CHAPTER 5

REPLENISHMENT POLICY FOR INVENTORY SYSTEMS WITH TIME-DEPENDENT DEMAND

5.1 INTRODUCTION

Some inventory models are formulated in a static environment where the demand is assumed to be constant and steady over a finite horizon. Many inventoried items such as electronic products, fashionable clothes, tasty food products and domestic goods generate a boom in sales after gaining consumers’ acceptance. The sales for other products may decline drastically due to the launching of more and more competitive products or due to a shift in consumers’ preferences. Therefore, the demand for the product during its growth and decline phases can be well approximated by continuous-time-dependent functions such as linear or exponential.

When shortages occur, it may be lost or completely backordered. The backlogging phenomenon was modeled using the approach in which customers are considered impatient. Hence when the stock out situation occurs, only a fraction of demand occurring at a given time is backordered. In this chapter, we assume that the length of waiting time for the next replenishment would determine, whether the backlogging will be accepted or not. Therefore, the backlogging rate is variable and it is dependent on the waiting time for the next replenishment.
To the best of our knowledge there is no model for non-instantaneous deteriorating items with time-dependent demand and partial backlogging. This chapter is organized in different sections. In section 5.2, the problem description is given. In section 5.3, we present the mathematical model formulation. In section 5.4, the necessary and sufficient conditions for the existence and uniqueness of the optimal solutions are derived theoretically. In section 5.5, numerical examples are presented to illustrate the model. Sensitivity analysis of the optimal solution with respect to various parameters of the system is carried out and the managerial implications are discussed in detail in section 5.6. This is followed by concluding remarks.

5.2 PROBLEM DESCRIPTION

For fitting into realistic circumstances, the problem of determining the optimal replenishment policy for non-instantaneous deteriorating items with time-dependent demand is considered in this chapter. Here shortages are allowed and are partially backlogged. The backlogging rate is variable and it is dependent on the waiting time for the next replenishment. The necessary and sufficient conditions for the existence and uniqueness of the optimal solutions are shown. The model and the solution procedure are demonstrated by numerical examples.
5.2.1 ASSUMPTIONS

1) Replenishment rate is infinite and the lead time is zero.

2) The demand rate $D(t)$ at time $t$ is assumed to be

$$D(t) = \begin{cases} ae^{bt} & 0 \leq t \leq t_1 \\ a & t_1 \leq t \leq T \end{cases}$$

where $a$ and $b$ are positive constants and $0 \leq b \leq 1$.

3) Shortages are allowed to occur. It is assumed that only a fraction of the demand is backlogged. Besides, the longer the waiting time is, the smaller the backlogging rate will be. Let $B(t)$ denote this fraction, where $t$ is the waiting time up to the next replenishment. We take $B(t) = \frac{1}{1 + \delta t}$, where the backlogging parameter $\delta$ is a positive constant.

4) It is assumed that during a certain period of time the product has no deterioration (i.e., fresh product time). After this period, a constant fraction, $\theta$ ($0 < \theta < 1$), of the on-hand inventory deteriorates and there is no repair or replacement of the deteriorated units.

5) Product transactions are followed by instantaneous cash flow.

6) $t_1, T$ and $Q$ are decision variables.

7) Time horizon is infinite.

5.2.2 NOTATIONS

Despite the common notations given in section 1.8, the following notations are used only in this chapter.

$h$ per unit inventory holding cost per unit time

$d$ per unit deterioration cost
\( I_1(t) \) the inventory level at time \( t \ (0 \leq t \leq t_d) \) in which the product has no deterioration

\( I_2(t) \) the inventory level at time \( t \ (t_d \leq t \leq t_1) \) in which the product has deterioration

\( I_3(t) \) the inventory level at time \( t \ (t_1 \leq t \leq T) \) in which the product has shortage

\( TVC(t_1, T) \) the total inventory cost per unit time of inventory system

5.3 MODEL FORMULATION

The inventory system evolves as follows: \( I_m \) units of item arrive at the inventory system at the beginning of each cycle. During the time interval \([0, t_d]\), the inventory level is decreasing only owing to demand and the inventory level is dropping to zero due to demand and deterioration during the time interval \([t_d, t_1]\). Then shortage interval keeps to the end of the current order cycle. The whole process is repeated. As described above the inventory level \( I(t) \) of an end-product at time \( t \) is governed by the following differential equation,

\[
\frac{dI(t)}{dt} = \begin{cases} 
-ae^{bt} & \text{if } 0 \leq t \leq t_d \\
-\theta I(t) - ae^{bt} & \text{if } t_d \leq t \leq t_1 \\
\frac{-a}{1+\delta(T-t)} & \text{if } t_1 \leq t \leq T 
\end{cases}
\]  \hspace{1cm} (40)

with the boundary conditions \( I(0) = I_{m0} \), \( I(t_1) = 0 \).
Solution of equation (40) with the above boundary conditions and considering the continuity of \( I(t) \) at \( t = t_d \), is

\[
I(t) = \begin{cases} 
I_1(t) & \text{if } 0 \leq t \leq t_d \\
I_2(t) & \text{if } t_d \leq t \leq t_1 \\
I_3(t) & \text{if } t_1 \leq t \leq T 
\end{cases}
\]

(41)

where,

\[
I_1(t) = \frac{a}{b} \left[ e^{bt_d} - e^{bt} \right] + \frac{a}{b + \theta} \left[ e^{\rho t} e^{(b+\theta) t_d} - e^{bt_d} \right], \quad 0 \leq t \leq t_d,
\]

\[
I_2(t) = \frac{a}{b + \theta} \left[ e^{\rho (t_1-t)} - e^{bt} \right], \quad t_d \leq t \leq t_1,
\]

\[
I_3(t) = \frac{-a}{\delta} \left\{ \ln \left[ 1 + \delta (T - t_1) \right] - \ln \left[ 1 + \delta (T - t) \right] \right\}, \quad t_1 \leq t \leq T.
\]

Using the fact that \( I_1(t_d) = I_2(t_d) \), the maximum inventory level for each cycle is

\[
I_m = \frac{a}{b + \theta} \left[ e^{\rho t_d} e^{(b+\theta) t_d} - e^{bt_d} \right] - \frac{a}{b} \left[ 1 - e^{bt_d} \right].
\]

(42)

The maximum amount of demand backlogged per cycle is given by

\[
I_b = -I_3(T) = \frac{a}{\delta} \left\{ \ln \left[ 1 + \delta (T - t_1) \right] \right\}.
\]

(43)

The order quantity \( Q \) is given by

\[
Q = I_m + I_b
\]

\[
= \frac{a}{b + \theta} \left[ e^{\rho t_d} e^{(b+\theta) t_d} - e^{bt_d} \right] - \frac{a}{b} \left[ 1 - e^{bt_d} \right] + \frac{a}{\delta} \left\{ \ln \left[ 1 + \delta (T - t_1) \right] \right\}.
\]

(44)

The total relevant inventory cost per unit time is given by,

\[
TVC(t_1, T) = \{ \text{Ordering cost} + \text{inventory holding cost per cycle} + \text{the deterioration cost per cycle} + \text{shortage cost per cycle due to backlog} + \text{opportunity cost per cycle due to lost sales} \}/ T.
\]
Based on equation (41), we can obtain the total average cost per unit time as, (refer to Appendix B for total cost calculations)

\[
TVC(t_1, T) = \frac{a}{T} \left\{ \frac{A + h t_d}{a + b \theta} \left( \frac{t_d + 1}{b + \theta} \right) + \frac{e^{b t_d}}{b} \left( t_d + \frac{1}{b + \theta} \right) \right. \\
- e^{b t_d} \left( \frac{t_d + 1}{b + \theta} \right)^2 \left( \frac{e^{b t_d}}{b + \theta} + \frac{1}{b} \right) \\
+ \left[ \frac{1}{b + \theta} [ e^{(b+\theta) t_d} - e^{b t_d} ] + \frac{1}{b} [ e^{b t_d} - e^{b t_d} ] \right] \\
+ \frac{s + \delta \pi}{\delta} \left[ (T - t_1) - \ln \left( 1 + \delta (T - t_1) \right) \right] \left\} \right. \\
\left\} \right. \\
(45)

5.4 THEORETICAL RESULTS

The necessary conditions for the total cost per unit time in equation (45) to be minimum are \( \partial TVC(t_1, T)/\partial t_1 = 0 \) and \( \partial TVC(t_1, T)/\partial T = 0 \), which gives,

\[
\frac{\partial TVC(t_1, T)}{\partial t_1} = \frac{a}{T} \left\{ e^{(b+\theta) t_d} h t_d + \frac{h}{\theta} + d - e^{b t_1} \left[ d + \frac{h}{\theta} \right] \\
+ \frac{s + \delta \pi}{\delta} \left[ -1 + \frac{1}{1 + \delta (T - t_1)} \right] \right\} = 0 \\
(46)
\]

and
\[
\frac{\partial \text{TVC}(t_1, T)}{\partial T} = \frac{a}{T^2} \left\{ \frac{s + \delta \pi}{\delta} \left[ \ln \left[ 1 + \delta (T - t_1) \right] + \frac{(T - t_1)(\delta t_1 - 1)}{1 + \delta (T - t_1)} \right] - \frac{A}{a} - \frac{e^{\eta (b+\theta)-\theta t_1}}{b + \theta} \left[ h \left( t_d + \frac{1}{\theta} \right) + d \right] - \frac{e^{b_{td}}}{b} \left( t_d h + \frac{h}{b + \theta} + d \right) + e^{b_{td}} \left( \frac{t_d h}{b + \theta} + \frac{h}{b^2} + \frac{d}{b + \theta} + \frac{d}{b^2} \right) \right\} = 0.
\]

(47)

For notational convenience, let
\[N = \frac{s + \delta \pi}{\delta}; L = \frac{h}{\theta} + d; M = h t_d + h / \theta + d; S = \frac{h t_d}{b + \theta} + \frac{h}{b^2} + \frac{b}{b + \theta}; \]
\[R = \frac{h t_d}{b} + \frac{h}{b(b + \theta)} + \frac{d}{b}.\]

Clearly \(N, L, M, S, R\) are \(> 0\). Equations (46) and (47) become,
\[T = t_1 + \frac{Me^{\eta (b+\theta)-\theta t_1} - Le^{b_{td}}}{N + Le^{b_{td}} - Me^{\eta (b+\theta)-\theta t_1}} \]

and
\[N \left[ \ln \left[ 1 + \delta (T - t_1) \right] + \frac{(T - t_1)(\delta t_1 - 1)}{1 + \delta (T - t_1)} \right] - \frac{A}{a} - \frac{e^{\eta (b+\theta)-\theta t_1}}{b + \theta} M - e^{b_{td}} R + e^{b_{td}} S + \frac{e^{b_{td}}}{b} L - \frac{h}{b^2} = 0.\]

(49)

Substituting equation (48) into equation (49), we have
\[\left( M e^{\eta (b+\theta)-\theta t_1} - Le^{b_{td}} \right) \left( \frac{\delta t_1 - 1}{\delta} - \frac{N}{\delta} \ln \left[ \frac{L e^{b_{td}} + 1 - M e^{\eta (b+\theta)-\theta t_1}}{N} \right] \right) - \frac{A}{a} - \frac{e^{\eta (b+\theta)-\theta t_1}}{b + \theta} M - e^{b_{td}} R + e^{b_{td}} S + \frac{e^{b_{td}}}{b} L - \frac{h}{b^2} = 0.\]

(50)
Now, we give the following results.

**Theorem 5.1**

(a) If \( \frac{\delta t_d}{\delta} - \frac{1}{N} e^{b t_d} (M - L) - \frac{N}{\delta} \ln \left[ \frac{L}{N} e^{b t_d} + 1 - \frac{M}{N} e^{b t_d} \right] \)
\[ - \frac{A}{a} e^{b t_d} \left( \frac{M}{b + \theta} + R - S - \frac{L}{b} \right) - \frac{h}{b^2} \leq 0 \), then the solution of \((t_1, T)\) which satisfies (48) and (49) not only exists but also is unique.

(b) If \( \frac{\delta t_d}{\delta} - \frac{1}{N} e^{b t_d} (M - L) - \frac{N}{\delta} \ln \left[ \frac{L}{N} e^{b t_d} + 1 - \frac{M}{N} e^{b t_d} \right] \)
\[ - \frac{A}{a} e^{b t_d} \left( \frac{M}{b + \theta} + R - S - \frac{L}{b} \right) - \frac{h}{b^2} > 0 \), then the solution of \((t_1, T)\) which satisfies (48) and (49) does not exist.

**Proof of part (a)**

Consider \( M e^{t (b + \theta) - \theta t_d} - L e^{b t_d} \). Now \( M e^{t (b + \theta) - \theta t_d} - L e^{b t_d} > 0 \).

\[ \Rightarrow M e^{t (b + \theta) - \theta t_d} - L > 0 \]
\[ \Rightarrow t_d < 1 - \frac{1}{\theta} \ln \left( \frac{L}{N} \right) + \frac{1}{\theta} \]

From equation (50), we let

\[ F(x) = \frac{\delta x}{\delta} - \frac{1}{N} e^{b x} \left( M e^{x (b + \theta) - \theta t_d} - L e^{b x} \right) \]
\[ - \frac{N}{\delta} \ln \left[ \frac{L}{N} e^{b x} + 1 - \frac{M}{N} e^{x (b + \theta) - \theta t_d} \right] - \frac{A}{a} e^{x (b + \theta) - \theta t_d} \]
\[ - e^{b t_d} R + e^{b t_d} S + \frac{e^{b x} L}{b} - \frac{h}{b^2} ; x \geq t_d . \] 

\[ (51) \]
Taking first order derivative of \( F(x) \) with respect to \( x \in (t_d, t_1^b) \), we have
\[
\frac{dF(x)}{dx} = \left( Me^{x(b+\theta) - \theta t_d} (b + \theta) - Le^{bx} b \right) \cdot \left( x + \frac{Me^{x(b+\theta) - \theta t_d} - Le^{bx}}{\delta \left( Le^{bx} + N - Me^{x(b+\theta) - \theta t_d} \right)} \right) > 0 \quad \forall \ x \in (t_d, t_1^b).
\]

Thus, \( F(x) \) is a strictly increasing function with respect to \( x \) in the interval \( [t_d, t_1^b] \). Furthermore, by using assumption we have,
\[
F(t_d) = \frac{\delta t_d - 1}{\delta} \left( Me^{bt_d} - Le^{bt_d} \right) - \frac{N}{\delta} \ln \left[ \frac{L}{N} e^{bt_d} + 1 - \frac{M}{N} e^{bt_d} \right] - \frac{A}{a} - \frac{e^{bt_d}}{b + \theta} M - e^{bt_d} R + e^{bt_d} S + \frac{e^{bt_d} L}{b} \cdot \frac{h}{b^2} \leq 0
\]

and \( F(t_1^b) \geq 0 \). Therefore by using the intermediate value theorem, there exists an unique \( t_1^* \in [t_d, t_1^b] \) such that \( F(t_1^*) = 0 \). That is, \( t_1^* \) is the unique solution of (50). Once we obtain the value of \( t_1^* \), then the value of \( T \) (denoted by \( T^* \)) can be found from equation (48) and is given by
\[
T^* = t_1^* + \frac{Me^{t_1^*(b+\theta) - \theta t_d} - Le^{b t_1^*}}{\delta \left( N + Le^{b t_1^*} - Me^{t_1^*(b+\theta) - \theta t_d} \right)}.
\]

**Proof of part (b)**

If \( \frac{\delta t_d - 1}{\delta} e^{bt_d} (M - L) - \frac{N}{\delta} \ln \left[ \frac{L}{N} e^{bt_d} + 1 - \frac{M}{N} e^{bt_d} \right] - \frac{A}{a} \)
\[
- e^{bt_d} \left( \frac{M}{b + \theta} + R - S - \frac{L}{b} \right) \cdot \frac{h}{b^2} > 0,
\]
then from (51) we have \( F(t_d) > 0 \). Since,
\( F(x) \) is a strictly increasing function of \( x \in [t_d, t_1^*] \), we cannot find a value

\[ t_1 \in [t_d, t_1^*] \]

such that \( F(t_1) = 0 \). Hence the proof.

**Theorem 5.2**

(a) If

\[
\frac{\delta t_d}{\delta} - \frac{1}{e^{b_{id}}(M - L) - \frac{N}{\delta} \ln \left[ \frac{L}{N} e^{b_{id}} + 1 - \frac{M}{N} e^{b_{id}} \right] - \frac{A}{a} - e^{b_{id}} \left( \frac{M}{b + \theta} + R - S - \frac{L}{b} \right) \frac{h}{b^2} < 0 ,
\]

then the total cost per unit time \( TVC(t_1, T) \) is convex and reaches its global minimum at the point \((t_1^*, T^*)\), where \((t_1^*, T^*)\) is the point which satisfies (48) and (49).

(b) If

\[
\frac{\delta t_d}{\delta} - \frac{1}{e^{b_{id}}(M - L) - \frac{N}{\delta} \ln \left[ \frac{L}{N} e^{b_{id}} + 1 - \frac{M}{N} e^{b_{id}} \right] - \frac{A}{a} - e^{b_{id}} \left( \frac{M}{b + \theta} + R - S - \frac{L}{b} \right) \frac{h}{b^2} > 0 ,
\]

then the total cost per unit time \( TVC(t_1, T) \) has a minimum value at the point \((t_1^*, T^*)\), where \( t_1^* = t_d \) and \( T^* = t_d + \frac{Me^{b_{id}} - Le^{b_{id}}}{\delta (N + Le^{b_{id}} - Me^{b_{id}})} \).

**Proof of part (a)**

Taking the second derivative of \( TVC(t_1, T) \) with respect to \( t_1 \) and \( T \) and then finding the values of these functions at point \((t_1^*, T^*)\), we obtain

\[
\frac{\partial^2 TVC(t_1, T)}{\partial t_1^2} \bigg|_{(t_1^*, T^*)} = \frac{a}{T^*} \left( b + \theta \right) \left( Me^{b_1(b + \theta) - \theta t_1} - Lb e^{b_1^*} + \frac{N\delta}{1 + \delta(T^* - t_1^*)} \right) > 0,
\]
\[ \frac{\partial^2 \text{TVC}(t_1, T)}{\partial t_1 \partial T} \left|_{(t_1^*, T^*)} \right. = -\frac{a}{T^*} \left( \frac{N\delta}{1 + \delta(T^* - t_1^*)} \right). \]

\[ \frac{\partial^2 \text{TVC}(t_1, T)}{\partial T^2} \left|_{(t_1^*, T^*)} \right. = \frac{a}{T^*} \left( \frac{N\delta}{1 + \delta(T^* - t_1^*)} \right) > 0 \quad (52) \]

and \[ \left( \frac{\partial^2 \text{TVC}(t_1, T)}{\partial t_1^2} \right) \left( \frac{\partial^2 \text{TVC}(t_1, T)}{\partial T^2} \right) - \left[ \frac{\partial \text{TVC}(t_1, T)}{\partial t_1 \partial T} \right]^2 \left|_{(t_1^*, T^*)} \right. \]

\[ = \left( \frac{a}{T^*} \right)^2 M(b + \theta) e^{\eta(b+\theta) - \eta_t} - L b e^{b t_1} \left( \frac{N\delta}{1 + \delta(T^* - t_1^*)} \right) > 0. \quad (53) \]

From equations (52), (53) and Theorem 5.1, we can see that \((t_1^*, T^*)\) is the global minimum of \(\text{TVC}(t_1, T)\).

**Proof of part (b)**

If \( \frac{\delta t_d - 1}{\delta} e^{b t_d} (M - L) - \frac{N}{\delta} \ln \left[ \frac{L}{N} e^{b t_d} + 1 - \frac{M}{N} e^{b t_d} \right] \)

\[-\frac{A}{a} - e^{b t_d} \left( \frac{M}{b + \theta} + R - S - \frac{L}{b} \right) - \frac{h}{b^2} > 0, \] then we know that \(F(x) > 0, \) for all \(x \in [t_d, t_1^*]. \)

Thus,

\[ \frac{\partial \text{TVC}(t_1, T)}{\partial T} = \frac{a}{T^2} \left[ \left( \frac{\delta t_1 - 1}{\delta} \right) \left[ (M e^{\eta(b+\theta) - \eta_t} - L e^{b t_1}) \right] \right. \]

\[-\frac{N}{\delta} \ln \left[ \frac{L}{N} e^{b t_1} + 1 - \frac{M}{N} e^{b t_1} \right] - \frac{A}{a} \]

\[-\frac{b + \theta}{b + \theta} M - e^{b t_d} R + e^{b t_d} S + \frac{b t_1}{b} L - \frac{h}{b^2} \]

\[= \frac{a}{T^2} F(t_1) > 0, \forall t_1 \in [t_d, t_1^*]. \]
This implies \( TVC(t_1, T) \) is a strictly increasing function of \( T \). Thus, \( TVC(t_1, T) \) has a minimum value when \( T \) is minimum. On the other hand, from equation (48), we can see that \( T \) has a minimum value of \( t_d + \frac{e^{b t_d} \cdot h t_d}{(s + \delta t_d) - \delta h t_d e^{b t_d}} \) as \( t_1 = t_d \). Therefore, \( TVC(t_1, T) \) has a minimum value at the point \((t_1^*, T^*)\), where \( t_1^* = t_d \) and \( T^* = t_d + \frac{e^{b t_d} h t_d}{(s + \delta t_d) - \delta h t_d e^{b t_d}} \).

Hence the proof.

**Note:** The convexity of the total inventory cost function \( TVC(t_1, T) \) can also be established **graphically** (refer to Appendix C).

When we obtain the optimal solution \((t_1^*, T^*)\) from equations (48) and (49), we can substitute \((t_1^*, T^*)\) into equation (44) and get the optimal order quantity \( Q^* \) (which is denoted by \( Q^* \)) as follows:

\[
Q^* = \frac{a}{b + \theta} \left[ e^{\theta (b + \theta) - \theta t_1} e^{b t_d} \right] + \frac{a}{b} \left[ e^{b t_d} - 1 \right] + \frac{a}{\delta} \left[ 1 + \delta (T^* - t_1^*) \right]
\]  \hspace{1cm} (54)

Moreover, from equation (45), we get the minimum total cost per unit time and it is given by

\[
TVC^* = TVC(t_1^*, T^*) = \frac{a}{T^*} \left\{ \frac{A}{a} + \frac{e^{\theta (b + \theta) - \theta t_1} e^{b t_d} M + e^{b t_d} R - e^{b t_d} S - e^{b t_d} L}{b} \right. \\
+ \left. \frac{h}{b^2} + N \left[ (T^* - t_1^*) - \frac{\ln[1 + \delta (T^* - t_1^*)]}{\delta} \right] \right\}.
\]  \hspace{1cm} (55)
5.5 NUMERICAL EXAMPLES

Example 5.1
In order to illustrate the above solution procedure, let us consider an inventory system with the following data.

\[ A = 250, \quad h = 0.5, \quad d = 1.2, \quad s = 2.5, \quad \pi = 2, \quad \delta = 2, \quad a = 600, \quad b = 0.5, \quad \theta = 0.08, \]

\[ t_d = 0.0833 \] in appropriate units. We first check the condition

\[ \frac{\delta t_d - 1}{\delta} e^{b t_d} (M - L) - \frac{N}{\delta} \ln \left[ \frac{L}{N} e^{b t_d} + 1 - \frac{M}{N} e^{b t_d} \right] \]

\[ - \frac{A}{a} e^{b t_d} \left[ \frac{M}{b + \theta} + R - S - \frac{L}{b} \right] - \frac{h}{b^2} = -0.4147 < 0. \]

Thus, the optimal length of inventory with positive inventory \( i_t \) and the optimal length of order cycle \( T^* \) can be obtained by solving equations (48) and (49), and are given by \( t^*_i = 0.8312, \quad T^* = 0.9847 \). Hence, the optimal order quantity per cycle is \( Q^* = 717 \) and the minimum total cost per unit time is \( TVC^* = 457.85 \).

Example 5.2
We consider another inventory system with the following data:

\[ A = 250, \quad h = 0.5, \quad d = 1.2, \quad s = 2.5, \quad \pi = 2, \quad \delta = 2, \quad a = 1000, \quad b = 0.5, \quad \theta = 0.08, \]

\[ t_d = 0.75 \] in appropriate units. Similarly, we check the condition

\[ \frac{\delta t_d - 1}{\delta} e^{b t_d} (M - L) - \frac{N}{\delta} \ln \left[ \frac{L}{N} e^{b t_d} + 1 - \frac{M}{N} e^{b t_d} \right] \]

\[ - \frac{A}{a} e^{b t_d} \left[ \frac{M}{b + \theta} + R - S - \frac{L}{b} \right] - \frac{h}{b^2} = 0.0038 > 0. \]
Thus, from part b of Theorem 5.2, the optimal length of inventory interval with positive inventory is \( t_i^* = t_d = 0.75 \) and the optimal length of order cycle is \( T^* = 0.8509 \). Hence, the optimal order quantity per cycle is \( Q^* = 1001 \) and the minimum total cost per unit time is \( TVC^* = 541.17 \).

5.6 MANAGERIAL IMPLICATIONS

Now, we study the effects of changes in the system parameters \( A, h, d, S, \pi, a, b, \) and \( \theta \) on the optimal length of inventory interval with positive inventory \( t_i^* \), the optimal length of order cycle \( T^* \), the optimal order quantity per cycle \( Q^* \) and the minimum total cost per unit time \( TVC^* \) of the Example 5.1. The sensitivity analysis is performed by changing each of the parameters by +50%, +25%, -25% and -50%, taking one parameter at a time and keeping the remaining parameters unchanged. The results are shown in Table 7.
Table 7. Effects of changing various parameters on the optimal values

<table>
<thead>
<tr>
<th>Parameter</th>
<th>% change</th>
<th>( t_1^* )</th>
<th>( T^* )</th>
<th>( TVC )</th>
<th>( Q )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( A )</td>
<td>-50%</td>
<td>-23.70</td>
<td>-25.92</td>
<td>-31.78</td>
<td>-28.97</td>
</tr>
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On the basis of the result shown in Table 7, the following observations can be made:

(a) The optimal length of inventory interval with positive inventory $t_i^*$, the optimal length of order cycle $T^*$, the optimal order quantity per cycle $Q^*$ and the minimum total cost per unit time $\text{TVC}^*$ increase with increase in the value of parameter $A$. It implies that if the ordering cost per order could be reduced effectively, the total cost per unit time could be improved.

(b) $t_i^*, T^*$ and $Q^*$ decrease while $\text{TVC}^*$ increases with increase in the values of parameters $h, d, \text{ and } \theta$.

(c) With increase in the shortage cost $s$, $t_i^*$ and $\text{TVC}^*$ increase but $T^*$ and $Q^*$ decrease.

(d) With increase in the value of parameter $\pi, t_i^*$ and $\text{TVC}^*$ increase but $T^*$ and $Q^*$ decrease.

(e) $t_i^*$ and $T^*$ decrease while $Q^*$ and $\text{TVC}^*$ increase with increase in the value of the demand parameter $a$.

(f) $t_i^*, T^*$ and $Q^*$ decrease while $\text{TVC}^*$ increases with increase in the values of the demand parameter $b$.

(g) $t_i^*$ and $\text{TVC}^*$ increases while $T^*$ and $Q^*$ decreases with the increase in the value of $\delta$.

(h) With increase in the value of $t_d$, $t_i^*$ and $T^*$ increases while $\text{TVC}^*$ and $Q^*$ decrease.
Thus, when the fresh product time increases, the optimal value of the total inventory cost decreases. This implies that if the retailer can extend the fresh product time ($t_d$) for a few days or months, the total cost per unit time will be reduced effectively.

Further as $\theta$ increases, the total cost increases whereas the order quantity and the cycle time decreases. Hence if the retailer can effectively reduce the deteriorating rate of an item by improving equipment of store house, the retailer could very well lower the total inventory cost. Further, when the deterioration rate of a product is high the retailer should order less quantity more frequently.
5.7 CONCLUSION

This model incorporates some realistic features that are likely to be associated with some kinds of inventory. First, non-instantaneous deterioration over time is a natural feature for goods. Secondly, occurrence of shortages in inventory is a natural phenomenon in real situations. Further, time-dependent demand is also realistic as the demand of certain products generates increasing sales after gaining consumers acceptance. The sales for other products may decline due to change in consumers preferences. In keeping with this reality, these functions are incorporated into the present model. The necessary and sufficient conditions for the existence and uniqueness of the optimal solution are shown. Furthermore, sensitivity analysis shows that, the increase in the backlogging parameter (or equivalently, decrease in the backlogging rate) decreases the order quantity. This proposed model can be used in inventory control of certain non-instantaneous deteriorating items such as food items, electronic components, fashionable commodities and others whose demand depends on time.