Chapter 4

Head-on collision of dust acoustic solitary waves in a four-component dusty plasma with nonthermal ions.
4.1 INTRODUCTION

There has been a rapidly growing interest to study different types of nonlinear phenomena processes in dusty plasma because of its presence in comet tails, asteroid zones, planetary ring, interstellar medium, and the lower part of the Earth’s ionosphere and magnetosphere [25],[34],[35],[36],[37],[196],[197]. It has also application in different fields like ratio frequency plasma discharge[38], plasma crystal [198]-[201] etc. The presence of charged dust grains in plasma the collective behavior of a plasma is modified and generates new modes like dust ion acoustic (DIA) Waves[39], dust-acoustic(DA) waves[29], dust-lattic(DL) waves[40]. DIA solitary waves, DA solitary waves, DL solitary waves and the envelope DIA/DA solitary waves [50],[202] have been extensively studied [3],[5],[37],[203],[204].

Many researchers observed DASWs [27],[29],[205]-[206] in one dimensional and unmagnetized plasma by using three component dusty plasma with electrons, ions and negatively charged dust grains. But positively charged dust grains are also observed in space. (i)Photo emission in the presence of flux of ultra-violate photons, (ii) thermionic emission induced by radiative heating, and (iii) secondary emission of electrons from the surface of dust grains are the principal mechanisms behind the presence of positively charged dust grains in space. There are direct evidences for co-existence of positively and negatively charged dust grains in different regions in space viz cometary tails [34], [36], [47], [49] upper and lower mesosphere [29], [50] Jupiter’s magnetosphere [47]-[49]. Chow et al [49] have investigated theoretically that due to the size effect on secondary emission insulating dust grains with different sizes in space plasma can have opposite charges. Recently, Mamun and Shukla [51] have considered a very simple dusty plasma system with positively and negatively charged dust grains only under the theoretical prediction and satellite observation and have
investigated the properties of linear/nonlinear electrostatic waves that may propagate in such a dusty plasma system. F. sayed and A. A. Mamun [52] have generalized the model of Mamun and Shukla [51] under the complete depletion of background electrons and ions and have investigated finite amplitude solitary potential structures in four component dusty plasma. They also observed that the presence of positive dust grains modified the solitary potential structures. The Vela satellite [207] observed nonthermal i.e. energetic ions in these region. From recent observation [208]-[209] it has been found that ion distribution does not follow the Boltzmann distribution, in these cases the nonthermal distribution of ions is suggested. Chatterjee et al [210] investigated Large amplitude solitary waves in a four-component dusty plasma with nonthermal ions.

Solitons are solitary waves with the remarkable property that the solitons preserve the form asymptotically even when undergo a collision. The term was first coined by Zabusky and Kruskal [143]. In a one dimensional system, the solitons may interact between them in two different ways. One is overtaking collision which can be studied by the inverse scattering transformation method [145]. An important collision is the head-on collision [146] where the angle between two propagation directions of two solitons is equal to $\pi$. It has been observed by many researchers. The solution of two solitary waves in the form of two Korteweg-de vries(KdV) equations can explain resonance phenomena and this has been observed in shallow water wave experiments [187], in plasmas experiments [188], in two core optical fiber [189] and in fluid filled elastic tubes [190] among others. For a head-on collision between two solitary waves travelling to positive and negative directions, one must search their evolution and hence one need to employ a suitable asymptotic expansion to solve the equations. The unique effects due to the collision are their phase shift and trajectories. Many authors investigated the head-on collision of two solitary waves in different plasma models using extended Poincaré-Lighthill- Kuo (PLK) method [147]-[161]. For exam-

To the best of our knowledge, there is no detailed investigation about the interaction of charged dust grains in collision less unmagnetized plasma. Two opposite directional solitary waves are propagated and head-on collision of these two waves takes place. The phase shifts and trajectories of the two waves are to be determined. The phase shifts and trajectories of the two solitons after collision are deduced. Thus in this paper we study the head-on collision of dust acoustic solitary waves by an extended version of the PLK method where the effects of nonthermal distribution parameter on the phase shifts on the characteristics of head-on collisions are included.
4.2 Basic Equations and Derivation of KdV Eqs and phase shifts

Let us consider a four component dusty plasma with Boltzmann distributed electrons, nonthermal ions negatively charged dust grains and also positively charged dust grains.

The basic equations are

\[
\begin{align*}
\frac{\partial n_1}{\partial t} + \frac{\partial (n_1 u_1)}{\partial x} &= 0; \\
\frac{\partial u_1}{\partial t} + u_1 \frac{\partial (u_1)}{\partial x} &= \frac{\partial \psi}{\partial x}; \\
\frac{\partial n_2}{\partial t} + \frac{\partial (n_2 u_2)}{\partial x} &= 0; \\
\frac{\partial u_2}{\partial t} + u_2 \frac{\partial (u_2)}{\partial x} &= -\alpha \beta \frac{\partial \psi}{\partial x}; \\
\frac{\partial^2 \psi}{\partial^2 x} &= n_1 - (1 - \mu_i + \mu_e)n_2 + \mu e^\psi \\
&- \mu_i (1 + \beta_1 \psi + \beta_1 \psi^2)e^{-\psi};
\end{align*}
\]

where \( n_1 \) and \( n_2 \) are negatively charged and positively charged dust number density normalized the equilibrium value \( n_{10} \) and \( n_{20} \) respectively. \( u_1 \) and \( u_2 \) are negatively charged and positively dust fluid speed normalized by \( z_1 K_B T_i/m_i \). The electric potential \( \Psi \) is normalized \( K_B T_i/e \). space \( x \) and time \( t \) are normalized by \( \lambda_D = \sqrt{\frac{K_B T_i}{4\pi z_1 e^2 n_{10}}} \) and \( \omega_{p1}^{-1} = \sqrt{\frac{m_1}{4\pi z_1 e^2 n_{10}}} \) respectively. We let \( \alpha = z_2/z_1, \beta = m_1/m_2, \mu_e = n_{e0}/z_1 n_{10}, \mu_i = n_{i0}/z_1 n_{10} \) \( \sigma = T_i/T_e \) with \( z_1 \) and \( z_2 \) are the number of electrons and protons residing on a negative and positive dust particles. Moreover \( \beta_1 = 4\alpha_1/(1+3\alpha_1) \) where \( \alpha_1 \) is the ion nonthermal parameter that determines the number of fast (energetic) ions, \( m_1 \) is the mass of negative dust particles, \( T_i \) and \( T_e \) are ion and electron temperatures, \( K_B \) is the Boltzmann constant and \( e \) is the electron charge.

Now we assume that two solitons \( \phi_\alpha \) and \( \phi_\beta \) in the plasma which are asymptotically
far apart in the initial state travel toward each other. After some time they interact, collide, and then depart. We also assume that the solitons have small amplitudes $\varepsilon$ (where $\varepsilon$ is the small parameter characterizing the strength of nonlinearity) and the interaction between two solitons is weak. Hence we expect that the collision will be quasi elastic, so it will only cause shifts of the post collision trajectories (phase shift).

In order to analyze the effects of collision, we employ an extended PLK method. According to this method, the dependent variables are expanded as

$$n_1 = 1 + \varepsilon^2 n_1^1 + \varepsilon^3 n_1^2 + \varepsilon^4 n_1^3 + \cdots; \quad (4.6)$$

$$n_2 = 1 + \varepsilon^2 n_2^1 + \varepsilon^3 n_2^2 + \varepsilon^4 n_2^3 + \cdots; \quad (4.7)$$

$$u_1 = 0 + \varepsilon^2 u_1^1 + \varepsilon^3 u_1^2 + \varepsilon^4 u_1^3 + \cdots; \quad (4.8)$$

$$u_2 = 0 + \varepsilon^2 u_2^1 + \varepsilon^3 u_2^2 + \varepsilon^4 u_2^3 + \cdots; \quad (4.9)$$

$$\psi = 0 + \varepsilon^2 \psi^1 + \varepsilon^3 \psi^2 + \varepsilon^4 \psi^3 + \cdots; \quad (4.10)$$

The independent variables are given by

$$\xi = \varepsilon(x - \lambda t) + \varepsilon^2 P_0(\eta, \tau) + \varepsilon^3 P_1(\eta, \xi, \tau) + \cdots; \quad (4.11)$$

$$\eta = \varepsilon(x + \lambda t) + \varepsilon^2 Q_0(\xi, \tau) + \varepsilon^3 Q_1(\eta, \xi, \tau) + \cdots; \quad (4.12)$$

$$\tau = \varepsilon^3 t; \quad (4.13)$$

where $\xi$ and $\eta$ denote the trajectories of two solitons traveling toward to each other, and $\lambda$ is the unknown phase velocity of DASWs. The variables $P_0(\eta, \tau)$ and $Q_0(\xi, \tau)$ are to be determined.

Substituting Eqs (4.6) – (4.13) into Eqs (4.1) – (4.5) and equating the quantities with equal power of $\varepsilon$, we obtain a set of coupled equations for each order of $\varepsilon$. To the leading order, we have
\[-\lambda \frac{\partial n_1^{(1)}}{\partial \xi} + \lambda \frac{\partial n_1^{(1)}}{\partial \eta} + \frac{\partial u_1^{(1)}}{\partial \xi} + \frac{\partial u_1^{(1)}}{\partial \eta} = 0; \quad (4.14)\]
\[-\lambda \frac{\partial n_1^{(1)}}{\partial \xi} + \lambda \frac{\partial u_1^{(1)}}{\partial \eta} - \left( \frac{\partial \psi^{(1)}}{\partial \xi} + \frac{\partial \psi^{(1)}}{\partial \eta} \right) = 0; \quad (4.15)\]
\[-\lambda \frac{\partial n_2^{(1)}}{\partial \xi} + \lambda \frac{\partial n_2^{(1)}}{\partial \eta} + \frac{\partial u_2^{(1)}}{\partial \xi} + \frac{\partial u_2^{(1)}}{\partial \eta} = 0; \quad (4.16)\]
\[-\lambda \frac{\partial u_2^{(1)}}{\partial \xi} + \lambda \frac{\partial u_2^{(1)}}{\partial \eta} + \alpha \beta \left( \frac{\partial \psi^{(1)}}{\partial \xi} + \frac{\partial \psi^{(1)}}{\partial \eta} \right) = 0; \quad (4.17)\]
\[n_1^{(1)} - (1 - \mu_i + \mu_e) n_2^{(1)} + (\sigma \mu_e + \mu_i - \beta_1 \mu_i) \psi^{(1)} = 0; \quad (4.18)\]

Solving the above we get

\[\psi^{(1)} = \psi_\xi(\xi, \tau) + \psi_\eta(\eta, \tau); \quad (4.19)\]
\[n_1^{(1)} = -\frac{1}{\lambda^2} \psi_\xi - \frac{1}{\lambda^2} \psi_\eta; \quad (4.20)\]
\[u_1^{(1)} = -\frac{1}{\lambda} \psi_\xi + \frac{1}{\lambda} \psi_\eta; \quad (4.21)\]
\[n_2^{(1)} = \frac{\alpha \beta}{\lambda^2} \psi_\xi + \frac{\alpha \beta}{\lambda^2} \psi_\eta; \quad (4.22)\]
\[u_2^{(1)} = \frac{\alpha \beta}{\lambda} \psi_\xi - \frac{\alpha \beta}{\lambda} \psi_\eta; \quad (4.23)\]

and with the solvability condition [i.e. the condition to obtain a uniquely defined \(n_1, u_1, n_2\) and \(u_2\) from Eqs (4.20 – 4.23) when \(\psi^1\) is given by (4.19)], the phase velocity \(\lambda\) is found to be \(\pm \sqrt{1 + \alpha \beta (1 - \mu_i + \mu_e) / (\sigma \mu_e + \mu_i - \beta_1 \mu_i)}\) is obtained. The unknown function \(\psi_\xi\) and \(\psi_\eta\) will be determined from the next orders. Relations (4.13 – 4.15) imply that, at the leading order, we have two waves, one of which \(\psi_\xi(\xi, \tau)\) traveling to the right, and the other one \(\psi_\eta(\eta, \tau)\) travelling to the left. At the next order, we have a system of equations given in the following by

\[-\lambda \frac{\partial n_1^{(2)}}{\partial \xi} + \lambda \frac{\partial n_1^{(2)}}{\partial \eta} + \frac{\partial u_1^{(2)}}{\partial \xi} + \frac{\partial u_1^{(2)}}{\partial \eta} = 0; \quad (4.24)\]
\[-\lambda \frac{\partial n_1^{(2)}}{\partial \xi} + \lambda \frac{\partial u_1^{(2)}}{\partial \eta} - (\frac{\partial \psi^{(2)}}{\partial \xi} + \frac{\partial \psi^{(2)}}{\partial \eta}) = 0; \quad (4.25)\]

\[-\lambda \frac{\partial n_2^{(2)}}{\partial \xi} + \lambda \frac{\partial n_2^{(2)}}{\partial \eta} + \frac{\partial u_2^{(2)}}{\partial \xi} + \frac{\partial u_2^{(2)}}{\partial \eta} = 0; \quad (4.26)\]

\[-\lambda \frac{\partial u_2^{(2)}}{\partial \xi} + \lambda \frac{\partial u_2^{(2)}}{\partial \eta} + \alpha \beta (\frac{\partial \psi^{(2)}}{\partial \xi} + \frac{\partial \psi^{(2)}}{\partial \eta}) = 0; \quad (4.27)\]

\[n_1^{(2)} - (1 - \mu_i + \mu_e) n_2^{(2)} + (\sigma \mu_e + \mu_i - \beta_1 \mu_i) \psi^{(2)} = 0; \quad (4.28)\]

the solutions of these equations are

\[\psi^{(2)} = \phi_\xi(\xi, \tau) + \phi_\eta(\eta, \tau); \quad (4.29)\]

\[n_1^{(2)} = -\frac{1}{\lambda^2} \phi_\xi - \frac{1}{\lambda^2} \phi_\eta; \quad (4.30)\]

\[u_1^{(2)} = -\frac{1}{\lambda} \phi_\xi + \frac{1}{\lambda} \phi_\eta; \quad (4.31)\]

\[n_2^{(2)} = \frac{\alpha \beta}{\lambda^2} \phi_\xi + \frac{\alpha \beta}{\lambda^2} \phi_\eta; \quad (4.32)\]

\[u_2^{(2)} = \frac{\alpha \beta}{\lambda} \phi_\xi - \frac{\alpha \beta}{\lambda} \phi_\eta; \quad (4.33)\]

Furthermore, the next higher order leads to the equations

\[F_1 \frac{\partial^2 u_1^3}{\partial \xi \partial \eta} - F_2 \frac{\partial^2 u_2^3}{\partial \xi \partial \eta} = \frac{\partial}{\partial \xi} (\frac{\partial \psi_\xi}{\partial \tau} + A \psi_\xi \frac{\partial \psi_\xi}{\partial \xi} + B \frac{\partial^3 \psi_\xi}{\partial \xi^3}) + \frac{\partial}{\partial \eta} (\frac{\partial \psi_\eta}{\partial \tau} - A \psi_\eta \frac{\partial \psi_\eta}{\partial \eta} - B \frac{\partial^3 \psi_\eta}{\partial \eta^3}) + (C \frac{\partial P_0}{\partial \eta} - D \psi_\eta) \frac{\partial^2 \psi_\xi}{\partial \xi^2} - (C \frac{\partial Q_0}{\partial \xi} - D \psi_\xi) \frac{\partial^2 \psi_\eta}{\partial \eta^2}; \quad (4.34)\]

Integrating the above equation with respect to the variable $\xi$ and $\eta$ yields

\[F_1 u_1^3 - F_2 u_2^3 = \int \left( \frac{\partial \psi_\xi}{\partial \tau} + A \psi_\xi \frac{\partial \psi_\xi}{\partial \xi} + B \frac{\partial^3 \psi_\xi}{\partial \xi^3} \right) d\eta + \int \left( \frac{\partial \psi_\eta}{\partial \tau} - A \psi_\eta \frac{\partial \psi_\eta}{\partial \eta} - B \frac{\partial^3 \psi_\eta}{\partial \eta^3} \right) d\xi + \int \int \left( C \frac{\partial P_0}{\partial \eta} - D \psi_\eta \right) \frac{\partial^2 \psi_\xi}{\partial \xi^2} d\xi d\eta. \]
\[- \int \int (C \frac{\partial Q_0}{\partial \xi} - D \psi_\xi) \frac{\partial^2 \psi_\eta}{\partial \eta^2} \, d\xi d\eta; \]  

(4.35)

where

\[
A = \frac{1}{A_1} \left[ \frac{(\sigma \mu_e + \mu_i - \beta_1 \mu_i)3\alpha^2 \beta^2(1 - \mu_i + \mu_e) - 1}{[1 + \alpha \beta(1 - \mu_i + \mu_e)]^2} - \frac{\mu_e \sigma^2 - \mu_i}{\sigma \mu_e + \mu_i - \beta_1 \mu_i} \right];
\]

\[
B = \frac{1}{(\sigma \mu_e + \mu_i - \beta_1 \mu_i)A_1};
\]

\[
C = \frac{4}{A_1};
\]

\[
D = \frac{1}{A_1} \left[ \frac{(\mu_e \sigma^2 - \mu_i)}{[1 + \alpha \beta(1 - \mu_i + \mu_e)]^2} - \frac{(\sigma \mu_e + \mu_i - \beta_1 \mu_i)\{1 - \alpha^2 \beta^2(1 - \mu_i + \mu_e)\}}{[1 + \alpha \beta(1 - \mu_i + \mu_e)]^2} \right];
\]

\[
A_1 = \frac{1}{[1 + \alpha \beta(1 - \mu_i + \mu_e)]^{1/2}} + \frac{\alpha \beta(1 - \mu_i + \mu_e)}{[1 + \alpha \beta(1 - \mu_i + \mu_e)]^{3/2}} \right];
\]

\[
F_1 = 4A_1^{-1}(\sigma \mu_e + \mu_i - \beta_1 \mu_i)^{-1/2}[1 + \alpha \beta(1 - \mu_i + \mu_e)]^{-1/2};
\]

\[
F_2 = 4A_1^{-1}(1 - \mu_i + \mu_e)(\sigma \mu_e + \mu_i - \beta_1 \mu_i)^{-1/2}[1 + \alpha \beta(1 - \mu_i + \mu_e)]^{-1/2};
\]

The first term in the right hand side of Eqs (4.35) will be proportional to \(\eta\) because the integrand is independent of \(\eta\) and the second term in the right hand side of Eqs (4.35) will be proportional to \(\xi\) because the integrand is independent of \(\xi\). These two terms of Eqn (4.35) are secular terms, which must be eliminated in order to avoid spurious resonances. Hence, we have

\[
\frac{\partial \psi_\xi}{\partial \tau} + A \psi_\xi \frac{\partial \psi_\xi}{\partial \xi} + B \frac{\partial^3 \psi_\xi}{\partial \xi^3} = 0; \quad (4.36)
\]

\[
\frac{\partial \psi_\eta}{\partial \tau} - A \psi_\eta \frac{\partial \psi_\eta}{\partial \eta} - B \frac{\partial^3 \psi_\eta}{\partial \eta^3} = 0; \quad (4.37)
\]

The third and fourth terms in Eqs (4.35) are not secular terms at this order, they could be secular for the next order. Hence we have

\[
C \frac{\partial P_0}{\partial \eta} = D \psi_\eta; \quad (4.38)
\]

\[
C \frac{\partial Q_0}{\partial \xi} = D \psi_\xi; \quad (4.39)
\]
Equation (4.36) is a KdV equation. This wave is travelling in the $\xi$ direction. Eqn (4.37) is also a KdV equation. This wave is propagating in the $\eta$ direction which is opposite to $\xi$. One of their special solutions are

$$
\psi_\xi = \psi_A \text{sech}^2\left(\frac{A\psi_A}{12B}\frac{1}{2}(\xi - \frac{1}{3}A\psi_A\tau)\right);
$$
(4.40)

$$
\psi_\eta = \psi_B \text{sech}^2\left(\frac{A\psi_B}{12B}\frac{1}{2}(\eta + \frac{1}{3}A\psi_B\tau)\right);
$$
(4.41)

where $\psi_A$ and $\psi_B$ are the amplitudes of the two solitons $\phi_\alpha$ and $\phi_\beta$ at their initial positions. The leading phase changes due to the collision can be calculated from Eqs (4.38), (4.39), (4.40) and (4.41) are as follows

$$
P_0(\eta, \tau) = \frac{D}{C}\left(\frac{12B\psi_B}{A}\right)^{1/2}\left[\tanh\left(\frac{A\psi_B}{12B}\frac{1}{2}(\eta + \frac{1}{3}A\psi_B\tau)\right) + 1\right];
$$
(4.42)

$$
Q_0(\xi, \tau) = \frac{D}{C}\left(\frac{12B\psi_B}{A}\right)^{1/2}\left[\tanh\left(\frac{A\psi_B}{12B}\frac{1}{2}(\xi - \frac{1}{3}A\psi_A\tau)\right) - 1\right];
$$
(4.43)

Therefore, up to $O(\varepsilon^2)$, the trajectories of the two solitary waves for head on interactions are

$$
\xi = \varepsilon(x - \lambda t) + \varepsilon^2\frac{D}{C}\left(\frac{12B\psi_B}{A}\right)^{1/2}\left[\tanh\left(\frac{A\psi_B}{12B}\frac{1}{2}(\eta + \frac{1}{3}A\psi_B\tau)\right) + 1\right] + \cdots;
$$
(4.44)

$$
\eta = \varepsilon(x + \lambda t) + \varepsilon^2\frac{D}{C}\left(\frac{12B\psi_B}{A}\right)^{1/2}\left[\tanh\left(\frac{A\psi_B}{12B}\frac{1}{2}(\xi - \frac{1}{3}A\psi_A\tau)\right) - 1\right] + \cdots;
$$
(4.45)

To obtain the phase shifts after a head-on collision of the two solitons, we assume that the solitons $\phi_\alpha$ and $\phi_\beta$ are, asymptotically, far from each other at the initial time $(t = -\infty)$ i.e. soliton $\phi_\alpha$ is at $\xi = 0, \eta = -\infty$ and soliton $\phi_\beta$ is at $\eta = 0, \xi = +\infty$, respectively. After the collision $(t = +\infty)$, soliton $\phi_\alpha$ is far to the right of soliton $\phi_\beta$, i.e.soliton $\phi_\alpha$ is at $\xi = 0, \eta = +\infty$ and soliton $\phi_\beta$ is at $\eta = 0, \xi = -\infty$. Using Eqs (4.44) and (4.45) we obtain the corresponding phase shift $\Delta P_0$ and $\Delta Q_0$ as follows

$$
\Delta P_0 = \varepsilon(x - \lambda t) \mid_{\xi=0,\eta=+\infty} - \varepsilon(x - \lambda t) \mid_{\xi=0,\eta=-\infty};
$$

$$
\Delta Q_0 = \varepsilon(x + \lambda t) \mid_{\eta=0,\xi=-\infty} - \varepsilon(x + \lambda t) \mid_{\eta=0,\xi=+\infty};
$$
that implies

\[ \Delta P_0 = -2\varepsilon^2 \frac{D}{C} \left( \frac{12B\psi_B}{A} \right)^{1/2}; \quad (4.46) \]
\[ \Delta Q_0 = 2\varepsilon^2 \frac{D}{C} \left( \frac{12B\psi_A}{A} \right)^{1/2}; \quad (4.47) \]

### 4.3 Discussion and Conclusion

In this paper we have investigated the head-on collision phenomenon between two DASWs in a four-component unmagnetized dusty plasma consisting nonthermal ions, Boltzmann distributed electrons, negatively charged dust grains and positively charged dust grains by using the extended version of PLK method. It is well known that soliton like solutions are formed due to the balance between the nonlinearity and dispersion in a nonlinear dispersive media. The soliton like solutions are obtained from Eqs (4.36) and (4.37). The coefficients \( A \) and \( B \) which are taking place in the soliton like solutions represented by Eqs (4.36) and (4.37) may creat two circumstances (i) \( AB > 0 \) and (ii) \( AB < 0 \).

Case (i) : When \( AB > 0 \), Eqs (4.40) and (4.41) give the DASW solutions with \( \psi_A > 0 \) and \( \psi_B > 0 \).

Case (ii) : When \( AB < 0 \), Eqs (4.40) and (4.41) give the DASW solutions with \( \psi_A < 0 \) and \( \psi_B < 0 \).

The positive or negative phase shift does not depend on the type of mode (i.e, ion-acoustic, dust-ion-acoustic, dust-acoustic and electrostatic waves). It is well known that the positive or negative phase shift depends on the co-efficient \( D \) in Eqs (4.38) and (4.39). For illustration, several authors [149],[151]-[153] illustrated that for ion-acoustic solitary waves, due to collision, each soliton has a negative phase shift in its traveling direction. By contrast, Han et al [157] demonstrated that due to collision,
each IASW has a positive phase shift in its traveling direction. On the other hand, Ju-xue [147] and Liang et al [158] showed that each soliton has a positive phase shift for dust acoustic solitary waves due to collision. El-Labany et al [165] illustrated that due to collision, each dust acoustic solitary wave has a negative phase shift in its traveling direction.

It is seen from Eqs (4.46) and (4.47) that the magnitude of the phase shifts are directly related to the physical parameters i.e $\varepsilon = 0.1$, $\psi_A = 1$, $\psi_B = 1$, $\alpha = 0.01$, $\mu_i = 1$ and $\mu_e = 0.2$. Here all physical quantities are assumed dimensionless. The chosen numerical values of the physical parameters [210] in our manuscript give A, B, C, D positive. Accordingly [160], since soliton $\phi_\alpha$ is traveling to the right and soliton $\phi_\beta$ is traveling to the left, it is from Eqs (4.46) and (4.47) that due to the collision, each soliton has a negative phase shift in its traveling direction. The negative phase shift means that the solitons reduce their velocities during the interaction stage [161].
Fig. 9. Graphs of phase shift $\Delta Q_0$ vs the non thermal parameter $\beta_1$.

$\varepsilon = 0.1$, $\psi_A = 1, \psi_B = 1$, $\alpha = 0.01$, $\mu_1 = 1$ and $\mu_2 = 0.2$, $\sigma = 0.5$.

For $\beta = 150$ (solid line) and $\beta = 50$ (dashed line)

To illustrate this idea we have plotted the positive magnitude of phase shift $\Delta Q_0$ against the parameters. Fig. 9 represents the variation of phase shift with mass ratio ($\beta$) of dust. This shows that for a given $\sigma = 0.5$ and $\beta = 150$ (solid line) the phase shifts decreases first and then increases, but for $\sigma = 0.5$ and $\beta = 50$ (dashed line) the phase shift increases monotonically.
which gives us the important feature that the phase shift increases monotonically in both cases.

$$\psi$$

The time evolution of head-on collision. It is seen that solitary wave solution $\psi$ decreases, which agrees with our analytical results.

Fig. 10. Graphs of phase shift $\Delta Q_0$ vs the non thermal parameter $\beta_1$.

$\varepsilon = 0.1, \psi_A = 1, \psi_B = 1, \alpha = 0.01, \mu_\gamma = 1$ and $\mu_\epsilon = 0.2, \beta = 150$.

For $\sigma = 0.4$ (solid line) and $\sigma = 0.2$ (dashed line)

In Fig. 10, we have plotted the phase shift $\Delta Q_0$ with $\beta_1$ for $\sigma = 0.4$ and $\sigma = 0.2$ which gives us the important feature that the phase shift increases monotonically in both cases.

In Fig. 11, we have plotted special solution $\psi_\xi$ and $\psi_\eta$ for several values of $\tau$ for time evolution of head-on collision. It is seen that solitary wave solution $\psi_\xi$ shifted towards right as time increases where as solution $\psi_\eta$ shifted towards left as time increases, which agrees with our analytical results.
Fig. 11. Plot of $\psi_\xi$ (solid line) and $\psi_\eta$ (dashed line) against $\xi$ and $\eta$ respectively, for $\tau = 2$ and $\tau = 10$ and the $\sigma = 0.3$, $\beta_1 = 0.1$, $\beta = 50$ and the other are same as above.
To see the effect of nonthermal distributed ions on the phase shifts Fig. 12 is plotted. In this figure the positive magnitude of phase shift $\Delta Q_0$ vs $\sigma$ for $\beta_1 = 0$ (dashed line) and for particular $\beta_1 = 0.25$ (solid line) is plotted. $\beta_1 = 0$ represents Maxwellian distributed ions. $\beta_1 = 0.25$ represents non-Maxwellian distributed ions.

Fig. 12. Graphs of phase shift $\Delta Q_0$ vs the temperature ratio $\sigma$.

$\varepsilon = 0.1, \psi_A = 1, \psi_B = 1, \alpha = 0.01, \mu_i = 1$ and $\mu_e = 0.2, \beta = 100$.

For $\beta_1 = 0.3$ (solid line) and $\beta_1 = 0$ (dashed line)
It is seen that the phase shifts increases monotonically for Maxwellian distributed ions with the increase of temperature ratio and with the contrary the phase shifts decreases monotonically for non-Maxwellian distributed ions with increase of temperature ratio. Here the magnitude of phase shift are positive in the specified region.