CHAPTER I

PROPAGATION OF VISCO-ELASTIC WAVES AND DISTURBANCES
The development of propagation of waves in various problems of science and engineering is enormous. A thorough and exhaustive literature in this direction will be found in the standard text-books of Eringin and Suhubi [49,50], Kolsky [69], Nowacki [95], Bullen [9], Ewing, Jardetzky and Press [52], and in the review article of Davies [34]. The problems of waves and vibrations in visco-elastic solid are comparatively limited, although a good amount of literature will be found in the monographs of Flugge [57,58], Bland [5], and in the article of Hunter [63]. In a large variety of problems of theoretical and practical interests of visco-elasticity the stress rate behaviour of visco-elasticity is generally considered. However, a limited number of problems are investigated taking into account the strain rate and stress rate simultaneously. As such it will be highly interesting to consider such problems of practical and theoretical interests.

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Rotatory vibration as well as radial vibration of a sphere of general visco-elastic solid of stress rate variation have been investigated by Sengupta and Roy [115,126]. The said authors have further developed similar problems. Radial motion of a conducting visco-elastic sphere in a magnetic field was studied by Roy and Sengupta [116] and this is followed by the problem of vibration of an infinite circular cylinder of visco-elastic higher order solid [117]. Dasgupta [33] has investigated the problem of disturbances due to radial impulsive force on the spherical cavity. Goodier and Jahsman [60] considered similar problems. Propagation of waves in a general visco-elastic medium due to impulsive forces on a spherical cavity was studied by Bhattacharyya and Sengupta [7]. Further the propagation of waves in a visco-elastic plate immersed in a liquid under the influence of gravity was investigated by Bhattacharyya and Sengupta [6]. A good number of basic interesting problems in the field of visco-elasticity had been investigated by Hunter [63]. Mukherjee [91-94] studied disturbances in simple models of visco-elastic medium due to various forces on the internal boundary of the solid.

In this paper an attempt has been made by the author to investigate the propagation of disturbances in a first order infinite visco-elastic medium by transient radial forces and a given twist on the surface of the cylinder. In all these cases the stress rate and strain rate of visco-elasticity have been considered. From
this general investigation particular cases of asymptotic solutions are derived. These solutions are very convenient and presented in finite closed form. The interesting feature of the investigation is that the solutions are in well agreement with the corresponding results derived previously $\int_{92}^{\text{f92j}}$.

2. DISTURBANCE DUE TO TRANSIENT RADIAL FORCE: FORMULATION, BOUNDARY CONDITIONS AND ASYMPTOTIC SOLUTION OF THE FIRST PROBLEM

Due to disturbance, the wave is produced by a transient radial force on the surface of the hole. The only non-vanishing displacement $u_r$ must be a function of $r$ and $t$ only. Following Voigt $\int_{159}^{\text{f159j}}$, the visco-elastic operators are

$$\lambda'' = \lambda + \lambda' \frac{\partial}{\partial t}, \quad \mu'' = \mu + \mu' \frac{\partial}{\partial t}, \quad \eta'' = \eta + \eta' \frac{\partial}{\partial t} \quad (1)$$

where $\lambda$, $\mu$, $\eta$ are elastic parameters and $\lambda'$, $\mu'$, $\eta'$ are constants of viscosity.

Non-vanishing stresses in cylindrical symmetry in terms of the only non-vanishing displacement component $u_r$ are given by

$$\left(\eta + \eta' \frac{\partial}{\partial t}\right) \sigma_r = \left[\left(\lambda+2\mu\right) + \left(\lambda'+2\mu'\right) \frac{\partial}{\partial t}\right] \frac{\partial u_r}{\partial r} + \left[\lambda + \lambda' \frac{\partial}{\partial t}\right] \frac{u_r}{r} \quad (2.1)$$

$$\left(\eta + \eta' \frac{\partial}{\partial t}\right) \sigma_\theta = \left[\lambda + \lambda' \frac{\partial}{\partial t}\right] \frac{\partial u_r}{\partial r} + \left[\left(\lambda+2\mu\right) + \left(\lambda'+2\mu'\right) \frac{\partial}{\partial t}\right] \frac{u_r}{r} \quad (2.2)$$

$$\sigma_z = \tau_{\theta r} = \tau_{\theta z} = \tau_{rz} = 0 \quad (2.3)$$
Two equations of motion are identically satisfied. The only non-vanishing equation of motion is

$$\frac{\partial r}{\partial t} + \frac{\partial r}{\partial t} - \hat{\theta} = \rho \frac{\partial^2 u_r}{\partial t^2} \quad (3)$$

where $\rho$ is the density of the material. With the help of equations (2.1) and (2.2), equation (3) takes the form

$$[(\lambda + 2\mu) + (\lambda + 2\mu) \frac{\partial}{\partial t}] \left[ \frac{\partial^2 u_r}{\partial r^2} + \frac{1}{r} \frac{\partial u_r}{\partial r} - \frac{u_r}{r^2} \right] = \eta'' \rho \frac{\partial^2 u_r}{\partial t^2} \quad (4)$$

where $\eta''$ is given by (1).

**BOUNDARY CONDITIONS**

We have now to apply the following boundary conditions

$$u_r \text{ is finite at } r \to 0 \quad (5.1)$$

$$\sigma_r \bigg|_{r=a} = -\rho e^{-\omega t} \quad (5.2)$$

where $\rho$ is a constant.

Now our object is to solve equation (4) with the help of the boundary conditions (5.1) and (5.2).

**SOLUTION OF THE PROBLEM**

To obtain the solution of equation (4), satisfying the boundary condition (5.2), we take the displacement $u_r$ in the following
Using (6), equation (4) becomes

\[ \frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} - (\kappa^2 + \frac{1}{r^2}) u = 0 \]  

(7)

where

\[ \kappa^2 = \frac{(\eta - \eta' \omega) P \omega^2}{[(\lambda + 2\mu) - (\lambda' + 2\mu') \omega]} \]

The solution of equation (7) is

\[ u = C_1 I_1(\kappa r) + D_1 K_1(\kappa r) \]  

(8)

where \( I_1 \) and \( K_1 \) are modified Bessel functions of order one and \( C_1, D_1 \) are constants independent of \( r \). Now with the help of (6) and (8), we have the radial displacement

\[ u_r = \left[ C_1 I_1(\kappa r) + D_1 K_1(\kappa r) \right] e^{-\omega t} \]  

(9)

Using boundary conditions (5.1) and (5.2), we obtain

\[ C_1 = 0 \quad \text{and} \quad D_1 = \frac{Pa \cdot (\eta - \eta' \omega)}{F(\alpha)} \]  

(10)
where

\[ F(x) = \left( \frac{\eta - \eta' \omega}{K} \right) \cdot \rho \omega^2 x K_0(Kx) + 2(\mu - \mu' \omega) K_1(Kx) \]  

(11)

With the help of (2.1), (9), (10) and (11), we get displacement and stress

\[ u_r = \frac{\rho a(\eta - \eta' \omega) K_1(kr)}{F(a)} \cdot e^{-\omega t} \]  

(12)

\[ \sigma_r = -\rho \cdot \frac{\sigma}{r} \cdot \frac{F(r)}{F(a)} \cdot e^{-\omega t} \]  

(13)

**ASYMPTOTIC SOLUTION**

From McLachlan [82] we have the relation

\[ K_0(z) \approx \left( \frac{\pi}{2z} \right)^{\frac{1}{2}} e^{-z} \]

and

\[ K_0(z) = \left( \frac{\pi}{2z} \right)^{\frac{1}{2}} e^{-z} \left( 1 + \frac{4 \nu^2 - 1^2}{1 \cdot 8z} + \frac{(4 \nu^2 - 1^2)(4 \nu^2 - 3^2)}{2! (8z)^2} + \ldots \right) \]  

(14)
Introducing (14) to (12) and (13), we obtain the following expressions for displacement and stress

\[ \mathcal{U}_r = p(\eta - \eta') \left( \frac{a}{a_r} \right)^{\frac{3}{2}} \left( \frac{a}{\eta} \right)^{\frac{3}{2}} \frac{3 + 8k(r-a)}{4(\eta - \eta')\rho \omega^2 a^2 + (\mu - \mu')(3 + 8ka)} e^{\{-k(r-a) + \omega t\}} \] (15)

\[ \sigma_r = -p \left( \frac{a}{\eta} \right)^{\frac{5}{2}} \frac{4(\eta - \eta')\rho \omega^2 a^2 + (\mu - \mu')(3 + 8ka)}{4(\eta - \eta')\rho \omega^2 a^2 + (\mu - \mu')(3 + 8ka)} e^{\{-k(r-a) + \omega t\}} \] (16)

### 3. DISTURBANCE DUE TO TRANSIENT TWIST: FORMULATION, BOUNDARY CONDITIONS AND ASYMPTOTIC SOLUTION OF THE SECOND PROBLEM

In the present problem, the non-vanishing displacement \( \mathcal{U}_\theta \) is a function of \( r \) and \( t \). The only stress component in terms of \( \mathcal{U}_\theta \) is

\[ (\eta + \eta' \frac{\partial}{\partial t}) \tau_{r\theta} = (\mu + \mu' \frac{\partial}{\partial t})(\frac{\partial \mathcal{U}_\theta}{\partial r} - \frac{\mathcal{U}_\theta}{r}) \] (17.1)

\[ \tau_{\theta z} = \tau_{r z} = \sigma_r = \sigma_\theta = \sigma_z = 0 \] (17.2)

The only non-vanishing equation of motion is

\[ \frac{\partial \tau_{r \theta}}{\partial r} + 2 \cdot \frac{\tau_{r \theta}}{r} = \rho \cdot \frac{\partial^2 \mathcal{U}_\theta}{\partial t^2} \] (18)
Using (17.1), equation (18) simplifies to

\[
(\mu + \mu' \frac{\partial}{\partial t}) \left[ \frac{\partial^2 \Theta}{\partial r^2} + \frac{1}{r} \frac{\partial \Theta}{\partial r} - \frac{\Theta}{r^2} \right] = \rho \eta'' \frac{\partial^2 \Theta}{\partial t^2}
\]

(19)

where \( \eta'' \) is given by (1).

BOUNDARY CONDITIONS

We have the following boundary conditions to solve the problem:

\( \frac{\partial \varphi}{\partial r} \) is finite at \( r \to \infty \) \hspace{1cm} (20.1)

and

\[
\tau \varphi \bigg|_{r=a} = -(\mu - \mu' \omega) S e^{-\omega t}
\]

(20.2)

where \( S \) is a constant.

Now we are to seek the solution of (19) using the boundary conditions (20.1) and (20.2).

SOLUTION OF THE PROBLEM

To find the solution of equation (19), satisfying the boundary
condition (20.2), we assume the displacement $u_\theta$ in the form

$$u_\theta = \varphi(r)e^{-\omega t} \tag{21}$$

Substituting the value of $u_\theta$ from (21) in the equation (19), we have

$$\frac{\partial^2 \varphi}{\partial r^2} + \frac{1}{r} \frac{\partial \varphi}{\partial r} - \left( k'^2 + \frac{1}{r^2} \right) \varphi = 0 \tag{22}$$

where

$$k'^2 = \frac{p(\eta - \eta'\omega)\omega^2}{(\mu - \mu'\omega)}.$$ 

The solution of (22) is

$$\varphi = C'_1 I_1(k'r) + D'_1 K_1(k'r) \tag{23}$$

where $C'_1$ and $D'_1$ are constants independent of $r$. With the help of (23), we have from (21)

$$u_\theta = [ C'_1 I_1(k'r) + D'_1 K_1(k'r) ] e^{-\omega t} \tag{24}$$

Applying the boundary conditions (20.1) and (20.2), we have

$$C'_1 = 0 \quad \text{and} \quad D'_1 = \frac{s(\eta - \eta'\omega)}{k'k_2(k'\omega)} \tag{25}$$
With the help of (17), (24) and (25), we obtain displacement and stress as

\[ U_g = \frac{S}{k'} \cdot \frac{(\eta - \eta'|\omega) k_1(k'|\eta)}{k_2(k'|\alpha)} \cdot e^{-\omega t} \]  

(26)

\[ \tau_{\theta} = -(\mu - \mu'|\omega) \cdot \frac{S k_2(k'|\eta)}{k_2(k'|\alpha)} \cdot e^{-\omega t} \]  

(27)

**ASYMPTOTIC SOLUTION**

Using the relation (14) in equations (26) and (27), we have the expressions for displacement and stress as

\[ U_g = \frac{S}{k'} \cdot \left(\eta - \eta'|\omega\right) x \left(\frac{a}{r}\right)^{\frac{3}{2}} \]  

\[ x \cdot \left(\frac{3 + 8k'|\eta}{15 + 8k'|\alpha}\right) \cdot e^{-\left\{k'|r - a + \omega t\right\}} \]  

(28)

\[ \tau_{\theta} = -(\mu - \mu'|\omega) \cdot S \left(\frac{a}{r}\right)^{\frac{3}{2}} x \]  

\[ x \cdot \left(\frac{15 + 8k'|\eta}{15 + 8k'|\alpha}\right) \cdot e^{-\left\{k'|r - a + \omega t\right\}} \]  

(29)
ON DISTURBANCES IN A HIGHER ORDER INFINITE VISCO-ELASTIC MEDIUM BY TRANSIENT RADIAL FORCES AND TWIST ON THE SURFACE OF A CYLINDER

1. INTRODUCTION

The problems of propagation of waves in an isotropic elastic solid medium is highly interesting and useful from the point of view of theoretical and practical applications in various directions. As such the theory of waves and vibrations in solids has developed tremendously and its literature is in plenty. The concept of visco-elasticity is also very useful to accommodate the viscous behaviour in elastic solids and it has also developed in various directions. The monographs of Bland \[5\], Flugge \[57,58\] can be referred and the article of Hunter \[63\] is also noteworthy.

Disturbances in a general visco-elastic medium due to impulsive forces on a spherical cavity was investigated by Bhattacharyya and Sengupta \[7\]. The propagation of waves in a visco-elastic plate immersed in a liquid under the influence of gravity has also been investigated by Bhattacharyya and Sengupta \[6\]. Rotatory vibration as well as radial vibration of a sphere

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of general visco-elastic solid have been studied by Sengupta and Roy [115, 126]. Further development in this field has been derived by the said authors. Radial motion of a conducting visco-elastic sphere in a magnetic field was studied by Roy and Sengupta [116], and this is followed by the problem of vibration of an infinite circular cylinder of visco-elastic higher order solid by Roy and Sengupta [117]. The problem of disturbances has been studied by Dasgupta [33] for radial impulsive force on spherical cavity. Goodier and Jahsman [60] studied similar problems. Recently Chakraborty [20] studied the disturbances in an elastic medium due to impulses on a cylindrical hole. Mukherjee [91-94] studied disturbances in visco-elastic medium due to various forces on the internal boundary of the solid.

In this paper an attempt has been made by the author to find the propagation of disturbances in a higher order infinite visco-elastic medium by transient radial forces as well as twist on the surface of a cylinder.

2. DISTURBANCE DUE TO TRANSIENT RADIAL FORCE: FORMULATION, BOUNDARY CONDITIONS AND ASYMPTOTIC SOLUTION OF THE FIRST PROBLEM

Due to disturbance, the wave is produced by a transient radial force on the surface of the hole. The only non-vanishing displacement $u_r$ must be a function of $r$ and $t$ only. Following Voigt [159], the general visco-elastic operators are
\[
\begin{align*}
\sigma_r &= (D_\lambda + 2D_\mu) \frac{\partial u_r}{\partial r} + D_\lambda \frac{u_r}{r} \\
\sigma_\theta &= D_\lambda \frac{\partial u_r}{\partial r} + (D_\lambda + 2D_\mu) \frac{u_r}{r} \\
\sigma_z &= \tau_{r\theta} = \tau_{\theta z} = \tau_{rz} = 0
\end{align*}
\]

Two equations of motion are identically satisfied. The only non-vanishing equation of motion is

\[
\frac{\partial \sigma_r}{\partial r} + \frac{\sigma_r - \sigma_\theta}{r} = \rho \frac{\partial^2 u_r}{\partial t^2}
\]

where \(\rho\) is the density of the material.

Substituting the values of \(\sigma_r\) and \(\sigma_\theta\), equation (3) takes the form

\[
(D_\lambda + 2D_\mu) \left[ \frac{\partial^2 u_r}{\partial r^2} + \frac{1}{r} \frac{\partial u_r}{\partial r} - \frac{u_r}{r^2} \right] = \rho \frac{\partial^2 u_r}{\partial t^2}
\]
BOUNDARY CONDITIONS

Boundary conditions of the present problem are

\[ U \text{ is finite at } r \to \infty \]  \hspace{1cm} (5.1)

\[ \delta r \bigg|_{r=a} = -p e^{-\omega t} \]  \hspace{1cm} (5.2)

where \( p \) is a constant.

Now our object is to solve equation (4) subject to the boundary conditions (5.1) and (5.2).

SOLUTION OF THE PROBLEM

To obtain the solution of equation (4), satisfying the boundary condition (5.2), we take the displacement \( u_r \) in the form

\[ u_r = u(r)e^{-\omega t} \]  \hspace{1cm} (6)

Substituting this in equation (4), it transforms into

\[ \frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} - \left( \kappa^2 + \frac{1}{r^2} \right) u = 0 \]  \hspace{1cm} (7)

where

\[ \kappa^2 = \sum_{m=0}^{\infty} \frac{\rho \omega^2}{\lambda_m + 2\mu_m (-\omega)^m} \]
The solution of equation (7) is

\[ U = C_1 I_1(\kappa r) + D_1 K_1(\kappa r) \]  

where \( I_1 \) and \( K_1 \) are modified Bessel functions of order one and \( C_1 \) and \( D_1 \) are constants independent of \( r \).

Now introducing (8) into (6), the radial displacement \( u_r \) takes the form

\[ u_r = [C_1 I_1(\kappa r) + D_1 K_1(\kappa r)] e^{-\omega t} \]  

Applying boundary conditions (5.1) and (5.2), we have

\[ C_1 = 0 \quad , \quad D_1 = \frac{aP}{F(a)} \]  

where

\[ F(x) = -\frac{\omega^2}{\kappa} x K_0(\kappa x) + 2 \sum_{m=0}^{\infty} \mu_m (-\omega)^m K_1(\kappa x) \]  

With the help of (2.1), (9), (10) and (11), we obtain displacement and stress

\[ u_r = \rho \frac{a K_1(\kappa r)}{F(a)} e^{-\omega t} \]  

\[ \sigma_r = -\rho \frac{a}{r} \frac{F(r)}{F(a)} e^{-\omega t} \]
ASYMPTOTIC SOLUTION

Now introducing McLachlan's relations $\left[82\right]$

\[
K_0(z) \simeq \left(\frac{\pi}{2z}\right)^{\frac{1}{2}} e^{-z}
\]

and

\[
K_n(z) = \left(\frac{\pi}{2z}\right)^{\frac{1}{2}} \sum_{\kappa=0}^{\infty} \frac{(i+iKr)^{\kappa}}{\kappa!} z^{-\kappa} + \ldots
\]

\[
x \left[ 1 + \frac{4v^2-1}{1! 8z} + \frac{(4v^2-1)(4v^2-3)}{2! (8z)^2} + \ldots \right]
\]

(14)

to (12) and (13), we obtain the expressions for displacement and stress as

\[
U_r = p \frac{\alpha}{\pi} \left(\frac{a}{r}\right)^{\frac{3}{2}} x
\]

\[
x \frac{(3+8Kr)}{4\rho^2} e^{-\left\{k(r-a)+\omega t\right\}}
\]

\[
\sum_{m=0}^{\infty} \frac{\mu m (-\omega)^m (3+8Kn)}{4\rho \omega^2 \rho^2 + \sum_{m=0}^{\infty} \mu m (-\omega)^m x (3+8Kn)}
\]

\[
= - p \left(\frac{\alpha}{\pi}\right)^{\frac{3}{2}} x
\]

\[
x \frac{[4\rho^2 \rho^2 + \sum_{m=0}^{\infty} \mu m (-\omega)^m x (3+8Kn)]}{[4\rho^2 \rho^2 + \sum_{m=0}^{\infty} \mu m (-\omega)^m x (3+8Kn)]} e^{-\left\{k(r-a)+\omega t\right\}}
\]

(15)

3. DISTURBANCE DUE TO TRANSIENT TWIST: FORMULATION, BOUNDARY CONDITIONS AND ASYMPTOTIC SOLUTION OF THE SECOND PROBLEM

In this present problem, the non-vanishing displacement $U_\theta$

is a function of $r$ and $t$. The only stress component in terms of
\[ \tau_{r\theta} = D_\mu \left( \frac{\partial \mu \theta}{\partial r} - \frac{\mu \theta}{r} \right) \]  \hfill (17.1)

\[ \tau_{\theta z} = \tau_{r z} = \sigma_r = \delta_{\theta} = \delta_z = 0 \]  \hfill (18.2)

where \( D_\mu \) is given by (1).

The only non-vanishing equation of motion is given by

\[ \frac{\partial \tau_{r\theta}}{\partial r} + 2. \frac{\tau_{r\theta}}{r} = \rho \frac{\partial^2 \mu \theta}{\partial t^2} \]  \hfill (18)

Putting the value of \( \tau_{r\theta} \), equation (18) becomes

\[ D_\mu \left[ \frac{\partial^2 \mu \theta}{\partial r^2} + \frac{1}{r} \frac{\partial \mu \theta}{\partial r} - \frac{\mu \theta}{r^2} \right] = \rho \frac{\partial^2 \mu \theta}{\partial t^2} \]  \hfill (19)

**BOUNDARY CONDITIONS**

Boundary conditions may be written as

\[ \mu_\theta \text{ is finite at } r \to \infty \]  \hfill (20.1)

and

\[ \tau_{r \theta} \bigg|_{r=a} = - \sum_{m=0}^{\infty} \mu_m (-\omega)^m \cdot S e^{-\alpha t} \]  \hfill (20.2)
where $S$ is a constant.

Now we seek the solution of (19) using the boundary conditions (20.1) and (20.2).

SOLUTION OF THE PROBLEM

To obtain the solution of equation (19), satisfying the boundary condition (20.2), we take the displacement $U_\theta$ in the form

$$U_\theta = \psi(r)e^{-\omega t} \quad (21)$$

Putting this in equation (19), it takes the form

$$\frac{\partial^2 \psi}{\partial r^2} + \frac{1}{r} \frac{\partial \psi}{\partial r} - \left( K^{12} + \frac{1}{r^2} \right) \psi = 0 \quad (22)$$

where

$$K^{12} = \frac{\rho \omega^2}{\sum_{m=0}^{\infty} \frac{\mu_m (-\omega)^m}{m}}$$

The solution of equation (22) is

$$\psi = C'_1 I_1 (K'r) + D'_1 K_1 (K'r) \quad (23)$$

where $C'_1$ and $D'_1$ are constants independent of $r$. 
Introducing (23) in (21), the non-vanishing displacement becomes

\[ U_\theta = \left[ C'_i I_1(k'r) + D'_i K_1(k'r) \right] e^{-\omega t} \quad (24) \]

Now with the help of boundary conditions (20.1) and (20.2), we obtain

\[ C'_i = 0, \quad D'_i = \frac{S}{k'_1 k'_2(k'a)} \quad (25) \]

Using (17), (24) and (25), we get displacement and stress which are given by

\[ U_\theta = \frac{S}{k'_i} \frac{K_1(k'_r)}{K_2(k'_a)} e^{-\omega t} \quad (26) \]

\[ \tau_{\rho \theta} = -\frac{n}{m=0} \mu_m (-\omega)^m \frac{S k_2(k'_r)}{k_2(k'_a)} e^{-\omega t} \quad (27) \]

ASYMPTOTIC SOLUTION

Putting the relations (14) in equations (26) and (27), we get the expressions for displacement and stress as follows

\[ U_\theta = \frac{S}{k'_i} \left( \frac{a}{r} \right)^{\frac{3}{2}} \frac{3 + 8k'_r}{15 + 8k'_a} \frac{1}{e^{-\left\{ k'_i (r-a) + \omega t \right\}}} \quad (28) \]

\[ \tau_{\rho \theta} = -\sum_{m=0}^{n} \mu_m (-\omega)^m \left( \frac{a}{r} \right)^{\frac{3}{2}} \frac{15 + 8k'_r}{15 + 8k'_a} e^{-\left\{ k'_i (r-a) + \omega t \right\}} \quad (29) \]
1. INTRODUCTION

The interaction of Maxwell electro-magnetic field or of an externally applied magnetic field on the deformation and motion with the electrically conducting continua has become a subject of investigation from various points of view. It has developed two-fold. If the continua is a conducting liquid, the interaction of the electro-magnetic field will substantially change the motion of the liquid and the motion of the liquid will also change the existing electro-magnetic field. This is interlocking in character and is very pronounced in the case of liquid flow. This is called magneto-hydrodynamics. It has developed enormously in various directions including the discovery of Alfven waves.

However, its counter part magneto-elasticity, originated from the interaction of electro-magnetic fields with the motion and deformation of an elastic solids has not developed so much due to its small motion and deformation. The interaction of an externally applied magnetic field and thermal field on the elastic motion and the deformation of the solid, termed as magneto-thermo-elasticity has developed to a considerable extent at the hands of

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Willson \(161\), Paria \(105\), Nowacki \(96\), Purushottama \(111\), Mal and Sengupta \(127-129\). Although the effect of small and moderate magnetic field on the elastic and thermo-elastic field is small, the theory of magneto-elasticity and magneto-thermo-elasticity in case of considerable magnetic field are developed owing to its importance in geophysical, seismological, cosmological problems and to a greater extent to practical problems when the magnetic field is large \(1,10,22\). Coupled fields in elasticity, in particular to thermo-elastic, magneto-elastic and magneto-thermo-elastic coupling were investigated in a concise form by Nowacki \(97\).

To consider the dynamical problem of visco-elasticity we can refer here the problem of radial vibration of a sphere of general visco-elastic solid by Sengupta and Roy \(126\). Further, the rotatory vibration of a sphere of a general visco-elastic solid and vibration of an infinite circular cylinder of higher order visco-elastic solid were also investigated by Roy and Sengupta \(115,117\). Recently, two-dimensional wave propagation in a micropolar thermo-visco-elastic layer was investigated by Roy and Sengupta \(118\). In this direction some recent works may also be cited \(11,21,23,39,70-74,98,106,107\). Dutta \(35-38\) also has investigated a good number of problems in this field.

In this paper an attempt has been made by the author to investigate some aspects of dispersion of coupled waves in a thermo-visco-elastic solid acted upon by a magnetic field. In this attempt the frequency equation of one-dimensional wave has been derived by solving the field equations including various effects such as magnetic, thermal, viscous fields. The nature of frequency has been analysed thoroughly
and it is shown in particular that if the frequency is sufficiently smaller than the certain critical frequency, the attenuation and the phase velocity is changed retaining the same frequency as in the purely thermo-elastic wave. This is also asserted when the coupling factor is properly changed. Besides, a mathematically analysis has also been presented on group velocity of the wave. For small values of frequency the group velocity of the waves equals the phase velocity of the wave. Moreover, the mathematical analysis of the frequency equation and its roots exhibit the dispersion of the waves in the general form.

2. GENERAL THEORY AND FUNDAMENTAL EQUATIONS

We study the interaction of the electromagnetic field, thermal field, and the visco-elastic field on the propagation of waves and for this we attempt to solve the appropriate equations displaying the above fields. Maxwell equations governing the electro-magnetic field in absence of the displacement current and charge density are

\[
\text{Curl } \vec{H} = \vec{J} \quad \text{div } \vec{B} = 0
\]

\[
\text{Curl } \vec{E} = -\frac{\partial \vec{B}}{\partial t} \quad \vec{B} = \mu \varepsilon \vec{H}
\]  \hspace{1cm} (2.1)

while for a perfectly conducting medium, the generalised Ohm's law is taken in the form
\[
\mathbf{f} = \sigma \left[ \mathbf{E} + \left( \frac{\partial \mathbf{u}}{\partial t} \times \mathbf{B} \right) \right] \tag{2.2}
\]

neglecting the small effect of temperature gradient on the current.

Where \( \mathbf{H} \Rightarrow \) magnetic intensity,
\( \mathbf{B} \Rightarrow \) magnetic induction,
\( \mathbf{E} \Rightarrow \) electric intensity,
\( \mathbf{J} \Rightarrow \) current density vector,
\( \mu_e \Rightarrow \) magnetic permeability of the body,
\( \mathbf{u} \Rightarrow \) displacement vector in the strained solid,
\( \sigma \Rightarrow \) electric conductivity.

If the deformation due to temperature distribution is taken into account, the stress-strain relations in visco-elastic medium of first order is determined by the modified Hooke's law (Hooke-Duhamel-Neumann) in the form

\[
\tau_{ij} = 2(\mu_0 + \mu_1 \frac{\partial}{\partial t}) \delta_{ij} + \left\{ (\lambda_0 + \lambda_1 \frac{\partial}{\partial t}) \delta_{ij} - (\beta_0 + \beta_1 \frac{\partial}{\partial t}) T \right\} \delta_{ij} \tag{2.3}
\]

where \( \tau_{ij} \) and \( \delta_{ij} \) are the stress tensor and strain tensor respectively, \( \lambda_0, \mu_0, \lambda_1, \mu_1 \) are visco-elastic parameter and
\( \beta_0 = (3\lambda_0 + 2\mu_0) \kappa_T, \beta_1 = (3\lambda_1 + 2\mu_1) \kappa_T, \kappa_T \) is the co-efficient of linear thermal expansion.
Due to the rise of temperature of the material it has been observed that the visco-elastic parameters, written as
\[ \mu_0 + \mu_1 \frac{\partial}{\partial t}, \quad \lambda_0 + \lambda_1 \frac{\partial}{\partial t} \]
and the thermal parameter \( \beta_0 + \beta_1 \frac{\partial}{\partial t} \)
are ultimately time dependent due to the fact that these parameters depend on temperature and temperature is a function of time. This implies the condition that the thermal parameters \( \beta_0, \beta_1 \) are needed to describe the state of affairs in the thermo-visco-elastic solids.

Neglecting the effect of current on the temperature, the generalised law of heat conduction in absence of heat source is
\[
K \nabla^2 T = \rho c_v \frac{\partial T}{\partial t} + \tau_0 (\beta_0 + \beta_1 \frac{\partial}{\partial t}) \frac{\partial \varphi}{\partial t}
\]
(2.4)

where \( K \Rightarrow \) thermal conductivity,
\( c_v \Rightarrow \) specific heat at constant strain,
\( \tau_0 \Rightarrow \) a certain reference temperature over which the perturbed temperature is \( T \),
\( \varphi \Rightarrow \) \( \text{div} \, \mathbf{u} \).

The strain displacement relations are
\[
e_{ij} = \frac{1}{2} (u_{i,j} + u_{j,i})
\]
(2.5)
The stress equation of motion are

\[ \tau_{i,j} + F_i = \rho \ddot{u}_i \]  \hspace{1cm} (2.6)

where \( \ddot{u} \) \( \Rightarrow \) \((u_1, u_2, u_3) \), the displacement vector, 
\( \rho \) \( \Rightarrow \) density of the material, 
\( \vec{F} \) \( \Rightarrow \) \((F_1, F_2, F_3) \), the body force per unit volume.

If there are no body force apart from Lorentz force, we have

\[ \vec{F} = \vec{J} \times \vec{B} \]  \hspace{1cm} (2.7)

Clearly, the set of equations (2.1) to (2.7) implies that the electro-magnetic, visco-elastic and thermal fields interact and they are coupled with each other.

3. REDUCTION TO THE CASE OF A UNI-DIRECTIONAL MOTION

Let us consider the case of the propagation of disturbance in one direction, say x-direction only. The z-axis is taken in the direction of original magnetic field (primary) \( H_z \). Then the displacement \( \vec{u} \) has the components \((u,0,0)\) where \( u(x,t) \) and all the above vector quantities depend on \( x \) and \( t \) only. Let the modified magnetic field be

\[ \vec{H} = \vec{H}_0 + \vec{h} \]  \hspace{1cm} (3.1)
where \( \vec{H}_0 = (0, 0, H_z) \) is the initial magnetic field acting parallel to z-axis and \( \vec{h} = (0, 0, h_z) \) is small perturbation in the field.

Using the first, third and fourth equations of (2.1) and (2.2), we obtain

\[
\frac{\partial h_z}{\partial t} = \nu_H \frac{\partial^2 h_z}{\partial x^2} - H_z \frac{\partial^2 u}{\partial x \partial t}
\]  

(3.2)

where

\[
\nu_H = \frac{1}{\mu e \sigma}.
\]

From the first and last equations of (2.1), neglecting the products of small quantities of \( \vec{u} \) and \( \vec{h} \), we obtain

\[
\vec{J} \times \vec{B} = \left[ -\mu e H_z \frac{\partial h_z}{\partial x}, 0, 0 \right]
\]

(3.3)

The second equation of (2.1) is then identically satisfied.

Eliminating \( \varepsilon_{ij} \) from equations (2.3) and (2.5) and putting the resulting components of stresses in the equations of motion (2.6) and using the equations (2.7) and (3.3), we obtain

\[
\left\{ \frac{\lambda_0 + 2\mu_0}{\lambda_1 + 2\mu_1} \frac{\partial^2 u}{\partial x^2} \right\} - (\beta_0 + \beta_1 \frac{\partial}{\partial t}) \frac{\partial T}{\partial x} - \mu e H_z \frac{\partial h_z}{\partial x} = \rho \frac{\partial^2 u}{\partial t^2}
\]

(3.4)
Also equation (2.4) reduces to
\[ K \frac{\partial^2 T}{\partial x^2} = \rho c_\gamma \frac{\partial T}{\partial t} + T_0 (\beta_0 + \beta_\gamma \frac{\partial}{\partial t}) \frac{\partial^2 T}{\partial x \partial t} \]  
(3.5)

4. SOLUTION OF THE PROBLEM AND DISCUSSION OF THE RESULTS

Here we consider the case when electrical conductivity of the medium is finite. In this case we take

\[ u = u^1 \exp [i(\gamma x - \omega t)] \]
\[ \phi = \phi^1 \exp [i(\gamma x - \omega t)] \]
\[ T = T^1 \exp [i(\gamma x - \omega t)] \]  
(4.1)

where \( u^1, \phi^1, T^1 \) are constants, \( \gamma \) is the velocity and \( \omega \) is the frequency of the wave.

Putting the values of \( u, \phi, T \) in the equations (3.2), (3.4) and (3.5) and then eliminating \( u^1, \phi^1, T^1 \) from the resulting equations, we obtain the wave velocity equation given by

\[
\begin{vmatrix}
H_\omega \omega & (\gamma \omega^2 - i \omega) & 0 \\
(\lambda^2 - i \omega \lambda^1)^2 - \omega^2 & i \frac{\mu e}{\rho} H_\omega \gamma & i (\frac{\beta_0}{\rho} - i \omega \beta_1) \gamma \\
T_0 (\beta_0 - i \omega \beta_1) \gamma & 0 & (\kappa \gamma^2 - i \rho c_\gamma \omega)
\end{vmatrix} = 0
\]  
(4.2)
where \( \lambda^2 = \frac{\lambda_0 + 2\mu_0}{p} \), \( \chi^2 = \frac{\lambda_1 + 2\mu_1}{p} \), are the longitudinal visco-elastic wave velocity.

Now the equation (4.2) can be written as

\[
\rho \omega^4 + i\alpha \omega^3 - R \omega^2 - i S \omega + T = 0
\]

(4.3)

where

\[
\rho = p c_0 + \frac{\beta_1^2}{p} \omega^2 T_0
\]

\[
\alpha = \nu_h (\omega^4 \frac{\beta_1}{p} T_0 + \nu_h \omega^2 p c_0
\]

\[+ 2 \beta_0 \beta_1 \frac{\omega^2}{p} T_0 + \kappa \omega^2 + \lambda^2 \omega^2 p c_0
\]

\[+ \frac{\beta_0^2}{p} \omega^2 T_0 + \omega^4 \kappa + \lambda^2 \omega^2 p c_0
\]

\[+ \frac{\beta_0^2}{p} \omega^2 + \omega^4 \kappa + \lambda^2 \omega^2 p c_0
\]

\[S = \nu_h \omega^6 \frac{\mu}{p} \kappa + \nu_h \omega^4 \frac{\beta_0}{p} \omega^2 T_0
\]

\[+ \nu_h \omega^6 \kappa + \nu_h \omega^4 \lambda^2 p c_0
\]

\[+ \lambda^2 \omega^4 \kappa
\]

\[T = \nu_h \omega^6 \lambda^2 \kappa
\]

(4.4)

Now we will study two different cases:

(1) when the wave length \( l = \frac{2\pi}{\lambda} \) is real, i.e., \( \lambda \) is real.

(2) when the frequency \( \omega \) is real.

Case I. Assuming \( \lambda \) to be real, the coefficients of the equations (4.3) will follow and the different possibilities of the roots of \( \omega \), firstly all the four roots of the
equation are imaginary, secondly two real and two imaginary and thirdly all the four roots are real. It is clear from equation (4.1) that imaginary part of $\omega$ only contributes to the amplitudes of the waves and hence the four real values of $\omega$ produces four additional modes in the wave. Further, if one considers an imaginary root of the form $\alpha + i\beta$ any equation of (4.1) will be of the form

$$A\exp\left[i\left(\alpha x - (\alpha + i\beta)t\right)\right] = A\exp\left[i(\alpha x - \alpha t)\right] \cdot \exp(\beta t).$$

This type of root generates a variety of interpretations depending on the nature of $\alpha$ and $\beta$. They express damping characteristics in time only when $\beta < 0$. Further, one can see from equations (4.3) and (4.4) that the frequency of the magneto-thermo-visco-elastic coupled waves depend on the wave length non-linearly and therefore, various types of wave lengths will propagate with different types of phase velocities. In general one may conclude that magneto-thermo-visco-elastic waves are dispersive in nature and the dispersion takes place in various ways according to the nature of the problem and material.

Case II. Assuming $\omega$ to be real, we develop the determinantal equation (4.2) in the form

$$L \beta^6 - M \beta^4 - N \beta^2 + R_1 = 0 \quad (4.5)$$
where
\[ L = \lambda^2 K \nu_H - i \lambda^2 K \nu_H \omega \]
\[ M = i \omega \frac{H_0}{P} H_2 K + i \lambda^2 K \omega + i \omega \nu_H T_0 \frac{\beta_0^2}{P} + \omega^2 \nu_H K + i PC \nu \lambda^2 \nu_H \omega \\
+ \lambda^2 \omega^2 PC \nu_H + \lambda^2 \omega^2 K + 2 \nu_H \frac{\beta_0 \beta_1}{P} T_0 \omega^2 - i \nu_H \omega^3 \frac{\beta_1^2}{P} T_0 \]
\[ N = \mu_0 H_2^2 \omega^2 PC \nu + \omega^2 PC \nu \lambda^2 + \omega^2 \frac{\beta_0^2}{P} T_0 - i \omega^3 PC \nu_H \]
\[ - i \omega^3 K - i \omega^3 PC \nu \lambda^2 - 2 i \omega^3 \frac{\beta_0 \beta_1}{P} T_0 - \omega^4 \frac{\beta_1^2}{P} T_0 \]
\[ R_1 = PC \nu \omega^4 \quad (4.6) \]

We find that \( \gamma \) has three pairs of equal and opposite imaginary roots.

If the roots be of the form \( (\alpha_1^0 \pm i \beta_1^0), (\alpha_2^0 \pm i \beta_2^0), (\alpha_3^0 \pm i \beta_3^0) \), then the amplitude of the waves contains the terms \( \exp(-\beta_1^0 x), \exp(-\beta_2^0 x) \) and \( \exp(-\beta_3^0 x) \). This clearly indicates that the amplitudes of motion die out in the x-direction provided \( \beta_1^0, \beta_2^0 \) and \( \beta_3^0 \) are positive.

**Group Velocity:**

As already remarked, the waves are dispersive and as the group velocity of the wave is a characteristic velocity that represents the speed with which energy is propagated, we now calculate the group velocity of the dispersive waves.

Putting \( c = \frac{\omega}{\beta} \) (phase velocity) in the equation \((4.3)\), we have

\[ c^4 + \rho c^2 + \alpha_1 = 0 \quad (4.7) \]
where

\[ P_1 = \frac{1}{P_{C_{19}}} \left[ \frac{P_{C_{19}} T_0}{P_{C_{19}}} \omega^2 + i\omega \left( \frac{\psi_H \psi_2 \beta^2}{P_{C_{19}}} \right) \right. \\
\left. + P_{C_{19}} \psi_H + \kappa + P_{C_{19}} \lambda^2 + 2 \frac{P_{C_{19}}}{P_{C_{19}}} \right) \\
- \left( \frac{\mu e H \kappa}{P_{C_{19}}} + P_{C_{19}} \lambda^2 + \frac{T_0 \beta^2}{P_{C_{19}}} + \psi_H \kappa \lambda^2 \\
+ \lambda^2 \psi_2 \psi_H P_{C_{19}} + \lambda^2 \psi_2 \kappa + 2 \frac{\psi_H \lambda^2 \beta P_{C_{19}}}{P_{C_{19}}} \right) \right] \\
Q_1 = \frac{1}{P_{C_{19}}} \left[ \lambda^2 \kappa \psi_H \psi_2^2 - i\omega \left( \frac{\mu e H \beta^2}{P_{C_{19}}} + \lambda^2 \kappa \\
+ \lambda^2 P_{C_{19}} \psi_H + \psi_H \frac{\beta^2}{P_{C_{19}}} + \lambda^2 \psi_H \psi_2 \kappa \right) \right] \] 

(4.8)

Now group velocity \( C_g \) of the wave can be written as

\[ C_g = \frac{C}{1 - \omega \frac{\partial C}{\partial \omega}} \]

\[ = \frac{C \left( 2 P_{e^2} + 4 C^4 \right)}{\left( 2 P_{e^2} + 4 C^4 \right) + \omega C^2 \frac{\partial \alpha}{\partial \omega} + \omega \frac{\partial \alpha}{\partial \omega}} \] 

(4.9)

\[ \text{by (4.7) and (4.8)} \]

If we assume that the values of \( \omega \) are large then from equation (4.9) it at once conclude that \( C_g \to \infty \), which implies that the wave packet consisting of infinitesimally short wave lengths will propagate with infinite velocity which is physically absurd. Hence the possibility of infinite value of \( C_g \) with \( \omega \to \infty \).
is in admissible in our problem. However, for small values of \( \omega \) one may conclude that the group velocity of the waves is equal to the phase velocity of the waves.

On the other hand, we define the characteristic frequency and the magnetic pressure number given by

\[
\omega_1 = \frac{p_c u_0 \lambda_0^2}{\kappa},
\]

\[
\mathcal{R}_H = \frac{\mu_0 H_z^2}{p \lambda_0^2}
\]

and also introduce the dimensionless quantities

\[
\chi = \frac{\omega}{\omega_1}, \quad \xi = \frac{\lambda_0^2}{\omega_1}, \quad \varepsilon_T = \frac{T_0 B_0^2}{\rho^2 c^2 \lambda_0^2},
\]

\[
\varepsilon_H = \frac{\mathcal{V}_H \omega_1}{\lambda_0^2}, \quad \eta = \frac{\lambda_0}{\chi^2} \sqrt{\omega_1}, \quad \eta_1 = \frac{\beta_1}{\beta_0} \omega_1
\]

Then we find that the determinantal equation (4.2) reduces to

\[
(\chi + i \varepsilon_H \xi^2)(\xi^2 - \chi^2)(\chi + i \xi^2) + \varepsilon_T \xi^2 \chi + R_H \xi^2 \chi(\chi + i \xi^2)
\]

\[
+ \xi^2 \chi \eta^2 [\xi^2 \chi (\varepsilon_H + i) + i (\varepsilon_H \xi^4 - \chi^2)]
\]

\[
+ \xi^2 \chi^2 \eta [\varepsilon_T (2 \varepsilon_H \xi^2 - \chi^2) \eta_1 - i \varepsilon_T \chi (2 + \varepsilon_H \xi^2 \eta_1)] = 0
\]
In the equation (4.12) if we put $R_H = 0$ and $\varepsilon_H = 0$, we obtain

\[(\xi^2 - \chi^2)(\chi + i\xi^2) + \varepsilon_T \xi^2 \chi + i\xi^2 \eta^2 (\xi^2 - i\chi) = 0 \quad (4.13)\]

which is the equation of purely thermo-elastic wave propagation $\mathcal{L}^{23}$.

We rewrite the equation (4.12) in the form

\[
\chi^4 - \xi^2 \chi^2 \left[ (1 + R_H) + \varepsilon_T (1 - \xi^2 \eta^2) - i\chi (1 + \xi^2 + \eta^2 + 2\varepsilon_T \eta^2) \right] \\
- \xi^2 \eta^2 \left[ \chi \left( \varepsilon_H (1 + \eta^2 + 2\varepsilon_T \eta^2) \right) + \eta^2 i \right] \\
+ i \left( (1 + R_H) + \varepsilon_H (1 + \varepsilon_T + \xi^2 \eta^2 - \varepsilon_T \xi^2 \eta^2) \right) + \varepsilon_H \xi^4 = 0 \quad (4.14)
\]

LOW FREQUENCY

The characteristic frequency $\omega_1$ for some common metals at temperature 20°C is given below $\mathcal{L}^{23}$

<table>
<thead>
<tr>
<th>Material</th>
<th>$\omega_1$ (\text{rad s}^{-1})</th>
</tr>
</thead>
<tbody>
<tr>
<td>Aluminium</td>
<td>$4.66 \times 10^{11}$</td>
</tr>
<tr>
<td>Copper</td>
<td>$1.75 \times 10^{11}$</td>
</tr>
<tr>
<td>Iron</td>
<td>$1.75 \times 10^{12}$</td>
</tr>
<tr>
<td>Lead</td>
<td>$1.91 \times 10^{11}$</td>
</tr>
</tbody>
</table>

On this presentation one may find that the wave frequency is much less than the characteristic frequency $\omega_1$ $\mathcal{L}^{23}$ and it follows from the first definition of (4.11)

\[
\chi \ll 1. \quad (4.15)
\]
Neglecting the squares and higher powers of $\chi$ in view of the above condition (4.15), the equation (4.14) becomes

$$\xi = \pm (1+i)\left( \frac{\chi}{2} \left\{ (1 + \varepsilon_T) + \frac{(1 + R_H)}{\varepsilon_H} \right\} \right)^{\frac{1}{2}}$$

(4.16)

Again with the same approximation (4.15), the equation (4.13) for the purely thermo-elastic case leads to

$$\xi = \pm (1+i)\left[ \frac{\chi}{2} (1 + \varepsilon_T) \right]^{\frac{1}{2}}$$

(4.17)

It is seen from the relations (4.16) and (4.17) that the effect of the magnetic field is to increase the thermo-elastic coupling factor by the amount $\frac{(1 + R_H)}{\varepsilon_H}$. With this modification the attenuation factor and the phase velocity retain the same characteristic as in purely thermo-elastic waves.