we calculate the values of \( \Delta_1 \) and \( \Delta_2 \). We find that \( \Delta_1 =-0.1884 \) and \( \Delta_2 =0.0366 \). Here \( \Delta_1 \leq 0 \leq \Delta_2 \). Applying the algorithm given in section 6.4 we find that \( t_1^*= 0.1217, \quad T^* = 0.1721, \quad TC^*(t_1^*, T^*)=2019.90 \) and \( Q^*=802 \).

Example 6.2

Again the data are same as in Example 6.1 except that \( t_d = 0.0904; \quad M=0.1096 \). Here we find that \( M > t_d \). Therefore we calculate the values of \( \Delta_1 \) and \( \Delta_2 \). We find that \( \Delta_1 =-0.1316 \) and \( \Delta_2 =-0.0495 \). Here \( \Delta_2 <0 \). Therefore using the algorithm given in section 6.4 we find the optimal values as \( t_1^*=0.1272, \quad T^* =0.1824, \quad TC^*(t_1^*, T^*)=2094.60, \quad Q^*=613 \).

Example 6.3

The data are same as in Example 6.1 except that \( t_d = 0.5014; \quad M=0.0548 \). Here we find that \( M \leq t_d \). Therefore we calculate the value of \( \Delta \). We find that \( \Delta =5.2617 \). Hence applying the algorithm given in section 6.4 we find the optimal values as \( t_1^* = t_d =0.5014, \quad T^* = 0.8515, \quad TC^*(t_1^*, T^*)=6701.10 \) and \( Q^*=821 \).
Example 6.4

The data are same as in Example 6.1 except \( t_d = 0.0822 \) and \( M=0.0548 \). Here we find that \( M \leq t_d \). Therefore we calculate the value of \( \Delta \). We find that \( \Delta = -0.1334 \). Applying the algorithm given in section 6.4 we find that \( t_1^* = 0.1218 \), \( T^* = 0.1899 \) and \( TC^*(t_1^*, T^*) = 2514.70 \) and \( Q^* = 653 \).

6.6 MANAGERIAL IMPLICATIONS

In this section, the effects of changes in the major parameters of the system on the optimal length of inventory interval with positive inventory \( t_1^* \), the optimal length of order cycle \( T^* \), the optimal order quantity per cycle \( Q^* \) and the minimum annual total cost \( TC^* \) (based on Example 6.1) are discussed.

Based on the computational results shown in Tables 8 and 9 the following managerial insights are obtained.

(1) When the fresh product time increases and other parameters remain unchanged, the optimal total annual cost decreases. That is, the longer the fresh product time is, the lower total cost would be. It implies that the model with non-instantaneous deteriorating items always has smaller total annual inventory cost than with instantaneous deteriorating items. If the retailer can extend effectively the length of time the product has no deterioration for a few days or months, the total annual cost will be reduced obviously. Increasing the fresh product time \( (t_d) \) decreases the order quantity \( (Q) \). From the inventory point of view, the longer the fresh product time is, the lower order quantity would be. It can be found
that $t_1^*$ and $T^*$ increases with an increase in $t_d$. It implies that the longer the fresh product time is, the longer the replenishment cycle and the length of inventory interval with positive inventory.

(2) It can be found that each of $T^*$, $Q^*$ and $TC^*$ decreases with an increase in the credit period ($M$) (other parameters are kept fixed). It implies that, the longer the credit period is, the shorter the replenishment cycle, the lower the order quantity and the total annual cost will be. From economical point of view, if the supplier provides a permissible delay in payments, the retailer will order lower quantity in order to take the benefits of the permissible delay more frequently.

(3) Increasing the backlogging parameter ($\delta$) (or equivalently decreasing the backlogging rate) decreases the order quantity $Q^*$ and increases the total annual cost $TC^*$. It indicates that when shortages are completely backlogged, total cost per unit time becomes lower. Also it can be found that the replenishment cycle time decreases with an increase in the backlogging parameter ($\delta$).

(4) It can be seen that when the parameter $\theta$ increases, $T^*$, $t_1^*$ and $Q^*$ decrease while $TC^*$ increases. Hence, if the retailer can effectively reduce the deteriorating rate of item by improving equipment of storehouse, the total annual inventory cost will be lowered.

(5) $TC^*$, $Q^*$, $T^*$ and $t_1^*$ decrease with increase in the value of the parameter $I_e$. That is, total annual cost, order quantity, the length of replenishment
cycle and the length of inventory interval with positive inventory decreases with increase in $I_c$. This implies that when the interest earned per dollar is high, the total cost is low.

(6) Increase in $I_p$ results in a decrease in $T^*$, $Q^*$ and $t_1^*$ and an increase in $TC^*$. The total cost increases when the capital opportunity cost in stock per dollar is high. From managerial point of view, it implies that when the capital opportunity cost in stock per dollar is high the retailer should order less amount of inventory.
### Table 8
Sensitivity analysis with respect to the parameters $t_d$, $M$, $\delta$ and $\theta$

<table>
<thead>
<tr>
<th>Parameter</th>
<th>$T$</th>
<th>$t_1$</th>
<th>$TC$</th>
<th>$Q$</th>
</tr>
</thead>
<tbody>
<tr>
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<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>20</td>
<td>0.1664</td>
<td>0.1173</td>
<td>2047.00</td>
<td>861</td>
</tr>
<tr>
<td>25</td>
<td>0.1721</td>
<td>0.1217</td>
<td>2019.90</td>
<td>802</td>
</tr>
<tr>
<td>30</td>
<td>0.1774</td>
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<td>2000.30</td>
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Table 9

Sensitivity analysis with respect to the parameters $I_e$ and $I_p$

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6.7 CONCLUSION

The main purpose of this chapter is to frame a model that will help the retailer to determine the optimal replenishment policy for non-instantaneous deteriorating items when the supplier offers a permissible delay in payments. This model well suits to situations where shortages are allowed. Here the allowed shortages are partially backlogged. Practically some customers will wait for backorders and some others will satisfy their needs from other sellers. The length of the waiting time for next replenishment (the time, the customers have to wait) would determine whether the backlogging will be accepted by the customer or not. In order to fit with realistic circumstances, the backlogging rate is assumed to be variable and dependent on the waiting time for the next replenishment. So this will maintain an acceptable level of customer satisfaction. The model proposed here is a general framework that includes numerous previous models as special cases. In this chapter, some useful Theorems which characterize the optimal solutions are framed. Several numerical examples are provided to illustrate the theoretical results. Sensitivity analysis is also carried out. From the analysis carried out, some managerial insights are obtained. The following are the managerial implications: The retailer can reduce total annual inventory cost by ordering lower quantity when the supplier provides a permissible delay in payments, improving storage conditions for non-instantaneous deteriorating items and increasing the backlogging rate (or equivalently decreasing the backlogging parameter ($\delta$)).