3.1 Introduction

In probability and statistics, the inverse Rayleigh distribution is one of the most flexible distributions among the inverted scale family. It has been considered as a suitable model in life testing and reliability theory. Due to its diverse applications in different fields and special case of the Weibull distribution many statisticians consider it for different types of data sets. Trayer (1964) introduced the Inverse Rayleigh distribution (IRD) as a special case of Inverse weibull distribution and discussed its application in reliability theory. Voda (1972) mentioned that the distribution of lifetimes of several types of experimental units can be approximated by the inverse Rayleigh distribution while Mohsin and Shahbaz (2005) compared the negative moment estimator and Maximum Likelihood estimator of the inverse Rayleigh distribution.

The pdf of the Inverse Rayleigh distribution with scale parameter $\lambda$ is

$$g(x) = \frac{2\lambda}{x^3} e^{-\frac{\lambda}{x^2}}, \quad x > 0, \lambda > 0,$$  \hspace{1cm} (3.1.1)

and the corresponding cdf is

$$G(x) = e^{-\frac{\lambda}{x^2}}, \quad x > 0, \lambda > 0.$$  \hspace{1cm} (3.1.2)

Inferences for the Inverse Rayleigh distribution have been discussed by a number of authors. Gharraph (1993) derived five measures of the parameter of Inverse Rayleigh distribution and also obtained the estimators of the unknown parameter using different methods of estimation. Bayes estimators of the parameter of the Inverse Rayleigh distribution were considered by El-Helbawy and Abdel-Monem (2005) while Soliman et al. (2010) compared the Bayesian and non-Bayesian estimation of inverse Rayleigh distribution based on lower record values. Dey (2012) considered Bayesian analysis of the inverse Rayleigh distribution using symmetric and asymmetric loss functions under a non-informative prior. Sindhua et al. (2013) discussed the Bayesian estimation and corresponding risks for the parameter of the Inverse Rayleigh distribution when the data are left censored. Reshi et al. (2014) considered the estimation of scale parameter of Generalized Inverse Rayleigh distribution. Sajid Ali (2015) described the 2-component mixture of the inverse
Rayleigh distributions under Bayesian framework. Recently, Huda et al. (2016) obtained and discussed the Bayesian approach for estimating the scale parameter of Inverse Rayleigh distribution under different loss function.

3.2 Bayesian Approximation for Inverse Rayleigh Distribution

3.2.1 Bayes Estimate of Inverse Rayleigh Distribution Using Normal Approximation:

The likelihood function of (3.1.1) for a sample of size $n$ is given as

$$L(x \mid \lambda) = 2^n \lambda^n \prod_{i=1}^{n} x_i^{-3} e^{-\lambda x_i^{-2}}. \quad (3.2.1)$$

Under Jeffrey’s prior $g(\lambda) \propto \frac{1}{\lambda}$, the posterior distribution for $\lambda$ is as

$$P(\lambda \mid x) \propto \lambda^{n-1} e^{-\lambda T}, \text{ where } T = \sum_{i=1}^{n} x_i^{-2} \quad (3.2.2)$$

To construct the approximation, we need the second derivatives of the log-posterior density,

$$\log P(\lambda \mid x) = \log \text{constant} + (n-1) \log \lambda - \lambda T \quad (3.2.3)$$

Now

$$\frac{\partial}{\partial \lambda} \log P(\lambda \mid x) = 0 \quad \Rightarrow \quad \frac{(n-1)}{\lambda} - T = 0,$$

$$\Rightarrow \quad \lambda = \frac{(n-1)}{T}.$$ 

Also,

$$\frac{\partial^2}{\partial \lambda^2} \log P(\lambda \mid x) = -\frac{(n-1)}{\lambda^2}.$$ 

$$\therefore \quad I(\hat{\lambda}) = -\frac{\partial^2}{\partial \lambda^2} \log P(\lambda \mid x) = \frac{T^2}{(n-1)},$$

$$\Rightarrow \quad [I(\hat{\lambda})]^{-1} = \frac{(n-1)}{T^2}.$$ 

$$\therefore \quad p(\lambda \mid x) \sim N\left(\frac{(n-1)}{T}, \frac{(n-1)}{T^2}\right). \quad (3.2.4)$$

Under modified Jeffrey’s prior $g(\lambda) \propto \left(\frac{1}{\sqrt{\lambda^3}}\right)$, the posterior distribution for $\lambda$ is as

$$P(\lambda \mid x) \propto \lambda^{(n-3/2)} e^{-\lambda T}, \text{ where } T = \sum_{i=1}^{n} x_i^{-2} \quad (3.2.5)$$

$$\log P(\lambda \mid x) = \log \text{constant} + (n-3/2) \log \lambda - \lambda T. \quad (3.2.6)$$

Now

$$\frac{\partial}{\partial \lambda} \log P(\lambda \mid x) = 0,$$
\[ \Rightarrow \left( \frac{n-3/2}{\lambda} \right) - T = 0 \quad \Rightarrow \quad \hat{\lambda} = \frac{(n-3/2)}{T}. \]

Also, \[ \frac{\partial^2}{\partial \lambda^2} \log P(\lambda \mid \mathbf{x}) = -\frac{(n-3/2)}{\lambda^2}. \]

\[ \therefore \quad I(\hat{\lambda}) = \frac{\partial^2}{\partial \lambda^2} \log P(\lambda \mid \mathbf{x}) = \frac{T^2}{(n-3/2)}, \]

\[ \Rightarrow \quad [I(\hat{\lambda})]^{-1} = \frac{(n-3/2)}{T^2}. \]

\[ \therefore \quad p(\lambda \mid \mathbf{x}) \sim N\left(\frac{(n-3/2)}{T}, \frac{(n-3/2)}{T^2}\right). \quad (3.2.7) \]

Under Gamma prior \[ g(\lambda) \propto \lambda^{a-1} e^{-b\lambda}; a, b > 0; \lambda > 0, \] where \( a \) and \( b \) are hyper parameters, thus the posterior distribution for \( \hat{\lambda} \) is as

\[ P(\hat{\lambda} \mid \mathbf{x}) \propto \lambda^{(n+1)} e^{-\lambda(b+T)}, \quad \text{where} \quad T = \sum_{i=1}^{n} \mathbf{x}_i^{-2} \]

\[ \log P(\hat{\lambda} \mid \mathbf{x}) = \log \text{constant} + (n + a - 1) \log \lambda - \lambda(b + T) \quad (3.2.9) \]

Now \[ \frac{\partial}{\partial \lambda} \log P(\lambda \mid \mathbf{x}) = 0, \]

\[ \Rightarrow \quad \frac{(n + a - 1)}{\lambda} - (b + T) = 0 \quad \Rightarrow \quad \hat{\lambda} = \frac{(n + a - 1)}{(b + T)}. \]

Also, \[ \frac{\partial^2}{\partial \lambda^2} \log P(\lambda \mid \mathbf{x}) = -\frac{(n + a - 1)}{\lambda^2}. \]

\[ \therefore \quad I(\hat{\lambda}) = \frac{\partial^2}{\partial \lambda^2} \log P(\lambda \mid \mathbf{x}) = \frac{(b + T)^2}{(n + a - 1)} \]

\[ \Rightarrow \quad [I(\hat{\lambda})]^{-1} = \frac{(n + a - 1)}{(b + T)^2}. \]

\[ \therefore \quad p(\lambda \mid \mathbf{x}) \sim N\left(\frac{(n + a - 1)}{(b + T)}, \frac{(n + a - 1)}{(b + T)^2}\right). \quad (3.2.10) \]

As a special case, when \( a = 1 \), then the above Gamma prior reduces to exponential prior with density as:

\[ g(\lambda) \propto b e^{-b\lambda}; b > 0; \lambda > 0. \quad (3.2.11) \]

Under Exponential prior \( g(\lambda) \propto e^{-b\lambda}; b > 0; \lambda > 0 \), where \( b \) is hyper parameter. The posterior distribution for \( \hat{\lambda} \) is as

\[ P(\hat{\lambda} \mid \mathbf{x}) \propto \lambda^ne^{-\lambda(T+b)}, \quad \text{where} \quad T = \sum_{i=1}^{n} \mathbf{x}_i^{-2} \]

\[ \log P(\hat{\lambda} \mid \mathbf{x}) = \log \text{constant} + (n) \log \lambda - \lambda(b + T) \quad (3.2.13) \]
\[
\frac{\partial}{\partial \lambda} \log P(\lambda \mid x) = 0, \\
\frac{n}{b + T} = 0 \quad \Rightarrow \quad \hat{\lambda} = \frac{(n)}{(b + T)}.
\]

Also, \[
\frac{\partial^2}{\partial \lambda^2} \log P(\lambda \mid x) = \frac{-(n)}{\lambda^2}.
\]

\[
I(\hat{\lambda}) = -\frac{\partial^2}{\partial \lambda^2} \log P(\lambda \mid x) = \frac{(b + T)^2}{(n)}
\]

\[
[I(\hat{\lambda})]^{-1} = \frac{(n)}{(b + T)^2}.
\]

\[
p(\lambda \mid x) \sim N \left( \frac{(n)}{(b + T)}, \frac{(n)}{(b + T)^2} \right). \tag{3.2.14}
\]

### 3.2.2 Bayes Estimate of Inverse Rayleigh Distribution Using T-K Approximation:

Tierney and Kadane (1986) gave a method to evaluate \( E(h(\lambda) \mid x) \) as

\[
E(h(\lambda) \mid x) \equiv \frac{\hat{\phi}^* \exp \{-nh^*(\hat{\lambda}^*)\}}{\phi \exp \{-nh(\hat{\lambda})\}} \tag{3.2.15}
\]

Under Jeffrey’s prior \( g(\lambda) \propto \frac{1}{\lambda} \), the posterior distribution for \( \lambda \) is given in (3.2.2)

\[
-nh(\hat{\lambda}) = (n - 1) \log \lambda - \hat{\lambda} T \quad \text{and} \quad -nh'(\hat{\lambda}) = \frac{(n - 1)}{\lambda} - T
\]

\[
\Rightarrow \quad \hat{\lambda} = \frac{(n - 1)}{T}
\]

Also, \[-nh''(\hat{\lambda}) = -\frac{T^2}{(n - 1)}.\]

\[
\hat{\phi}^2 = \{[-nh''(\hat{\lambda})]^{-1}\} = \frac{(n - 1)}{T^2} \quad \text{or} \quad \hat{\phi} = \sqrt\frac{(n - 1)}{T},
\]

now, \[-nh^*(\hat{\lambda}^*) = -nh(\hat{\lambda}) + \ln h(\lambda) = (n) \ln(\lambda^*) - \hat{\lambda}^* T
\]

\[
-nh''(\hat{\lambda}^*) = \frac{(n)}{\hat{\lambda}^*} - T \quad \Rightarrow \quad \hat{\lambda}^* = \frac{(n)}{T}.
\]

Also, \[-nh''(\hat{\lambda}^*) = -\frac{T^2}{(n)} \quad \Rightarrow \quad \hat{\phi}^2 = \{[-nh''(\hat{\lambda})]^{-1}\} = \frac{(n)}{T^2} \quad \text{or} \quad \hat{\phi} = \sqrt\frac{(n)}{T}.\]
Thus, using (3.2.15) we have

\[
E(\lambda | x) = \frac{(n)^{1/2}}{(n-1)^{1/2}} \frac{\exp \{(n) \ln \hat{\lambda} - \hat{\lambda} T\}}{\exp \{(n-1) \ln \hat{\lambda} - \hat{\lambda} T\}} = \frac{(n)^{1/2}}{(n-1)^{1/2}} \frac{\hat{\lambda}^{(n)} e^{-\hat{\lambda} T + \hat{\lambda} T}}{\hat{\lambda}^{(n-1)}}
\]

\[
\Rightarrow E(\lambda | x) = \left(\frac{n-1}{T}\right) \left(\frac{n}{n-1}\right)^{n+1/2} e^{-1}. \quad (3.2.16)
\]

Note that the relative error (relative error to exact the posterior mean \(\left(\frac{n-1}{T}\right)\)) is

\[
\left(\frac{n}{n-1}\right)^{n+1/2} e^{-1}
\]

Similarly, \(E(\chi^2 | x) \equiv \hat{\phi}^* \exp \{-nh^*(\hat{\lambda}^*)\} \phi \exp \{-nh(\hat{\lambda})\}\)

Here, \(-nh^*(\hat{\lambda}^*) = \ln(\hat{\lambda}^2) - nh(\hat{\lambda}) = (n+1)\ln(\hat{\lambda}^* - \hat{\lambda}^* T)

\[
E(\chi^2 | x) = \left(\frac{n+1}{n-1}\right)^{n+1/2} \frac{(n+1)(n-1)}{T^2} e^{-2} \quad (3.2.17)
\]

\[
\therefore V(\lambda | x) = \left(\frac{n+1}{n-1}\right)^{n+1/2} \frac{(n+1)(n-1)}{T^2} e^{-2} - \left[\left(\frac{n-1}{T}\right) \left(\frac{n}{n-1}\right)^{n+1/2} e^{-1}\right]^2
\]

Under modified Jeffrey’s prior \(g(\lambda) \propto \left(\frac{1}{\sqrt{\lambda^3}}\right)\), the posterior distribution for \(\lambda\) is given in (3.2.5)

\[
\therefore -nh(\hat{\lambda}) = (n-3/2)\log \hat{\lambda} - \hat{\lambda} T \quad \therefore -nh' (\hat{\lambda}) = \left(\frac{n-3/2}{\hat{\lambda}}\right) - T;
\]

\[
\Rightarrow \hat{\lambda} = \left(\frac{n-3/2}{T}\right).
\]

Also, \(-nh^*(\hat{\lambda}) = -T^2 (n-3/2)\)

\[
\therefore \hat{\phi}^2 = \left[-nh'(\hat{\lambda})\right]^{-1} = \left(\frac{n-3/2}{T^2}\right) \text{ or } \hat{\phi} = \sqrt{\left(\frac{n-3/2}{T}\right)}.
\]

Now, \(-nh^*(\hat{\lambda}^*) = -nh(\hat{\lambda}) + nh(\hat{\lambda}) = (n-1/2)\ln(\hat{\lambda}^*) - \hat{\lambda}^* T

\[
-nh'^* (\hat{\lambda}^*) = \left(\frac{n-1/2}{\hat{\lambda}^*}\right) - T \quad \Rightarrow \hat{\lambda}^* = \left(\frac{n-1/2}{T}\right).
\]

\[
-nh^* (\hat{\lambda}^*) = -\left(\frac{T^2}{n-1/2}\right)
\]

\[45\]
\[ \hat{\phi}^{\gamma^2} = -[ -nh^* (\hat{\lambda}) ]^{-1} = \frac{(n-1/2)}{T^2} \] or \[ \hat{\phi} = \sqrt{(n-1/2)} \frac{T}{T} \]

Thus, using (3.2.15) we have

\[ E(\lambda | \mathbf{x}) = \frac{(n-1/2)^{1/2}}{(n-3/2)^{1/2}} \frac{\exp\left\{ (n-1/2)\ln \hat{\lambda} - \hat{\lambda} T \right\}}{\exp\left\{ (n-3/2)\ln \hat{\lambda} - \hat{\lambda} T \right\}} \]

\[ \Rightarrow \quad E(\lambda | \mathbf{x}) = \left( \frac{n-1/2}{n-3/2} \right)^{n+1/2} \left( \frac{n-1/2}{n-3/2} \right)^{n+1/2} \frac{e^{-1}}{T} \]

(3.2.18)

Note that the relative error is \[ \left( \frac{n-1/2}{n-3/2} \right)^{n+1/2} \frac{e^{-1}}{T} \]

Similarly, \[ E(\lambda^2 | \mathbf{x}) \equiv \frac{\hat{\phi}^* \exp\{-nh^* (\hat{\lambda}^{(n)})\}}{\hat{\phi} \exp\{-nh(\hat{\lambda})\}} \]

Here, \[ -nh^* (\hat{\lambda}) = \ln(\hat{\lambda}^2) - nh(\hat{\lambda}) = (n - 2c_1 + 2) \ln(\hat{\lambda}^2) - \hat{\lambda} T \]

\[ E(\lambda^2 | \mathbf{x}) = \left( \frac{n+1/2}{n-3/2} \right)^{n+1/2} \frac{\sqrt{(n+1/2)(n-3/2)}}{T^2} e^{-2} \]

(3.2.19)

\[ \therefore \quad \text{V}(\lambda | \mathbf{x}) = \left( \frac{n+1/2}{n-3/2} \right)^{n+1/2} \frac{\sqrt{(n+1/2)(n-3/2)}}{T^2} e^{-2} \left( \frac{\sqrt{(n-3/2)}}{\sqrt{(n-1/2)}} \left( \frac{n-1/2}{n-3/2} \right)^{n+1/2} \frac{e^{-1}}{T} \right)^2 \]

Under the Gamma prior \( g(\lambda) \propto \lambda^{a-1} e^{-b\lambda} ; a, b > 0 ; \lambda > 0 \), where \( a \) and \( b \) are the known hyper parameters, thus the posterior distribution for \( \hat{\lambda} \) is given in (3.2.8)

\[ -nh(\hat{\lambda}) = (n + a - 1) \log \lambda - \lambda (b + T) ; -nh^* (\hat{\lambda}) = \frac{(n + a - 1)}{\hat{\lambda}} \lambda - \lambda (b + T) \; \Rightarrow \hat{\lambda} = \frac{(n + a - 1)}{(b + T)} \]

Also, \[ -nh^* (\hat{\lambda}) = - \left( \frac{(b + T)^2}{(n + a - 1)} \right) \]

\[ \therefore \quad \hat{\phi}^{\gamma^2} = -[ -nh^* (\hat{\lambda}) ]^{-1} = \frac{(n + a - 1)}{(b + T)^2} \] or \[ \hat{\phi} = \sqrt{(n + a - 1)} \frac{(b + T)}{(b + T)} \]

now, \[ -nh' (\hat{\lambda}^*) = -nh(\lambda) + \ln h(\lambda) = (n + a) \ln(\hat{\lambda}^*) - \hat{\lambda}^* (b + T) \]

\[ -nh'^* (\hat{\lambda}^*) = \frac{(n + a)}{\hat{\lambda}^*} - (b + T) \; \Rightarrow \hat{\lambda}^* = \frac{(n + a)}{(b + T)} \]

\[ -nh'' (\hat{\lambda}^*) = - \left( \frac{(b + T)^2}{(n + a)} \right) \]

\[ \hat{\phi}^{\gamma^2} = -[ -nh'^* (\hat{\lambda}^*) ]^{-1} = \frac{(n + a)}{(b + T)^2} \] or \[ \hat{\phi} = \sqrt{(n + a)} \frac{(b + T)}{(b + T)} \]
Thus, using (3.2.15) we have
\[
E(\lambda | \chi) = \frac{(n + a)^{1/2}}{(n + a - 1)^{1/2}} \frac{\exp \{ (n + a) \ln \hat{\lambda} - \hat{\lambda} (b + T) \}}{\exp \{ (n + a - 1) \ln \hat{\lambda} - \hat{\lambda} (b + T) \}}. 
\]
\[
\Rightarrow E(\lambda | \chi) = \left( \frac{n + a}{n + a - 1} \right)^{n+a-1/2} e^{-1}. 
\]
(3.2.20)

Note that the relative error is
\[
\left( \frac{n + a}{n + a - 1} \right)^{n+a-1/2} e^{-1}. 
\]
Similarly, \[
E(\lambda^2 | \chi) \approx \frac{\hat{\phi}^* \exp \{-nh^*(\hat{\lambda}^*)\}}{\hat{\phi} \exp \{-nh(\hat{\lambda})\}}. 
\]
Here,
\[
-nh^*(\hat{\lambda}^*) = \ln(\hat{\lambda}^2) - nh(\hat{\lambda}) = (n + a + 1) \ln(\hat{\lambda}^*) - \hat{\lambda}^* (b + T) 
\]
\[
E(\lambda^2 | \chi) = \frac{(n + a + 1)(n + a - 1)}{(b + T)^2} \left( \frac{n + a + 1}{n + a - 1} \right) \left( \frac{n + a}{n + a - 1} \right)^{n+a+1/2} e^{-2} 
\]
(3.2.21)
\[
\Rightarrow V(\lambda | \chi) = \frac{(n + a + 1)(n + a - 1)}{(b + T)^2} \left( \frac{n + a + 1}{n + a - 1} \right) \left( \frac{n + a}{n + a - 1} \right)^{n+a+1/2} e^{-2} \cdot \left[ \frac{n + a}{b + T} \right]^2. 
\]

As a special case, when \( a = 1 \), then the above Gamma prior reduces to exponential prior with density as:
\[
g(\lambda) \propto be^{-b\lambda}, b > 0; \lambda > 0. 
\]
(3.2.22)

Under the Exponential prior \( g(\lambda) \propto e^{-b\lambda}, b > 0; \lambda > 0 \) where \( b \) is the known hyper parameter. The posterior distribution for \( \hat{\lambda} \) is given in (3.2.12)
\[
-nh(\hat{\lambda}) = (n) \log \hat{\lambda} - \hat{\lambda} (b + T) ; -nh'(\hat{\lambda}) = \frac{(n)}{\hat{\lambda}} - (b + T); 
\]
\[
\Rightarrow \hat{\lambda} = \frac{(n)}{(b + T)} 
\]
Also, \(-nh^*(\hat{\lambda}) = -\frac{(b + T)^2}{(n)} \)
\[
\Rightarrow \hat{\phi}^2 = -[nh''(\hat{\lambda})]^{-1} = \frac{(n)}{(b + T)^2} \text{ or } \hat{\phi} = \sqrt{(n)/(b + T)}, 
\]
now, \(-nh^*(\hat{\lambda}^*) = -nh(\hat{\lambda}) + \ln h(\hat{\lambda}) = (n + 1) \ln(\hat{\lambda}^*) - \hat{\lambda}^* (b + T) 
\]
\[
-nh''(\hat{\lambda}^*) = \frac{(n + 1)}{\hat{\lambda}^*} - (b + T) \Rightarrow \hat{\lambda}^* = \frac{(n + 1)}{(b + T)}. 
\]
\[-nh^*(\hat{\lambda}^*) = \frac{(b + T)^2}{(n + 1)}\]

\[\hat{\phi}^2 = -[nh^*(\hat{\lambda}^*)]^{-1} = \frac{(n + 1)}{(b + T)^2} \quad \text{or} \quad \hat{\phi} = \sqrt{\frac{(n + 1)}{(b + T)}}.\]

Thus using (3.2.15) we have

\[E(\lambda \mid x) = \frac{(n + 1)^{1/2}}{(n)^{1/2}} \exp \{(n + 1)\ln \hat{\lambda} - \hat{\lambda}^* (b + T)\}.\]

\[\Rightarrow E(\lambda \mid x) = \left(\frac{n + 1}{b + T}\right) \left(\frac{n + 1}{n}\right)^{n+1/2} e^{-1}.\]  \hspace{1cm} (3.2.23)

Note that the relative is \(\left(\frac{n + 1}{n}\right)^{n+1/2} e^{-1}\). Similarly,

\[E(\lambda^2 \mid x) \geq \frac{\hat{\phi}^* \exp \{-nh^*(\hat{\lambda}^*)\}}{\hat{\phi} \exp \{-nh(\hat{\lambda})\}}\]

Here, \(-nh^*(\hat{\lambda}^*) = \ln(\lambda^2) - nh(\hat{\lambda}) = (n + 2)\ln(\lambda^*) - \lambda^* (b + T)\)

\[E(\lambda^2 \mid x) = \left(\frac{n + 2}{b + T}\right) \left(\frac{n + 2}{n}\right)^{n+1/2} e^{-2}\]  \hspace{1cm} (3.2.24)

\[\therefore V(\lambda \mid x) = \left(\frac{n + 2}{b + T}\right) \left(\frac{n + 2}{n}\right)^{n+1/2} e^{-2} - \left[\left(\frac{n + 1}{b + T}\right) \left(\frac{n + 1}{n}\right)^{n+1/2} e^{-1}\right]^2.\]

### 3.3 Weighted Inverse Rayleigh Distribution

The concept of weighted distributions was provided by Fisher (1934) and Rao (1965). Fisher (1934) studied how the methods of ascertainment can influence the form of the distribution of recorded observations and after that Rao (1965) recognized a unifying method that can be used for several sampling situations and can be displayed by means of the weighted distributions. Many authors have employed the concept of weighted distribution for different purposes. Kishore and Roy (2011) studied the Length-biased weighted Weibull distribution. Mudasir and Ahmad (2015) proposed the Length-biased weighted Nakagami Distribution. Afaq et al. (2016) presented the Length-biased Weighted Lomax Distribution. Recently, Fatima and Ahmad (2017) introduced the weighted Inverse Rayleigh distribution.

In this section, we propose a new distribution which is a Weighted Inverse Rayleigh distribution. Firstly, we will give a common definition of the Weighted Inverse Rayleigh (WIR) distribution which will consequently expose its pdf.
**Definition 1**: If $X$ follows a lifetime distribution with pdf $g(x)$ and expected value, $E_g(x^k) < \infty$, then the pdf of Weighted distribution of $X$ can be defined as:

$$f(x) = \frac{x^k g(x)}{E_g(x^k)}, k \geq 0, x > 0.$$  \hspace{1cm} (3.3.1)

**Theorem 3.3.1**: Let $X$ be a random variable of an IR distribution with pdf $g(x)$. Then the pdf of WIR is given by:

$$f(x) = \frac{2\lambda^{1-k/2}}{\Gamma(1-k/2)} x^{k-3} e^{-\lambda x^2}; x > 0, \lambda > 0, 0 \leq k < 2.$$  \hspace{1cm} (3.3.2)

Where $\lambda$ and $k$ are scale and weight parameters respectively.

**Proof**: The $k^{th}$ moment of the IR distribution is given as the following:

$$E_g(x^k) = \int_0^\infty x^k \frac{2\lambda}{x^3} e^{-\lambda x^2} dx.$$  

Making the substitution $y = \frac{1}{x^2}$, we obtain

$$E_g(x^k) = \int_0^\infty y^{(1-k)/2-1} e^{-\lambda y} dy.$$  

Thus

$$E_g(x^k) = \lambda^{k/2} \Gamma(1-k/2).$$  \hspace{1cm} (3.3.3)

Then, the expected value of $X$ can be written as:

$$E_g(x) = \lambda^{1/2} \Gamma(1/2)$$  \hspace{1cm} (3.3.4)

By definition 1, substitute (3.1.1) and (3.3.3) into (3.1.1), then the pdf for the WIR distribution can be obtained by:

$$f(x) = \frac{x^k}{\lambda^{k/2} \Gamma(1-k/2) x^3} e^{-\lambda x^2}.$$  

We observe from Figure 3.1 that the density function of WIR distribution is positively skewed and that the curve decreases as the value of $\lambda$ increases. So, the shape of the proposed WIR distribution could be decreasing. Also, we observed that the shape of the proposed WIR distribution could be unimodal.
Figure 3.1: The probability density function of the WIR distribution for selected values of $\lambda$ and $k$.

**Theorem 3.3.2:** Let $X$ be a random variable of the WIR distribution with parameter $\lambda \& k$. The distribution function of the WIR distribution is written as:

$$F(x) = \frac{\Gamma\left(1 - \frac{k}{2}, \frac{\lambda}{x^2}\right)}{\Gamma\left(1 - \frac{k}{2}\right)},$$

(3.3.5)

where $\Gamma(s, x) = \int_x^\infty t^{s-1} e^{-t} dt$ is an upper incomplete gamma function.

**Proof:** Generally, the distribution function of lifetime distribution is defined as:

$$F(x) = \int_0^x f(x) \, dx$$

(3.3.6)

Substituting (3.3.2) into (3.3.6), we obtain:

$$F(x) = \int_0^x \frac{2\lambda^{(1-k/2)}}{\Gamma(1 - k/2)} x^{k-3} e^{-\frac{\lambda}{x^2}} \, dx$$

By setting $y = \frac{\lambda}{x^2} \Rightarrow -\frac{2\lambda}{x^2} \, dx = dy; \frac{\lambda}{x^2} < y < \infty$ the above integration becomes:

$$F(x) = \frac{1}{\Gamma(1 - k/2)} \int_{\frac{\lambda}{x^2}}^\infty \left(\frac{1}{2}\right)^{1-k/2} e^{-y} \, dy$$

$$F(x) = \frac{\Gamma\left(1 - \frac{k}{2}, \frac{\lambda}{x^2}\right)}{\Gamma\left(1 - \frac{k}{2}\right)}.$$
Theorem 3.3.3: Let $X$ be a random variable of the WIR distribution with parameters $\lambda$ and $k$. The survival function of the WIR distribution can be written as:

$$S(x) = \frac{\gamma \left( 1 - \frac{k}{2}, \frac{\lambda}{x^2} \right)}{\Gamma \left( 1 - \frac{k}{2} \right)},$$

where $\gamma(s, x) = \int_0^x t^{s-1} e^{-t} \, dt$ is a lower incomplete gamma function.

Proof: By definition, the survival function of the random variable $X$ is given by:

$$S(x) = 1 - F(x).$$

Using (3.3.5), the survival function of the WIR distribution can be expressed by:

$$S(x) = 1 - \frac{\Gamma \left( 1 - \frac{k}{2}, \frac{\lambda}{x^2} \right)}{\Gamma \left( 1 - \frac{k}{2} \right)} = \frac{\Gamma \left( 1 - \frac{k}{2}, \frac{\lambda}{x^2} \right) - \Gamma \left( 1 - \frac{k}{2}, \frac{\lambda}{x^2} \right)}{\Gamma \left( 1 - \frac{k}{2} \right)} = \frac{\gamma \left( 1 - \frac{k}{2}, \frac{\lambda}{x^2} \right)}{\Gamma \left( 1 - \frac{k}{2} \right)}.$$

Figure 3.2: The distribution function of the WIR distribution for selected values of $\lambda$ and $k$. The distribution curves show the increasing rate.
Chapter 3. Classical and Weighted Inverse Rayleigh Distribution

Theorem 3.3.4: Let $X$ be a random variable of the WIR distribution with Parameter $\lambda$ & $k$. The hazard rate of the WIR distribution takes the form:

$$h(x) = \frac{2^{\lambda-1/2} x^{k-3} e^{-\lambda x^2}}{\Gamma(1 - k/2)}.$$  

(3.3.8)

Proof: Let $X$ is a continuous random variable with pdf and survival function, $f(x)$ and $S(x)$, respectively, then the hazard rate is defined by:

$$h(x) = \frac{f(x)}{S(x)}. \quad (3.3.9)$$

Substituting (3.3.2) and (3.3.7) into (3.3.9), we obtain:

$$h(x) = \frac{2^{\lambda-1/2} x^{k-3} e^{-\lambda x^2}}{\Gamma(1 - k/2)}.$$  

Theorem 3.3.5: Let $X$ be a random variable of the WIR distribution with Parameter $\lambda$ & $k$. The reverse hazard rate of the WIR distribution takes the form:

$$\phi(x) = \frac{2^{\lambda-1/2} x^{k-3} e^{-\lambda x^2}}{\Gamma(1 - k/2)}.$$  

(3.3.10)

Proof: Let $X$ be a continuous random variable with pdf and cdf, $f(x)$ and $F(x)$, respectively, then the reverse hazard rate is defined by:

$$\phi(x) = \frac{f(x)}{F(x)}. \quad (3.3.11)$$

Substituting (3.3.2) and (3.3.5) into (3.3.11), we obtain:

$$\phi(x) = \frac{2^{\lambda-1/2} x^{k-3} e^{-\lambda x^2}}{\Gamma(1 - k/2)}.$$  

3.4 Some Special Cases of Weighted Inverse Rayleigh (WIR) Distribution

This section presents some special cases that deduced from equation (3.3.2) are
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Case 1: When $k = 0$, then WIRD (3.3.2) reduces to IRD with pdf as:

$$f(x) = \frac{2\lambda}{x^3} e^{-\frac{\lambda}{x}}.$$  \hspace{1cm} (3.4.1)

Case 2: When $k = 1$, then WIRD (3.3.2) reduces to LBIRD with pdf as:

$$f(x) = \frac{2\lambda^{1/2}}{\Gamma(1/2)} x^{-2} e^{-\frac{\lambda}{x}}.$$  \hspace{1cm} (3.4.2)

Case 3: If a random variable is such that $Y = 1/X$ in Equation (3.3.2) reduces to give the WRD with pdf as:

$$f(x) = \frac{2\lambda^{(k+1)}x^{k+1} e^{-\lambda x^2}}{\Gamma(k/2+1)}.$$  \hspace{1cm} (3.4.3)

Case 4: If a random variable is such that $Y = 1/X$ and $k = 1$ in Equation (3.3.2) reduces to give the LBRD with pdf as:

$$f(x) = \frac{4\lambda^{(3/2)} x^2 e^{-\lambda x^2}}{\Gamma(1/2)}.$$  \hspace{1cm} (3.4.4)

Case 5: If a random variable is such that $Y = 1/X$ and $k = 0$ in Equation (3.3.2) reduces to give the RD with pdf as:

$$f(x) = 2\lambda x e^{-\lambda x^2}.$$  \hspace{1cm} (3.4.5)

3.5 Statistical Properties of the Weighted Inverse Rayleigh (WIR) Distribution

This section provides some basic statistical properties of the WIR Distribution.

3.5.1 The $r^{\text{th}}$ Moment of the WIR Distribution

The following theorem of this section gives the $r^{\text{th}}$ moment of WIR distribution.

**Theorem 3.5.1:** If $X \sim \text{WIR}(\lambda)$, then $r^{\text{th}}$ moment of a continuous random variable $X$ is given as follow:

$$\mu'_r = E(X^r) = \frac{\lambda^{r/2}}{\Gamma(1-r/2)} \Gamma(1-(r+k)/2), r = 1,2$$

**Proof:** Let $X$ is an absolutely continuous non-negative random variable with pdf $f(x)$, then $r^{\text{th}}$ moment of $X$ can be obtained by:

$$\mu'_r = E(X^r) = \int_0^\infty x^r f(x)dx.$$
From the pdf of the WIR distribution in (3.3.2), then show that $E(X')$ can be written as:

$$E(X') = \int_0^{\infty} x^{k} e^{-\frac{x}{k}} \frac{2^{-\frac{k}{2}}}{\Gamma(1-k/2)} x^{-3} e^{-x} dx$$

Making the substitution $y = \frac{\lambda}{x^2} \Rightarrow \frac{-2\lambda}{x^3} dx = dy$, so that $x = \frac{\lambda}{y^{1/2}}$, we obtain

$$E(X') = \frac{\lambda^{\frac{1}{2}}}{\Gamma(1-k/2)} \int_0^{\infty} y^{(1-(r+k)/2)-1} e^{-y} dy.$$  

After some calculations, 

$$\mu' = E(X') = \frac{\lambda^{\frac{1}{2}}}{\Gamma(1-k/2)} \Gamma(1-(r+k)/2). \quad (3.5.1)$$

Substitute $r = 1, 2$, in (4.1) we get mean and variance of WIRD

$$E(X) = \frac{\lambda^{1/2}}{\Gamma(1-k/2)} \Gamma(1-(1+k)/2). \quad (3.5.2)$$

$$E(X^2) = \frac{\lambda}{\Gamma(1-k/2)} \Gamma(1-(2+k)/2) \quad (3.5.3)$$

$$V(X) = \mu_2 = \frac{\lambda}{\Gamma(1-k/2)} \Gamma(1-(2+k)/2) - \left\{ \frac{\lambda^{1/2}}{\Gamma(1-k/2)} \Gamma(1-(1+k)/2) \right\}^2. \quad (3.5.4)$$

### 3.5.2 Harmonic mean of WIR distribution

The harmonic mean (H) is given as:

$$H = \frac{1}{\frac{1}{E(X)}} = \frac{1}{\frac{1}{\int_0^{\infty} \frac{2^{-\frac{k}{2}}}{\Gamma(1-k/2)} x^{-3} e^{-x} dx}}$$

$$= \frac{1}{\Gamma(1-k/2) \lambda^{1/2}} \left[ \Gamma(1-(k+1)/2) \right].$$

$$\Rightarrow \quad H = \frac{\Gamma(1-k/2) \lambda^{1/2}}{\Gamma(1-(k+1)/2)}. \quad (3.5.5)$$

### 3.5.3 Mode

Consider the density of the WIR distribution given in (3.3.2), we take the logarithm of (3.3.2) as follows:

$$\log f(x) = \log(2) + \left( 1 - \frac{k}{2} \right) \log(\lambda) + (k - 3) \log(x) - \frac{\lambda}{x^2} - \log \Gamma(1-k/2). \quad (3.5.6)$$
Now, \( \frac{\partial}{\partial x} \log f(x) = \frac{(k-3)}{x} + \frac{2\lambda}{x^3}. \) (3.5.7)

Also, set equation (3.5.7) equal to 0 and solve for \( x \), to get

\[
x_0 = \sqrt[3]{\frac{2\lambda}{(3-k)}}.
\] (3.5.8)

### 3.5.4 Moment generating function

**Theorem 3.5.2:** Let \( X \) have a WIR distribution. Then the moment generating function of \( X \) denoted by \( M_X(t) \) is given by:

\[
M_X(t) = \sum_{r=0}^{\infty} t^r \frac{\lambda^{r/2}}{r! \Gamma(1-k/2)} \Gamma(1-(r+k)/2).
\] (3.5.9)

**Proof:** By definition

\[
M_X(t) = E(e^{itX}) = \int_0^{\infty} e^{itx} f(x)dx
\]

Using Taylor series

\[
M_X(t) = \int_0^{\infty} \left(1 + tx + \frac{(tx)^2}{2!} + \cdots \right) f(x)dx
\]

\[
\Rightarrow M_X(t) = \sum_{r=0}^{\infty} t^r \int_0^{\infty} x^r f(x)dx
\]

\[
\Rightarrow M_X(t) = \sum_{r=0}^{\infty} \frac{t^r}{r!} E(X^r)
\]

\[
\Rightarrow M_X(t) = \sum_{r=0}^{\infty} \frac{t^r}{r!} \frac{\lambda^{r/2}}{\Gamma(1-k/2)} \Gamma(1-(r+k)/2).
\]

### 3.5.5 Characteristic function

**Theorem 3.5.3:** Let \( X \) have a WIR distribution. Then the characteristic function of \( X \) denoted by \( \phi_X(t) \) is given by:

\[
\phi_X(t) = \sum_{r=0}^{\infty} \frac{(it)^r}{r!} \frac{\lambda^{r/2}}{\Gamma(1-k/2)} \Gamma(1-(r+k)/2). \] (3.5.10)

**Proof:** By definition

\[
\phi_X(t) = E(e^{itx}) = \int_0^{\infty} e^{itx} f(x)dx
\]
Using Taylor series
\[
\phi_X(t) = \int_0^\infty \left( 1 + itx + \frac{(itx)^2}{2!} + \cdots \right) f(x)dx
\]
\[
\Rightarrow \quad \phi_X(t) = \sum_{r=0}^\infty \frac{(it)^r}{r!} \int_0^\infty x^r f(x)dx
\]
\[
\Rightarrow \quad \phi_X(t) = \sum_{r=0}^\infty \frac{(it)^r}{r!} E(X^r)
\]
\[
\Rightarrow \quad \phi_X(t) = \sum_{r=0}^\infty \frac{(it)^r}{r!} \frac{\lambda^{1/2}}{\Gamma(1-k/2)} \Gamma(1-(r+k)/2).
\]

### 3.5.6 Shannon’s Entropy of Weighted Inverse Rayleigh Distribution

**Theorem 3.5.4:** Let \( x = (x_1, x_2, \ldots, x_n) \) be \( n \) positive i.i.d random samples drawn from a population having WIR density function as
\[
f(x) = \frac{2\lambda^{1-k/2}}{\Gamma(1-k/2)} x^{k-3} e^{-x^2/\lambda}
\]
Then Shannon’s entropy of weighted Inverse Rayleigh distribution is
\[
H(x) = \log \left( \frac{\Gamma(1-k/2)}{2\lambda^{1-k/2}} \right) - \frac{(k-3)}{2} \left[ \log(\lambda) - \psi(1-k/2) \right] + (1-k/2).
\]

**Proof:** Shannon’s entropy is defined as
\[
H(x) = -E[\log f(x)]
\]
\[
= -E \left[ \log \left( \frac{2\lambda^{1-k/2}}{\Gamma(1-k/2)} x^{k-3} e^{-x^2/\lambda} \right) \right]
\]
\[
\Rightarrow \quad H(x) = -\log \left( \frac{2\lambda^{1-k/2}}{\Gamma(1-k/2)} \right) - \frac{(k-3)}{2} E(\log(x)) + \lambda E \left( \frac{1}{x^2} \right). \quad (3.5.11)
\]

Now, \( E(\log(x)) = \int_0^\infty \log(x) f(x) dx. \)
\[
\Rightarrow \quad E(\log(x)) = \int_0^\infty \log(x) \frac{2\lambda^{1-k/2}}{\Gamma(1-k/2)} x^{k-3} e^{-x^2/\lambda} dx.
\]

By setting \( y = \frac{x^2}{\lambda} \Rightarrow dy = \frac{2\lambda}{x^2} dx, x = \sqrt{\frac{\lambda}{y}}, \) as \( x \to 0, y \to \infty, x \to \infty, y \to 0 \)
\[
E(\log(x)) = \frac{\lambda^{1/2}}{\Gamma(1-k/2)} \int_0^\infty \log \left( \frac{\lambda^{1/2}}{y^{1/2}} \right) \left( \frac{\lambda}{y} \right)^{k/2} e^{-y} dy
\]
\[
= \frac{1}{\Gamma(1-k/2)} \left\{ \frac{1}{2} \log \lambda \int_0^\infty y^{1-k/2} e^{-y} dy \right\} - \frac{1}{2} \left\{ \int_0^\infty \log y y^{1-k/2} e^{-y} dy \right\}
\]
\[ E(\log(x)) = \frac{1}{2} \left[ \log(\lambda) - \Psi\left(1 - \frac{k}{2}\right) \right] \] (3.5.12)

Also,
\[ E\left(\frac{1}{x^2}\right) = \int_0^\infty \frac{1}{x^2} \frac{2\lambda^{k-1}x^{k-3}e^{-\lambda x}}{\Gamma(1-k/2)} \, dx. \]

By setting \( y = \frac{x}{\lambda} \Rightarrow dy = \frac{2\lambda}{x^2} dx, x = \sqrt{\frac{x}{y}}, as \ x \to 0, y \to \infty, x \to \infty, y \to 0, \) we obtain
\[ E\left(\frac{1}{x^2}\right) = \frac{1}{\lambda^2 \Gamma(1-k/2)} \int_0^\infty y^{(2-k/2)-1}e^{-y} \, dy. \]

After solving the above expression, we get
\[ E\left(\frac{1}{x^2}\right) = \frac{(1-k/2)}{\lambda}. \] (3.5.13)

Substitute the values of equation (3.5.12) and (3.5.13) in equation (3.5.11), we get
\[ H(x) = \log\left(\frac{\Gamma(1-k/2)}{2\lambda^{k-1}x^{k-3}e^{-\lambda x}}\right) - \frac{(k-3)}{2} \left[ \log(\lambda) - \Psi(1-k/2) \right] + (1-k/2). \] (3.5.14)

The above relation (3.5.14) indicates the Shannon’s entropy of WIR distribution.

### 3.6 Estimation of Parameters by using MLE Method

Let \( x = (x_1, x_2, ..., x_n) \) be a random sample having probability density function (3.3.2), and then the likelihood function is given by
\[ L(x | \lambda) = \frac{2^n \lambda^{n(1-k/2)}}{\Gamma(1-k/2)^n} \prod_{i=1}^n x_i^{k-3} e^{-\lambda x_i} \] (3.6.1)

\[ \ln L(x | \lambda) = n \ln 2 + n(1-k/2) \ln \lambda - n \log \Gamma(1-k/2) + (k-3) \sum_{i=1}^n \ln x_i - \sum_{i=1}^n \left( \frac{\lambda}{x_i^2} \right). \] (3.6.2)

\[ \therefore \quad \frac{\partial}{\partial \lambda} \ln L(x | \lambda) = 0 \quad \Rightarrow \quad \frac{n(1-k/2)}{\lambda} - \sum_{i=1}^n \lambda x_i^{-2} = 0, \]

\[ \Rightarrow \quad \hat{\lambda} = \frac{n(1-k/2)}{T} \quad \text{where} \quad T = \sum_{i=1}^n x_i^{-2}. \] (3.6.3)

### 3.7 Bayesian Method of Estimation

#### 3.7.1 Posterior Distribution of WIR Distribution under Jeffery’s prior (JP)

The Jeffery’s prior is defined as \( g(\lambda) \propto \frac{1}{\lambda}. \) (3.7.1)
Combining the prior (3.7.1) and the likelihood function (3.6.1), then the posterior
distribution of $\lambda$ is defined by

$$p(\lambda \mid x) \propto g(\lambda) L(x \mid \lambda)$$

$$\Rightarrow p(\lambda \mid x) = \rho \lambda^{(n-1)/2} e^{-\sum_{i=1}^{n} x_i^{-2}}.$$ 

The normalizing constant $\rho$ is determined by the relation

$$\int_{0}^{\infty} p(\lambda \mid x) d\lambda = 1,$$

$$\Rightarrow \rho \lambda^{(n-1)/2} e^{-\lambda T} = 1, \text{ where } T = \sum_{i=1}^{n} x_i^{-2}.$$

Thus, the value of $\rho$ is given by

$$\rho = \frac{T^{n(1-k/2)}}{\Gamma(n(1-k/2))}.$$ 

Hence the posterior distribution of $\lambda$ is given as

$$p(\lambda \mid x) = \frac{T^{a_1}}{\Gamma a_1} \lambda^{a_1-1} e^{-\lambda T}.$$ 

So, the posterior distribution of $(\lambda \mid x) \sim G(a_1, T), \text{ where } a_1 = n\left(1 - \frac{k}{2}\right) \text{ and } T = \sum_{i=1}^{n} x_i^{-2}.$

### 3.7.2 Bayes’ estimators and risk functions under Jeffery’s prior using SELF, ABLF and PLF

Under SELF the risk function is given by

$$R(\hat{\lambda}, \lambda) = \int_{0}^{\infty} c(x - \hat{\lambda})^2 \frac{T^{a_1}}{\Gamma a_1} \lambda^{a_1-1} e^{-\lambda T} d\lambda$$

$$= c \left[ \hat{\lambda}^2 + \frac{\alpha_1(a_1 + 1)}{T^2} - \frac{2\lambda\alpha_1}{T} \right].$$

$$\therefore \frac{\partial}{\partial \lambda} R(\hat{\lambda}, \lambda) = 0. \Rightarrow \hat{\lambda}_{\text{SELF}} = \frac{\alpha_1}{T}.$$ 

Under ABLF the risk function is given by

$$R(\hat{\lambda}, \lambda) = \int_{0}^{\infty} \hat{\lambda}^{\alpha_1} (\hat{\lambda} - \lambda)^2 \frac{T^{a_1}}{\Gamma a_1} \lambda^{a_1-1} e^{-\lambda T} d\lambda.$$ 

$$= \frac{\hat{\lambda} \Gamma(\alpha_1 + c_2)}{\Gamma(a_1)T^{c_2}} + \frac{\Gamma(\alpha_1 + c_2 + 2)}{\Gamma(a_1)T^{c_2+2}} - \frac{2\hat{\lambda}\Gamma(\alpha_1 + c_2 + 1)}{\Gamma(a_1)T^{c_2+1}}.$$
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\[ \frac{\partial}{\partial \lambda} R(\hat{\lambda}, \lambda) = 0. \quad \Rightarrow \quad \hat{\lambda}_{\text{WIR}} = \frac{(\alpha_1 + c_2)}{T}. \quad (3.7.8) \]

Under PLF the risk function is given by

\[ R(\hat{\lambda}, \lambda) = \int_0^{\infty} (\hat{\lambda} - \lambda)^2 \frac{T^{\alpha_1} \lambda^{\alpha_1-1} e^{-\lambda T}}{\Gamma(\alpha_1)} d\lambda. \quad (3.7.9) \]

\[ = \hat{\lambda} + \frac{\alpha_1(\alpha_1 + 1)}{\lambda T^2} - \frac{2\alpha_1}{T}. \quad (3.7.10) \]

\[ \therefore \quad \frac{\partial}{\partial \lambda} R(\hat{\lambda}, \lambda) = 0. \quad \Rightarrow \quad \hat{\lambda}_{\text{PLF}} = \frac{\sqrt{\alpha_1(\alpha_1 + 1)}}{T}. \quad (3.7.11) \]

**Remark 3.1:**

- Replacing \( c_2 = 0 \) in (3.7.8) we get the same Bayes estimator as obtained in (3.7.5) corresponding to the SELF.

### 3.7.3 Posterior Distribution of WIR Distribution under Extension of Jeffery’s prior (Ext.JP)

The Extension of Jeffery’s prior is defined as \( g(\lambda) \propto \frac{1}{\lambda^{2c_1}}, c_1 \in R^+ \). (3.7.12)

The posterior distribution is obtained in a similar way as in case of Jeffery’s prior and is given by

\[ p(\lambda \mid x) = \rho \lambda^{n(1-k/2)-2c_1} e^{-\sum_{i=1}^n x_i^{-2}}, \]

\[ \Rightarrow \quad \rho = \frac{T^{n(1-k/2)-2c_1+1}}{\Gamma(n(1-k/2) - 2c_1 + 1)}, \text{where } T = \lambda \sum_{i=1}^n x_i^{-2}. \]

Thus, the posterior distribution of \( \lambda \) is given as

\[ p(\lambda \mid x) = \frac{T^{\alpha_2}}{\Gamma(\alpha_2)} \lambda^{\alpha_2-1} e^{-\lambda T}. \quad (3.7.13) \]

So, the posterior distribution of \( \lambda \mid x \sim G(\alpha_2, T) \), \( \alpha_2 = n \left(1 - \frac{k}{2}\right) - 2c_1 \) and \( T = \sum_{i=1}^n x_i^{-2} \).

### 3.7.4 Baye’s estimators and risk functions under Extension of Jeffery’s prior using SELF, ABLF and PLF

Under SELF the risk function is given by

\[ R(\hat{\lambda}, \lambda) = \int_0^{\infty} c(\hat{\lambda} - \lambda)^2 \frac{T^{\alpha_2}}{\Gamma(\alpha_2)} \lambda^{\alpha_2-1} e^{-\lambda T} d\lambda. \quad (3.7.14) \]
\[
= c \left[ \hat{\lambda}^2 + \frac{(\alpha_2 + 2)(\alpha_2 + 1)}{T^2} - \frac{2\hat{\lambda}(\alpha_2 + 1)}{T} \right].
\]

(3.7.15)

\[ \therefore \quad \frac{\partial}{\partial \lambda} R(\hat{\lambda}, \lambda) = 0. \quad \Rightarrow \quad \hat{\lambda}_{\text{SELF}} = \frac{(\alpha_2 + 1)}{T}. \quad \text{(3.7.16)} \]

Under ABLF the risk function is given by

\[
R(\hat{\lambda}, \lambda) = \int_0^\infty \left( \hat{\lambda} - \lambda \right)^2 \frac{\lambda^{\alpha_2} e^{-\lambda T}}{\Gamma(\alpha_2 + 1)} d\lambda.
\]

(3.7.17)

\[
= \hat{\lambda} \left[ \Gamma(\alpha_2 + c_2 + 1) \right] + \frac{\Gamma(\alpha_2 + c_2 + 3)}{\Gamma(\alpha_2 + 1) T^{c_2 + 2}} - \frac{2\hat{\lambda}}{\Gamma(\alpha_2 + 1) T^{c_2 + 1}}.
\]

(3.7.18)

\[ \therefore \quad \frac{\partial}{\partial \lambda} R(\hat{\lambda}, \lambda) = 0. \quad \Rightarrow \quad \hat{\lambda}_{\text{ABLF}} = \frac{(\alpha_2 + c_2 + 1)}{T}. \quad \text{(3.7.19)} \]

Under PLF the risk function is given by

\[
R(\hat{\lambda}, \lambda) = \int_0^\infty \frac{(\hat{\lambda} - \lambda)^2}{\lambda} \frac{T^{\alpha_2}}{\Gamma(\alpha_2)} \lambda^{\alpha_2 - 1} e^{-\lambda T} d\lambda.
\]

\[ = \hat{\lambda} + \frac{(\alpha_2 + 2)(\alpha_2 + 1) - 2(\alpha_2 + 1)}{\lambda T^2}.
\]

(3.7.21)

\[ \therefore \quad \frac{\partial}{\partial \lambda} R(\hat{\lambda}, \lambda) = 0. \quad \Rightarrow \quad \hat{\lambda}_{\text{PLF}} = \frac{\sqrt{(\alpha_2 + 2)(\alpha_2 + 1)}}{T}. \quad \text{(3.7.22)} \]

**Remark 3.2:**

- Replacing \( c_2 = 1/2 \) in (3.7.15) we get the same Bayes estimators as obtained in (3.7.5) corresponding to the Jeffrey’s prior and replace \( c_1 = 3/2 \) we get the Hartigan’s prior.

- Replacing \( c_1 = 1/2 \) in (3.7.19) we get the same Bayes estimators as obtained in (3.7.8) under ABLF using Jeffrey’s prior and replace \( c_1 = 3/2 \) we get the Hartigan’s prior.

- Replacing \( c_1 = 1/2 \) & \( c_2 = 0 \) in (3.7.19) we get the same Bayes estimators as obtained in (3.7.5) corresponding to the Jeffrey’s prior and replace \( c_1 = 3/2 \) we get the Hartigan’s prior.

- Replacing \( c_2 = 0 \) in (3.7.19) we get the same Bayes estimator as obtained in (3.7.8) under SELF using extension of Jeffrey’s prior.
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- Replacing $c_1 = 1/2$ in (3.7.22) we get the same Bayes estimators as obtained in (3.7.11) under PLF using Jeffery’s prior and replace $c_1 = 3/2$ we get the Hartigan’s prior.

3.7.5 Posterior Distribution of WIR Distribution under Pareto1 prior

The Pareto1 prior is defined as $g(\lambda) = bc^\lambda e^{-c^{\lambda b+1}}$, $c, b > 0; \lambda > 0$, where $b$ and $c$ are the known hyper parameters.

Hence the posterior distribution of $\lambda$ using the Pareto1 prior distribution is given by

$$p(\lambda \mid x) = \frac{T^{\alpha_3}}{\Gamma(\alpha_3)} \lambda^{\alpha_3 - 1} e^{-\lambda T}.$$  \hspace{1cm} (3.7.23)

So, the posterior distribution of $(\lambda \mid x) \sim G(\alpha_3, T)$, where $\alpha_3 = n \left(1 - \frac{k}{2}\right) - b & T = \sum_{i=1}^{n} x_i^{-2}$.

3.7.6 Baye’s estimators and risk functions under Pareto1 prior using SELF, ABLF and PLF

Under SELF the risk function is given by

$$R(\hat{\lambda}, \lambda) = \int_0^{\infty} c(\hat{\lambda} - \lambda)^2 \frac{T^{\alpha_3}}{\Gamma(\alpha_3)} \lambda^{\alpha_3 - 1} e^{-\lambda T} d\lambda$$  \hspace{1cm} (3.7.24)

$$= c \left[ \hat{\lambda}^2 + \frac{(\alpha_3 + 1)\alpha_3}{T^2} - \frac{2\lambda\alpha_3}{T} \right].$$  \hspace{1cm} (3.7.25)

Hence, $\frac{\partial^2}{\partial \hat{\lambda}^2} R(\hat{\lambda}, \lambda) = 0. \Rightarrow \hat{\lambda}_{\text{SEL}} = \frac{\alpha_3}{T}.$  \hspace{1cm} (3.7.26)

Under ABLF the risk function is given by

$$R(\hat{\lambda}, \lambda) = \int_0^{\infty} \lambda^2 (\hat{\lambda} - \lambda)^2 \frac{T^{\alpha_3}}{\Gamma(\alpha_3)} \lambda^{\alpha_3 - 1} e^{-\lambda T} d\lambda$$  \hspace{1cm} (3.7.27)

$$= \hat{\lambda}^2 \Gamma(\alpha_3 + c_2) + \frac{\Gamma(\alpha_3 + c_2 + 2)}{\Gamma(\alpha_3) \Gamma(c_2 + 1)} - \frac{2\lambda\Gamma(\alpha_3 + c_2 + 1)}{\Gamma(\alpha_3) T^{c_2+1}}.$$  \hspace{1cm} (3.7.28)

Hence, $\frac{\partial^2}{\partial \hat{\lambda}^2} R(\hat{\lambda}, \lambda) = 0. \Rightarrow \hat{\lambda}_{\text{ABLF}} = \frac{(\alpha_3 + c_2)}{T}.$  \hspace{1cm} (3.7.29)

Under PLF the risk function is given by

$$R(\hat{\lambda}, \lambda) = \int_0^{\infty} \left(\frac{\hat{\lambda} - \lambda}{\hat{\lambda}}\right)^2 \frac{T^{\alpha_3}}{\Gamma(\alpha_3)} \lambda^{\alpha_3 - 1} e^{-\lambda T} d\lambda$$  \hspace{1cm} (3.7.30)

$$= \hat{\lambda} + \frac{(\alpha_3 + 1)(\alpha_3)}{\hat{\lambda} T^2} - \frac{2\alpha_3}{T}.$$  \hspace{1cm} (3.7.31)
\[ \frac{d}{d\lambda} R(\tilde{\lambda}, \lambda) = 0. \quad \Rightarrow \quad \hat{\lambda}_{PLF} = \frac{\sqrt{\alpha_3(\alpha_3 + 1)}}{T}. \] (3.7.32)

**Remark 3.3:**

- Replacing \( c_2 = 0 \) in (3.7.29) we get the same Bayes estimator as obtained in (3.7.26) under SELF using Pareto1 prior.

### 3.7.7 Posterior Distribution of WIR Distribution under Inverse Levy prior

The Inverse Levy prior is defined as
\[ g(\lambda) = \sqrt{\frac{a}{2\pi}} \lambda^{-\frac{a+1}{2}} \exp\left(-\frac{a}{2\lambda}\right); \quad a > 0; \lambda > 0, \] where \( a \) is the known hyper parameter.

Hence the posterior distribution of \( \lambda \) using the Inverse Levy prior distribution is given by
\[ p(\lambda | x) = \frac{(T + a/2)^{\alpha_4}}{\Gamma(\alpha_4)} \lambda^{\alpha_4 - 1} e^{-\lambda(T + a/2)}. \] (3.7.33)

So, the posterior distribution of \( (\lambda | x) \sim G\left(\alpha_4, T + \frac{a}{2}\right) \), where \( \alpha_4 = \left(n\left(1 - \frac{k}{2}\right) + \frac{1}{2}\right) \) and \( T = \sum_{i=1}^{n} x_i^{-2} \).

### 3.7.8 Baye’s estimators and risk functions under Inverse Levy prior using SELF, ABLF and PLF

Under SELF the risk function is given by
\[ R(\tilde{\lambda}, \lambda) = \int_0^{\infty} c(\tilde{\lambda} - \lambda)^2 \frac{(T + a/2)^{\alpha_4}}{\Gamma(\alpha_4)} \lambda^{\alpha_4 - 1} e^{-\lambda(T + a/2)} d\lambda \] (3.7.34)
\[ = c \left[ \tilde{\lambda}^2 + \frac{(\alpha_4 + 1)\alpha_4}{(T + a/2)^2} - \frac{2\tilde{\lambda}\alpha_4}{(T + a/2)} \right]. \] (3.7.35)
\[ \therefore \quad \frac{d}{d\tilde{\lambda}} R(\tilde{\lambda}, \lambda) = 0. \quad \Rightarrow \quad \hat{\lambda}_{SELF} = \frac{\alpha_4}{(T + a/2)}. \] (3.7.36)

Under ABLF the risk function is given by
\[ R(\tilde{\lambda}, \lambda) = \int_0^{\infty} c(\tilde{\lambda} - \lambda)^2 \frac{(T + a/2)^{\alpha_4}}{\Gamma(\alpha_4)} \lambda^{\alpha_4 - 1} e^{-\lambda(T + a/2)} d\lambda \] (3.7.37)
\[ = \hat{\lambda}^2 \Gamma(\alpha_4 + c_2) \frac{\Gamma(\alpha_4 + c_2 + 2)}{\Gamma(\alpha_4)(T + a/2)^{2+2}} - \frac{2\hat{\lambda}\Gamma(\alpha_4 + c_2 + 1)}{\Gamma(\alpha_4)(T + a/2)^{1+1}}. \] (3.7.38)
\[ \therefore \quad \frac{d}{d\tilde{\lambda}} R(\tilde{\lambda}, \lambda) = 0. \quad \Rightarrow \quad \hat{\lambda}_{ABL} = \frac{(\alpha_4 + c_2)}{(T + a/2)}. \] (3.7.39)
Under PLF the risk function is given by

\[ R(\hat{\lambda}, \lambda) = \int_0^{\infty} \frac{(\hat{\lambda} - \lambda)^2}{\hat{\lambda}} \frac{(T + a/2)^{\alpha_4}}{\Gamma(\alpha_4)} \lambda^{\alpha_4-1} e^{-\lambda(T + a/2)} \, d\lambda \]  

(3.7.40)

\[ = \hat{\lambda} + \frac{(\alpha_4 + 1)(\alpha_4)}{\hat{\lambda}(T + a/2)^2} - \frac{2(\alpha_4)}{(T + a/2)}. \]  

(3.7.41)

\[ \therefore \frac{\partial}{\partial \lambda} R(\hat{\lambda}, \lambda) = 0. \quad \Rightarrow \quad \hat{\lambda}_{PLF} = \frac{\sqrt{\alpha_4(\alpha_4 + 1)}}{(T + a/2)}. \]  

(3.7.42)

Remark 3.4:

- Replacing \( c_z = 0 \) in (3.7.39) we get the same Bayes estimator as obtained in (3.7.36) under SELF using Inverse Levy prior.

3.8 Simulation Study and Real Data Analysis for Inverse Rayleigh distribution

3.8.1 Simulation study

We have generated a sample of size 25, 50 and 100 from inverse Rayleigh distribution by using R software. To examine the performance of Bayesian estimates for parameter of inverse Rayleigh distribution under different approximation techniques, estimates are presented along with posterior variances given in parenthesis in the tables 3.1 and 3.4.

Table 3.1: Posterior estimates and posterior variances (in parenthesis) under normal approximation based on simulated data sets

<table>
<thead>
<tr>
<th>( n )</th>
<th>( \lambda )</th>
<th>Jeffrey’s Prior</th>
<th>Modified Jeffrey’s prior</th>
<th>Exponential prior</th>
<th>Gamma prior</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>( b = 1.0 )</td>
<td>( b = 2.0 )</td>
</tr>
<tr>
<td>25</td>
<td>0.5</td>
<td>0.64972 (0.00685)</td>
<td>0.63618 (0.00700)</td>
<td>0.65895 (0.00689)</td>
<td>0.64203 (0.00667)</td>
</tr>
<tr>
<td></td>
<td>1.0</td>
<td>0.64118 (0.02752)</td>
<td>0.62783 (0.02695)</td>
<td>0.65052 (0.02682)</td>
<td>0.63402 (0.02514)</td>
</tr>
<tr>
<td></td>
<td>1.5</td>
<td>2.19093 (0.11485)</td>
<td>2.14528 (0.11245)</td>
<td>2.09130 (0.10465)</td>
<td>1.92986 (0.09232)</td>
</tr>
<tr>
<td>50</td>
<td>0.5</td>
<td>0.48712 (0.00761)</td>
<td>0.48214 (0.00753)</td>
<td>0.49216 (0.00757)</td>
<td>0.48737 (0.00739)</td>
</tr>
<tr>
<td></td>
<td>1.0</td>
<td>0.85331 (0.02231)</td>
<td>0.84460 (0.02208)</td>
<td>0.85582 (0.02183)</td>
<td>0.84142 (0.02094)</td>
</tr>
<tr>
<td></td>
<td>1.5</td>
<td>1.72948 (0.05586)</td>
<td>1.71184 (0.05529)</td>
<td>1.70462 (0.05334)</td>
<td>1.64842 (0.05002)</td>
</tr>
<tr>
<td>100</td>
<td>0.5</td>
<td>0.53987 (0.00268)</td>
<td>0.53704 (0.00267)</td>
<td>0.56234 (0.00268)</td>
<td>0.55919 (0.00266)</td>
</tr>
<tr>
<td></td>
<td>1.0</td>
<td>0.94228 (0.01513)</td>
<td>0.93752 (0.01506)</td>
<td>0.94282 (0.01492)</td>
<td>0.93402 (0.01456)</td>
</tr>
<tr>
<td></td>
<td>1.5</td>
<td>1.33719 (0.02414)</td>
<td>1.33043 (0.02402)</td>
<td>1.33269 (0.02364)</td>
<td>1.31517 (0.02293)</td>
</tr>
</tbody>
</table>
3.8.2 A real data example: In this section, a real data set is used to compare the performances of the Baye’s estimates obtained under different approximations using different priors. The data set is the failure times of 84 Aircraft Windshield. The windshield on a large aircraft is a complex piece of equipment, comprised basically of several layers of material, including a very strong outer skin with a heated layer just beneath it, all laminated under high temperature and pressure. Failures of these items are not structural failures. Instead, they typically involve damage or delamination of the nonstructural outer ply or failure of the heating system. These failures do not result in damage to the aircraft but do result in failure of the windshield. These data on failure times are reported in the book “Weibull Models” by Murthy et al. (2004, page 297).

The failure times of 84 Aircraft Windshield is

0.040, 1.866, 2.385, 3.443, 0.301, 1.876, 2.481, 3.467, 0.309, 1.899, 2.610, 3.478, 0.557, 1.911, 2.625, 3.578, 0.943, 1.912, 2.632, 3.595, 1.070, 1.914, 2.646, 3.699, 1.124, 1.981, 2.661, 3.779,1.248, 2.010, 2.688, 3.924, 1.281, 2.038, 2.823, 4.035, 1.281, 2.085, 2.890, 4.121, 1.303, 2.089, 2.902, 4.167, 1.432, 2.097, 2.934, 4.240, 1.480, 2.135, 2.962, 4.255, 1.505, 2.154, 2.964, 4.278, 1.506, 2.190, 3.000, 4.305, 1.568, 2.194, 3.103, 4.376, 1.615, 2.223, 3.114, 4.449, 1.619, 2.224, 3.117, 4.485, 1.652, 2.229, 3.166, 4.570, 1.652, 2.300, 3.344, 4.602, 1.757, 2.324, 3.376, 4.663.

<table>
<thead>
<tr>
<th>$n$</th>
<th>$\lambda$</th>
<th>Jeffrey’s prior</th>
<th>Modified Jeffrey’s prior</th>
<th>Exponential prior</th>
<th>Gamma prior</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$b = 1.0$</td>
<td>$b = 1.0$</td>
</tr>
<tr>
<td>25</td>
<td>0.5</td>
<td>0.74131 (0.00819)</td>
<td>0.72648 (0.00803)</td>
<td>0.74875 (0.00822)</td>
<td>0.72779 (0.00795)</td>
</tr>
<tr>
<td></td>
<td>1.0</td>
<td>1.10699 (0.02518)</td>
<td>1.08486 (0.02468)</td>
<td>1.10245 (0.02460)</td>
<td>1.05761 (0.02316)</td>
</tr>
<tr>
<td></td>
<td>1.5</td>
<td>2.16252 (0.07321)</td>
<td>2.11928 (0.07174)</td>
<td>2.06967 (0.06851)</td>
<td>1.91734 (0.06199)</td>
</tr>
<tr>
<td>50</td>
<td>0.5</td>
<td>0.43799 (0.00396)</td>
<td>0.43361 (0.00392)</td>
<td>0.44287 (0.00397)</td>
<td>0.43906 (0.00390)</td>
</tr>
<tr>
<td></td>
<td>1.0</td>
<td>1.31387 (0.01477)</td>
<td>1.30073 (0.01462)</td>
<td>1.30583 (0.01456)</td>
<td>1.27323 (0.01408)</td>
</tr>
<tr>
<td></td>
<td>1.5</td>
<td>1.69752 (0.03905)</td>
<td>1.68055 (0.03866)</td>
<td>1.67462 (0.03769)</td>
<td>1.62138 (0.03572)</td>
</tr>
<tr>
<td>100</td>
<td>0.5</td>
<td>0.54883 (0.00300)</td>
<td>0.54609 (0.00298)</td>
<td>0.55129 (0.00299)</td>
<td>0.54830 (0.00296)</td>
</tr>
<tr>
<td></td>
<td>1.0</td>
<td>0.85172 (0.01125)</td>
<td>0.84746 (0.01119)</td>
<td>0.85297 (0.01112)</td>
<td>0.84583 (0.01089)</td>
</tr>
<tr>
<td></td>
<td>1.5</td>
<td>1.75778 (0.02501)</td>
<td>1.72909 (0.02488)</td>
<td>1.72518 (0.02447)</td>
<td>1.69621 (0.02373)</td>
</tr>
</tbody>
</table>

Table 3.2: Posterior estimates and posterior variances (in parenthesis) under T-K approximation based on simulated data sets

$0.1\leq b\leq 0.3$
Chapter 3. Classical and Weighted Inverse Rayleigh Distribution

Table 3.3: Posterior estimates and posterior variances (in parenthesis) under normal Approximation for failure times of 84 Aircraft Windshield data

<table>
<thead>
<tr>
<th>$\lambda$</th>
<th>Jeffrey’s prior</th>
<th>Modified Jeffrey’s prior</th>
<th>Exponential prior</th>
<th>Gamma prior</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td>$b = 1.0$</td>
<td>$b = 2.0$</td>
</tr>
<tr>
<td>0.5</td>
<td>0.45454 (0.00258)</td>
<td>0.45181 (0.00256)</td>
<td>0.45751 (0.00258)</td>
<td>0.45504 (0.00255)</td>
</tr>
<tr>
<td>1.0</td>
<td>0.94716 (0.01591)</td>
<td>0.94145 (0.01582)</td>
<td>0.94775 (0.01567)</td>
<td>0.93718 (0.01525)</td>
</tr>
<tr>
<td>1.5</td>
<td>1.58454 (0.03928)</td>
<td>1.57499 (0.03905)</td>
<td>1.57359 (0.03808)</td>
<td>1.54465 (0.03651)</td>
</tr>
</tbody>
</table>

Table 3.4: Posterior estimates and posterior variances (in parenthesis) under T-K approximation for failure times of 84 Aircraft Windshield data

<table>
<thead>
<tr>
<th>$\lambda$</th>
<th>Jeffrey’s prior</th>
<th>Modified Jeffrey’s prior</th>
<th>Exponential prior</th>
<th>Gamma prior</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td>$b = 1.0$</td>
<td>$b = 2.0$</td>
</tr>
<tr>
<td>0.5</td>
<td>0.50363 (0.00295)</td>
<td>0.50063 (0.00293)</td>
<td>0.50659 (0.00295)</td>
<td>0.50359 (0.00292)</td>
</tr>
<tr>
<td>1.0</td>
<td>1.07857 (0.01432)</td>
<td>1.07215 (0.01423)</td>
<td>1.07757 (0.01412)</td>
<td>1.06408 (0.01376)</td>
</tr>
<tr>
<td>1.5</td>
<td>1.42167 (0.02596)</td>
<td>1.41321 (0.02581)</td>
<td>1.41465 (0.02537)</td>
<td>1.39149 (0.02452)</td>
</tr>
</tbody>
</table>

3.8.3 Results and Discussion for Inverse Rayleigh distribution

Here, we have obtained approximations to Bayesian integrals of Inverse Rayleigh distribution depending upon numerical integration and simulation study. We observe that under different priors, the T-K approximation behaves well than normal approximation, although the posterior variances in case of normal approximation are very close to that of T-K approximation.

From the findings of above tables, it can be observed that the large sample distribution could be improved when prior is taken into account. In both normal as well as T-K approximations, posterior variance under Gamma prior is less with respect to other assumed priors particularly at $b=3.0$. Further, it is also noticed that the posterior variance based on different priors tends to decrease with the increase in sample size which means that the estimates computed are consistent.

Also, the real-life data justifies the simulated data results concluding that the Gamma prior performs well in the Inverse Rayleigh distribution.
3.9 Applications for Weighted Inverse Rayleigh distribution

3.9.1 Application for special cases of weighted Inverse Rayleigh distribution

In this section, the flexibility and potentiality of the WIR distribution are examined using two real data sets to show that the WIR distribution can be a better model than its sub models. The analysis is performed with the help of R software.

**Data set I:** The first real data set represents the 72 exceedances for the years 1958–1984 (rounded to one decimal place) of flood peaks (in m$^3$/s) of the Wheaton River near Car cross in Yukon Territory, Canada. The data are as follows:

1.7, 2.2, 14.4, 1.1, 0.4, 20.6, 5.3, 0.7, 1.9, 13.0, 12.0, 9.3, 1.4, 18.7, 8.5, 25.5, 11.6, 14.1, 22.1, 1.1, 2.5, 14.4, 1.7, 37.6, 0.6, 2.2, 39.0, 0.3, 15.0, 11.0, 7.3, 22.9, 1.7, 0.1, 1.1, 0.6, 9.0, 1.7, 7.0, 20.1, 0.4, 2.8, 14.1, 9.9, 10.4, 10.7, 30.0, 3.6, 5.6, 30.8, 13.3, 4.2, 25.5, 3.4, 11.9, 21.5, 27.6, 36.4, 2.7, 64.0, 1.5, 2.5, 27.4, 1.0, 27.1, 20.2, 16.8, 5.3, 9.7, 27.5, 2.5 and 7.0. This data set is previously used by Merovci and Puka (2014).

**Data set II:** The second data set is regarding remission times (in months) of a random sample of 128 bladder cancer patients given in Lee and Wang (2003). The data set is given as follows: 0.08, 2.09, 2.73, 3.48, 4.87, 6.94, 8.66, 13.11, 23.63, 0.20, 2.22, 3.52, 4.98, 6.99, 9.02, 13.29, 0.40, 2.26, 3.57, 5.06, 7.09, 9.22, 13.80, 25.74, 0.50, 2.46, 3.64, 5.09, 7.26, 9.47, 14.24, 25.82, 0.51, 2.54, 3.70, 5.17, 7.28, 9.74, 14.76, 26.31, 0.81, 2.62, 3.82, 5.32, 7.32, 10.06, 14.77, 32.15, 2.64, 3.88, 5.32, 7.39, 10.34, 14.83, 34.26, 0.90, 2.69, 4.18, 5.34, 7.59, 10.66, 15.96, 36.66, 1.05, 2.69, 4.23, 5.41, 7.62, 10.75, 15.62, 43.01, 1.19, 2.75, 4.26, 5.41, 7.63, 17.12, 46.12, 1.26, 2.83, 4.33, 5.49, 7.66, 11.25, 17.14, 79.05, 1.35, 2.87, 5.62, 7.87, 11.64, 17.36, 1.40, 3.02, 4.34, 5.71, 7.93, 11.79, 18.10, 1.46, 4.40, 5.85, 8.26, 11.98, 19.13, 1.76, 3.25, 4.50, 6.25, 8.37, 12.02, 2.02, 3.31, 4.51, 6.54, 8.53, 12.03, 20.28, 2.02, 3.36, 6.93, 8.65, 12.63 and 22.69.

We fit the WIR distribution to above two data sets and compare the fitness with its special cases such as the IR, LBIR, WR, LBR and Rayleigh distributions. The MLEs of the parameters with standard errors in parentheses and the corresponding log-likelihood values, AIC, AICC and BIC are displayed in Table 3.5 and 3.6.
Table 3.5: MLEs (S.E in parentheses) for Wheat on River flood data

<table>
<thead>
<tr>
<th>Distribution</th>
<th>Parameter Estimates</th>
<th>-2Log L</th>
<th>AIC</th>
<th>AICC</th>
<th>BIC</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$k$</td>
<td>$\lambda$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>WIR</td>
<td>1.64725 (0.04472)</td>
<td>0.09136 (0.02813)</td>
<td>575.2203</td>
<td>579.2203</td>
<td>579.3942</td>
</tr>
<tr>
<td>LBIR</td>
<td>0.25899 (0.04316)</td>
<td>0.06105 (0.02813)</td>
<td>669.7216</td>
<td>671.7216</td>
<td>671.7787</td>
</tr>
<tr>
<td>IR</td>
<td>0.51799 (0.06105)</td>
<td></td>
<td>915.6792</td>
<td>917.6792</td>
<td>917.7363</td>
</tr>
<tr>
<td>WR</td>
<td>0.51008 (0.34756)</td>
<td>0.00421 (0.00067)</td>
<td>664.1989</td>
<td>668.1989</td>
<td>664.3728</td>
</tr>
<tr>
<td>LBR</td>
<td>0.00503 (0.00046)</td>
<td></td>
<td>682.7753</td>
<td>684.7753</td>
<td>684.807</td>
</tr>
<tr>
<td>Rayleigh(RD)</td>
<td>–</td>
<td>0.00335 (0.00036)</td>
<td>605.6757</td>
<td>607.6757</td>
<td>607.7074</td>
</tr>
</tbody>
</table>

Table 3.6: MLEs (S.E in parentheses) for bladder cancer data

<table>
<thead>
<tr>
<th>Distribution</th>
<th>Parameter Estimates</th>
<th>-2Log L</th>
<th>AIC</th>
<th>AICC</th>
<th>BIC</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$k$</td>
<td>$\lambda$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>WIR</td>
<td>1.61586 (0.03672)</td>
<td>0.11869 (0.02649)</td>
<td>975.4733</td>
<td>979.4733</td>
<td>979.5693</td>
</tr>
<tr>
<td>LBIR</td>
<td>–</td>
<td>0.30899 (0.03862)</td>
<td>1111.258</td>
<td>1113.258</td>
<td>1113.29</td>
</tr>
<tr>
<td>IR</td>
<td>–</td>
<td>0.61798 (0.05462)</td>
<td>1497.54</td>
<td>1499.54</td>
<td>1499.572</td>
</tr>
<tr>
<td>WR</td>
<td>0.51433 (0.27344)</td>
<td>0.00657 (0.00085)</td>
<td>1082.764</td>
<td>1086.764</td>
<td>1086.86</td>
</tr>
<tr>
<td>LBR</td>
<td>–</td>
<td>0.00782 (0.00055)</td>
<td>1101.261</td>
<td>1103.261</td>
<td>1103.293</td>
</tr>
<tr>
<td>Rayleigh(RD)</td>
<td>–</td>
<td>0.00521 (0.00044)</td>
<td>992.2544</td>
<td>994.2544</td>
<td>994.2861</td>
</tr>
</tbody>
</table>

Figure 3.3: Plots of the fitted WIR, IR, LBIR, WR, LBR and Rayleigh distributions for data sets 1 and 2.
3.9.2 Applications for different Bayes estimates and posterior risks of WIRD

In this section, we compare the performance of Bayes estimates and posterior risks under different loss functions using different priors for the scale parameter of WIRD, three real data sets are used and analysis performed with the help of R software.

Data set I: The first data set, strength data, which were originally reported by Badar and Priest (1982) and it represents the strength measured in GPA for single carbon fibers and impregnated 1000-carbon fiber tows. Single fibers were tested under tension at gauge lengths of 10 mm with sample size \( n = 63 \). This data set consists of observations: 1.901, 2.132, 2.203, 2.228, 2.257, 2.350, 2.361, 2.396, 2.397, 2.445, 2.454, 2.474, 2.518, 2.522, 2.525, 2.532, 2.575, 2.614, 2.616, 2.618, 2.624, 2.659, 2.675, 2.738, 2.740, 2.856, 2.917, 2.928, 2.937, 2.977, 2.996, 3.030, 3.125, 3.139, 3.145, 3.220, 3.223, 3.235, 3.243, 3.264, 3.272, 3.294, 3.332, 3.346, 3.377, 3.408, 3.435, 3.493, 3.501, 3.537, 3.554, 3.562, 3.628, 3.852, 3.871, 3.886, 3.971, 4.024, 4.027, 4.225, 4.395, 5.020. This data set is previously used by Mead et al (2017).

Table 3.7: Baye’s estimators and Baye’s risk under JP for the first data set

<table>
<thead>
<tr>
<th>( k )</th>
<th>( c = 0.5 )</th>
<th>( c = 1.0 )</th>
<th>( c_2 = 0.5 )</th>
<th>( c_2 = -0.5 )</th>
<th>( \text{PLF} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.5</td>
<td>6.25371 (0.41385)</td>
<td>6.25371 (0.82770)</td>
<td>6.31988 (2.08624)</td>
<td>6.18753 (0.33011)</td>
<td>6.31954 (0.13166)</td>
</tr>
<tr>
<td>1.0</td>
<td>4.16914 (0.27590)</td>
<td>4.16914 (0.55180)</td>
<td>4.23532 (1.14004)</td>
<td>4.10296 (0.26918)</td>
<td>4.23479 (0.13132)</td>
</tr>
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<td>2.08457 (0.27590)</td>
<td>2.15075 (0.40774)</td>
<td>2.01839 (0.18958)</td>
<td>2.149727 (0.13032)</td>
</tr>
</tbody>
</table>

Table 3.8: Baye’s estimators and Baye’s risk under Ext. JP for the first data set

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<tr>
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<th>( c_1 )</th>
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<th>( c_2 = 0.5 )</th>
<th>( c_2 = -0.5 )</th>
<th>( \text{PLF} )</th>
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<tbody>
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<td>6.18753 (2.02070)</td>
<td>6.05518 (0.32657)</td>
<td>6.18718 (0.13165)</td>
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<td>5.98900 (0.79267)</td>
<td>6.05518 (1.95587)</td>
<td>5.92282 (0.32301)</td>
<td>6.05482 (0.13163)</td>
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</tr>
<tr>
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<td>1.0</td>
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<td>4.03678 (0.53428)</td>
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<td>3.97061 (0.26483)</td>
<td>4.10243 (0.13128)</td>
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<td>3.90443 (0.51676)</td>
<td>3.97061 (1.03403)</td>
<td>3.83825 (0.26042)</td>
<td>3.97007 (0.13125)</td>
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<tr>
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<td>1.95222 (0.25838)</td>
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</tr>
<tr>
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<td>1.81986 (0.24086)</td>
<td>1.88604 (0.33370)</td>
<td>1.75368 (0.17693)</td>
<td>1.88487 (0.13003)</td>
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Table 3.9: Baye’s estimators and Baye’s risk under PP for the first data set

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<th>ABLF $c_2 = -0.5$</th>
<th>PLF</th>
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<td>6.18753 (0.40947)</td>
<td>6.18753 (0.81984)</td>
<td>6.25371 (2.05338)</td>
<td>6.12135 (0.32834)</td>
<td>6.25336 (0.13165)</td>
</tr>
<tr>
<td></td>
<td>1.5</td>
<td>6.05518 (0.40071)</td>
<td>6.05517 (0.80142)</td>
<td>6.12135 (1.98819)</td>
<td>5.98900 (0.32479)</td>
<td>6.12099 (0.13164)</td>
</tr>
<tr>
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<td>0.5</td>
<td>4.10296 (0.27152)</td>
<td>4.10296 (0.54304)</td>
<td>4.16914 (1.11322)</td>
<td>4.03678 (0.26701)</td>
<td>4.16861 (0.13130)</td>
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<td>3.97061 (0.52552)</td>
<td>4.03678 (1.06020)</td>
<td>3.96443 (0.26263)</td>
<td>4.03624 (0.13127)</td>
</tr>
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<td>1.81986 (0.19914)</td>
<td>1.95109 (0.13011)</td>
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Table 3.10: Baye’s estimators and Baye’s risk under ILP for the first data set

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<th>PLF</th>
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<td>6.11747 (0.78374)</td>
<td>6.18152 (1.95363)</td>
<td>6.05341 (0.31604)</td>
<td>6.18119 (0.12745)</td>
</tr>
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<td>5.74919 (0.34611)</td>
<td>5.74919 (0.69221)</td>
<td>5.80939 (1.88449)</td>
<td>5.68899 (0.31435)</td>
<td>5.80908 (0.11978)</td>
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<td>4.09966 (0.52522)</td>
<td>4.16372 (1.07586)</td>
<td>4.03561 (0.25839)</td>
<td>4.16323 (0.12713)</td>
</tr>
<tr>
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<td>3.85286 (0.23195)</td>
<td>3.85286 (0.46389)</td>
<td>3.91306 (0.92118)</td>
<td>3.79266 (0.23541)</td>
<td>3.91259 (0.11947)</td>
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<td>2.08186 (0.26672)</td>
<td>2.145918 (0.39364)</td>
<td>2.01780 (0.18344)</td>
<td>2.14496 (0.12620)</td>
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<td>1.95653 (0.11778)</td>
<td>1.95653 (0.23557)</td>
<td>2.016731 (0.33704)</td>
<td>1.89633 (0.16712)</td>
<td>2.01583 (0.11860)</td>
</tr>
</tbody>
</table>

Data set II: The second real data set is a subset of the data reported by Bekker et al. (2000), which corresponds to the survival times (in years) of a group of patients given chemotherapy treatment alone. The data consisting of survival times (in years) for 46 patients are: 0.047, 0.115, 0.121, 0.132, 0.164, 0.197, 0.203, 0.260, 0.282, 0.296, 0.334, 0.395, 0.458, 0.466, 0.501, 0.507, 0.529, 0.534, 0.540, 0.641, 0.644, 0.696, 0.841, 0.863, 1.099, 1.219, 1.271, 1.326, 1.447, 1.485, 1.553, 1.581, 1.589, 2.178, 2.343, 2.416, 2.444, 2.825, 2.830, 3.578, 3.658, 3.743, 3.978, 4.003, 4.033.
### Table 3.11: Baye’s estimators and Baye’s risk under JP for the data set II

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<td>0.04023 (0.03230)</td>
<td>0.04082 (0.02409)</td>
</tr>
<tr>
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<td>0.02682 (0.01077)</td>
<td>0.02682 (0.02154)</td>
<td>0.02741 (0.01318)</td>
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<tr>
<td>1.5</td>
<td>0.01341 (0.01981)</td>
<td>0.01341 (0.01077)</td>
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<table>
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<th>$c_2 = -0.5$</th>
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</thead>
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<td>0.03963 (0.01567)</td>
<td>0.03963 (0.03134)</td>
<td>0.03844 (0.02304)</td>
</tr>
<tr>
<td>1.0</td>
<td>0.03784 (0.01519)</td>
<td>0.03784 (0.03039)</td>
<td>0.03840 (0.02200)</td>
</tr>
<tr>
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<td>0.02562 (0.00981)</td>
<td>0.02562 (0.01962)</td>
<td>0.02502 (0.01148)</td>
</tr>
<tr>
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<td>0.01222 (0.01805)</td>
<td>0.01222 (0.00981)</td>
<td>0.01281 (0.00413)</td>
</tr>
<tr>
<td>1.0</td>
<td>0.01102 (0.01628)</td>
<td>0.01102 (0.00885)</td>
<td>0.01162 (0.00355)</td>
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<td>0.01102 (0.01628)</td>
<td>0.01102 (0.00885)</td>
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### Table 3.12: Baye’s estimators and Baye’s risk under Ext. JP for the data set II

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<td>0.03903 (0.03134)</td>
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</tr>
<tr>
<td>1.5</td>
<td>1.0</td>
<td>0.03784 (0.01519)</td>
<td>0.03784 (0.03039)</td>
<td>0.03840 (0.02200)</td>
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<tr>
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<td>1.0</td>
<td>0.02562 (0.00981)</td>
<td>0.02562 (0.01962)</td>
<td>0.02502 (0.01148)</td>
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<tr>
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<td>1.0</td>
<td>0.01222 (0.01805)</td>
<td>0.01222 (0.00981)</td>
<td>0.01281 (0.00413)</td>
</tr>
<tr>
<td>1.5</td>
<td>1.5</td>
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<td>0.01102 (0.00885)</td>
<td>0.01162 (0.00355)</td>
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### Table 3.13: Baye’s estimators and Baye’s risk under PP for the data set II

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<td>0.03963 (0.03183)</td>
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<tr>
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<td>0.03844 (0.03087)</td>
<td>0.03903 (0.02252)</td>
</tr>
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<td>0.02622 (0.02106)</td>
<td>0.02681 (0.01276)</td>
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<tr>
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<td>0.02503 (0.02010)</td>
<td>0.02562 (0.01191)</td>
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<tr>
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<td>0.01281 (0.01029)</td>
<td>0.01341 (0.00443)</td>
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<tr>
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<td>1.5</td>
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<td>0.01162 (0.00933)</td>
<td>0.01222 (0.00384)</td>
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Table 3.14: Baye’s estimators and Baye’s risk under ILP for the data set II

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<td>0.04078 (0.03272)</td>
<td>0.041381 (0.02458)</td>
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<tr>
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<td>0.02800 (0.01362)</td>
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<td>0.01400 (0.01124)</td>
<td>0.01459 (0.000505)</td>
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<tr>
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<td>0.01399 (0.01123)</td>
<td>0.01458 (0.000504)</td>
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</table>

Data set III: The third real data set represents the survival times, in weeks, of 33 patients suffering from acute Myelogeneous Leukaemia. The data has been previously used by Feigl and Zelen (1965) and M. E. Mead et al (2017). The data is as follows: 65, 156, 100, 134, 16, 108, 121, 4, 39, 143, 56, 26, 22, 1, 1, 5, 65, 56, 65, 17, 7, 16, 22, 3, 4, 2, 3, 8, 4, 3, 30, 4, and 43.

Table 3.15: Baye’s estimators and Baye’s risk under JP for the data set III

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<td>8.44621 (2.88236)</td>
<td>8.61684 (8.50301)</td>
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<tr>
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<td>5.63081 (1.92158)</td>
<td>5.80144 (4.66249)</td>
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<tr>
<td>1.5</td>
<td>2.81540 (0.48039)</td>
<td>2.81540 (0.96079)</td>
<td>2.98603 (1.68413)</td>
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Table 3.16: Baye’s estimators and Baye’s risk under Ext. JP for the data set III

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<td>$c_2 = 0.5$</td>
<td>$c_2 = -0.5$</td>
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</tr>
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<td>7.76369 (2.64944)</td>
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<td>7.59306 (0.94566)</td>
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<tr>
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<tr>
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<td>2.64477 (1.39542)</td>
<td>2.30351 (0.52761)</td>
</tr>
<tr>
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<td>2.13288 (0.36393)</td>
<td>2.13288 (0.72788)</td>
<td>2.30351 (1.12534)</td>
<td>1.96225 (0.48853)</td>
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</table>
Chapter 3. Classical and Weighted Inverse Rayleigh Distribution

Table 3.17: Baye’s estimators and Baye’s risk under PP for the data set III

<table>
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<th>b</th>
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<th>PLF</th>
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<td>8.27558</td>
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<td>8.44448</td>
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</tr>
<tr>
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<td>(9.7667)</td>
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<tr>
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<td>7.93432</td>
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<td>(7.7659)</td>
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<td>2.30351</td>
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<td>2.13288</td>
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<td></td>
<td>(0.39305)</td>
<td>(0.78609)</td>
<td>(1.25797)</td>
<td>(0.50844)</td>
<td>(0.3294)</td>
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Table 3.18: Baye’s estimators and Baye’s risk under ILP for the data set III

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<th>k</th>
<th>a</th>
<th>c</th>
<th>b</th>
<th>c</th>
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<td>7.93948</td>
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<td>7.78227</td>
<td>8.09517</td>
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<td></td>
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<td>(1.24964)</td>
<td>(7.13814)</td>
<td>(8.0161)</td>
<td>(3.1138)</td>
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<td>6.86084</td>
<td>6.99669</td>
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<td>6.99538</td>
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<td></td>
<td>(0.93210)</td>
<td>(1.86420)</td>
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<td>(8.0189)</td>
<td>(2.6908)</td>
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<td>5.34539</td>
<td>5.50261</td>
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<td>5.50037</td>
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<td>(1.68078)</td>
<td>(3.97097)</td>
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<td>(0.3099)</td>
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<td>4.61918</td>
<td>4.61918</td>
<td>4.75504</td>
<td>4.48332</td>
<td>4.75309</td>
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<td>(1.25511)</td>
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<td>(0.64601)</td>
<td>(1.03809)</td>
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<td>(0.2643)</td>
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3.9.3 Results and Discussion for weighted Inverse Rayleigh distribution

In this chapter, we have introduced Weighted Inverse Rayleigh (WIR) distribution, which acts as a generalization to many distributions viz. IRD, LBIRD, WRD, LBRD and RD. After introducing WIRD, we have derived its different mathematical properties. Two real data sets have been considered in order to make comparison between WIRD and its special cases in terms of fitting. The results are given in Table 3.5 and 3.6. It is obvious from tables that WIRD possesses minimum values of AIC, AICC and BIC. Therefore, we conclude that the WIRD will be treated as a best fitted distribution to the data sets as compared to its other special cases.

In addition, we have also calculated the Bayes estimator for the scale parameter of weighted Inverse Rayleigh distribution under squared error loss function, Al-Bayyati’s loss function and precautionary loss function using different
informative and non-informative priors. From the results, it is observed that the precautionary loss function provides the smallest Bayes posterior risk as compared to the other loss functions for each and every value of parameter $\lambda$. So, we can say that the precautionary loss function is better loss function as compared to the other loss functions used in this chapter. It is also observed that among all the priors, the Inverse Levy prior provides the minimum posterior risk as compared to the other assumed priors.