Chapter 2

Classical and Exponentiated Inverse Exponential Distribution
Chapter 2. Classical and Exponentiated Inverse Exponential Distribution

2.1 Introduction

The exponential distribution is the most widely used lifetime model in reliability theory, because of its simplicity and mathematical feasibility. The applicability of this distribution is restricted due to its constant failure rate but in practical experience, it seems to be meaningless when failure rate is non-constant. For such types of data, another extension of the exponential distribution has been proposed in statistical literature. That is known as one parameter inverse exponential or one parameter inverted exponential distribution (IED) which possess the inverted bathtub hazard rate and was proposed by Keller and Kamath (1982). If X is a non-negative Exponential random variable, then the distribution of a random variable \( Y = \frac{1}{X} \) follows an Inverse Exponential distribution.

The pdf and cdf of IED are respectively given by:

\[
\begin{align*}
    g(x) &= \frac{\lambda}{x^2} e^{-\frac{\lambda}{x}} \quad ; \quad x > 0, \lambda > 0 \\
    G(x) &= e^{-\frac{\lambda}{x}} \quad ; \quad x > 0, \lambda > 0 ,
\end{align*}
\]

where \( \lambda \) is the scale parameter.


2.2 Normal approximation of Inverse exponential distribution under different priors:

The likelihood function of (2.1.1) for a sample of size n is given as

\[
L(\mathbf{x} | \lambda) = \lambda^n \prod_{i=1}^{n} x_i^{-2} e^{-\lambda \sum_{i=1}^{n} x_i^{-1}} .
\]

Under Quasi prior \( g(\lambda) = \frac{1}{\lambda^d} , d \geq 0 \), the posterior distribution for \( \lambda \) is as

\[
P(\lambda | \mathbf{x}) \propto \lambda^{n-d} e^{-\tau}, \text{ where } \tau = \sum_{i=1}^{n} x_i^{-1}
\]
To construct the approximation, we need the second derivatives of the log-posterior density,

$$\log P(\lambda | \mathbf{x}) = \log \text{constant} + (n - d)\log \lambda - \lambda T.$$  \hspace{1cm} (2.2.3)

Now \( \frac{\partial}{\partial \lambda} \log P(\lambda | \mathbf{x}) = 0. \)

$$\Rightarrow \quad \frac{n - d}{\lambda} - T = 0 \quad \Rightarrow \quad \hat{\lambda} = \frac{(n - d)}{T}. \hspace{1cm} (2.2.4)$$

Under extension of Jeffrey’s prior \( g(\lambda) = \frac{1}{\lambda^{2c_1}}, c_1 \in R^+ \), the posterior distribution for \( \lambda \) is as

$$P(\lambda | \mathbf{x}) \propto \lambda^{-2c_1} e^{-\lambda T}, \quad \text{where } T = \sum_{i=1}^{n} x_i^{-1}. \hspace{1cm} (2.2.5)$$

$$\log P(\lambda | \mathbf{x}) = \log \text{constant} + (n - 2c_1)\log \lambda - \lambda T.$$  \hspace{1cm} (2.2.6)

Now \( \frac{\partial}{\partial \lambda} \log P(\lambda | \mathbf{x}) = 0, \)

$$\Rightarrow \quad \frac{n - 2c_1}{\lambda} - T = 0 \quad \Rightarrow \quad \hat{\lambda} = \frac{(n - 2c_1)}{T}. \hspace{1cm} (2.2.7)$$

Under the Pareto1 prior \( g(\lambda) = bc^b \lambda^{-(b+1)}, c, b > 0; \lambda > 0 \), where \( b \) and \( c \) are the known hyper parameters, the posterior distribution for \( \hat{\lambda} \) is as

$$P(\lambda | \mathbf{x}) \propto \lambda^{-b-1} e^{-\lambda T}, \quad \text{where } T = \sum_{i=1}^{n} x_i^{-1}. \hspace{1cm} (2.2.8)$$
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\[
\log P(\lambda \mid x) = \log \text{constant} + (n-b-1)\log \lambda - \lambda T. \tag{2.2.9}
\]

Now \[
\frac{\partial}{\partial \lambda} \log P(\lambda \mid x) = 0,
\]

\[
\Rightarrow \quad \frac{(n-b-1)}{\lambda} - T = 0 \quad \Rightarrow \quad \hat{\lambda} = \frac{(n-b-1)}{T}.
\]

Also, \[
\frac{\partial^2}{\partial \lambda^2} \log P(\lambda \mid x) = -\frac{(n-b-1)}{\lambda^2}.
\]

\[
\therefore \quad I(\hat{\lambda}) = -\frac{\partial^2}{\partial \lambda^2} \log P(\lambda \mid x) = \frac{(n-b-1)}{\lambda^2} = \frac{T^2}{(n-b-1)},
\]

\[
\Rightarrow \quad [I(\hat{\lambda})]^{-1} = \frac{(n-b-1)}{T^2}.
\]

\[
\therefore \quad p(\lambda \mid x) \sim N\left(\frac{(n-b-1)}{T}, \frac{(n-b-1)}{T^2}\right). \tag{2.2.10}
\]

Under the Inverse Levy prior \( g(\lambda) = \sqrt{\frac{a}{2\pi}} \lambda^{-a/2} e^{-a \lambda/2} ; a > 0; \lambda > 0 \), where \( a \) is the known hyper parameter, thus the posterior distribution for \( \lambda \) is as

\[
P(\lambda \mid x) \propto \lambda^{n-1/2} e^{-\lambda(T+a/2)}, \quad \text{where} \quad T = \sum_{i=1}^{n} x_i^{-1} \tag{2.2.11}
\]

\[
\log P(\lambda \mid x) = \log \text{constant} + (n-1/2)\log \lambda - \lambda(T+a/2). \tag{2.2.12}
\]

Now \[
\frac{\partial}{\partial \lambda} \log P(\lambda \mid x) = 0,
\]

\[
\Rightarrow \quad \frac{(n-1/2)}{\lambda} - \frac{a/2 + T}{\lambda} = 0 \quad \Rightarrow \quad \hat{\lambda} = \frac{(n-1/2)}{(a/2 + T)^2}.
\]

Also, \[
\frac{\partial^2}{\partial \lambda^2} \log P(\lambda \mid x) = -\frac{(n-1/2)}{\lambda^2}.
\]

\[
\therefore \quad I(\hat{\lambda}) = -\frac{\partial^2}{\partial \lambda^2} \log P(\lambda \mid x) = \frac{(a/2 + T)^2}{(n-1/2)},
\]

\[
\Rightarrow \quad [I(\hat{\lambda})]^{-1} = \frac{(n-1/2)}{(a/2 + T)^2}.
\]

\[
\therefore \quad p(\lambda \mid x) \sim N\left(\frac{(n-1/2)}{(a/2 + T)}, \frac{(n-1/2)}{(a/2 + T)^2}\right). \tag{2.2.13}
\]

2.3 T-K Approximation of Inverse exponential distribution under different priors:

Tierney and Kadane (1986) gave Laplace method to evaluate \( E(h(\lambda) \mid x) \) as

\[
E(h(\lambda) \mid x) \approx \frac{\hat{\phi}^* \exp\{\hat{\phi}^* (\hat{\lambda}^*)\}}{\hat{\phi} \exp\{-\hat{\phi} h(\lambda)\}}, \tag{2.3.1}
\]
where $-nh(\lambda) = \log P(\lambda | x)$ ; $-nh^*(\lambda^*) = \log P(\lambda | x) + \log h(\lambda)$

$$\hat{\phi}^2 = \frac{1}{[-nh(\lambda)]^{-1}}; \hat{\phi}^* = \frac{1}{[-nh^*(\lambda^*)]^{-1}}$$

Under Quasi prior $g(\lambda) = \frac{1}{\lambda^d}, \ d \geq 0$, the posterior distribution for $\hat{\lambda}$ is given in (2.2.2)

$$-nh(\lambda) = (n-d)\log \lambda - \lambda T,$$

$$\therefore \ -nh'(\lambda) = \frac{(n-d)}{\lambda} - T \Rightarrow \hat{\lambda} = \frac{(n-d)}{T}.$$ 

Also, $-nh^*(\lambda) = -\frac{T^2}{(n-d)}$.

$$\therefore \ \hat{\phi}^2 = \frac{1}{[-nh^*(\lambda)]^{-1}} = \frac{(n-d)}{T^2} \quad \text{or} \quad \hat{\phi} = \frac{\sqrt{(n-d)}}{T},$$

now, $-nh^*(\lambda^*) = -nh(\lambda) + \ln h(\lambda) = (n-d+1)\ln(\lambda^*) - \lambda^* T$

$$-nh''(\lambda^*) = \frac{(n-d+1)}{\lambda^*} - T \Rightarrow \hat{\lambda}^* = \frac{(n-d+1)}{T}.$$ 

Also, $-nh''^*(\lambda^*) = -\frac{T^2}{(n-d+1)}$

$$\hat{\phi}^{*2} = \frac{1}{[-nh''(\lambda^*)]^{-1}} = \frac{(n-d+1)}{T^2} \quad \text{or} \quad \hat{\phi}^* = \frac{\sqrt{(n-d+1)}}{T}.$$ 

Thus, using (2.3.1) we have

$$E(\lambda | x) = \frac{(n-d+1)}{(n-d)^{1/2}} \exp \left\{ (n-d+1)\ln \hat{\lambda} - \hat{\lambda} T \right\} = \frac{(n-d+1)}{(n-d)^{1/2}} \frac{\lambda^{(n-d+1)} e^{-\hat{\lambda}T+\hat{\lambda}T}}{\hat{\lambda}^{n-d}}$$

$$\Rightarrow \ E(\lambda | x) = \left( \frac{n-d+1}{T} \right) \left( \frac{n-d+1}{n-d} \right)^{n-d+1/2} e^{-1}. \tag{2.3.2}$$

Note that the relative error (relative error to exact the posterior mean $\left( \frac{n-d+1}{T} \right)$) is

$$\left( \frac{n-d+1}{n-d} \right)^{n-d+1/2} e^{-1}.$$ 

Similarly,

$$E(\lambda^2 | x) \approx \frac{\hat{\phi}^* \exp \{-nh^*(\lambda^*)\}}{\hat{\phi} \exp \{-nh(\lambda)\}}.$$ 

Here,

$$-nh^*(\lambda^*) = \ln(\lambda^*) - nh(\lambda) = (n-d+2)\ln(\lambda^*) - \lambda^* T$$

$$E(\lambda^2 | x) = \left( \frac{n-d+2}{T} \right)^2 \left( \frac{n-d+2}{n-d} \right)^{n-d+1/2} e^{-2}. \tag{2.3.3}$$
\[ \therefore V(\lambda | \chi) = \left( \frac{n-d+2}{T} \right)^2 \left( \frac{n-d+2}{n-d} \right)^{n-d+1/2} e^{ - \left[ \left( \frac{n-d+1}{T} \right)^2 \left( \frac{n-d+1}{n-d} \right)^{n-d+1/2} e^{-1} \right]^2} . \]

Under extension of Jeffrey’s prior \( g(\lambda) = \frac{1}{\lambda^2 c_1}, c_1 \in \mathbb{R}^+ \), the posterior distribution for \( \lambda \) is given in (2.2.5)

\[ -nh(\lambda) = (n-2c_1) \log \lambda - \lambda T \]

\[ \therefore -nh'(\lambda) = \frac{(n-2c_1)}{\lambda} - T \quad \Rightarrow \hat{\lambda} = \frac{(n-2c_1)}{T} . \]

Also,

\[ -nh^*(\hat{\lambda}) = -\frac{T^2}{(n-2c_1)} \]

\[ \therefore \hat{\phi}^2 = -[-nh^*(\hat{\lambda})]^{-1} = \frac{(n-2c_1)}{T^2} \text{ or } \hat{\phi} = \sqrt{\frac{n-2c_1}{T}} , \]

now,

\[ -nh^*(\lambda^*) = -nh(\lambda) + \ln h(\lambda) = (n-2c_1+1) \ln (\lambda^*) - \lambda T \]

\[ \therefore -nh^*(\lambda^*) = \frac{(n-2c_1+1)}{\lambda^*} - T \quad \Rightarrow \hat{\lambda}^* = \frac{(n-2c_1+1)}{T} , \]

and

\[ -nh^*(\hat{\lambda}^*) = -\frac{T^2}{(n-2c_1+1)} , \]

\[ \hat{\phi}^* = -[-nh^*(\hat{\lambda}^*)]^{-1} = \frac{(n-2c_1+1)}{T^2} \text{ or } \hat{\phi}^* = \sqrt{\frac{n-2c_1+1}{T}} . \]

Thus, using (2.3.1) we have

\[ E(\lambda | \chi) = \frac{(n-2c_1+1)^{1/2} \exp \{(n-2c_1+1) \ln \hat{\lambda} - \hat{\lambda}^2 T\}}{(n-2c_1)^{1/2} \exp \{(n-2c_1) \ln \hat{\lambda} - \hat{\lambda}^2 T\}} \]

\[ \Rightarrow E(\lambda | \chi) = \left( \frac{n-2c_1+1}{T} \right)^{n-2c_1+1/2} \left( \frac{n-2c_1+1}{n-2c_1} \right)^{e^{-1}} . \quad (2.3.4) \]

Note that the relative error is

\[ \left( \frac{n-2c_1+1}{n-2c_1} \right)^{n-2c_1+1/2} e^{1} . \]

Similarly,

\[ E(\lambda^2 | \chi) \equiv \hat{\phi}^* \exp \{-nh^*(\hat{\lambda}^*)\} \]

\[ \exp \{-nh(\hat{\lambda})\} \]

Here,

\[ -nh^*(\lambda^*) = \ln(\lambda^2) - nh(\lambda) = (n-2c_1+2) \ln (\lambda^*) - \lambda^2 T \]

\[ E(\lambda^2 | \chi) = \left( \frac{n-2c_1+2}{T} \right)^{n-2c_1+1/2} \left( \frac{n-2c_1+2}{n-2c_1} \right)^{e^{-2}} . \quad (2.3.5) \]
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\[ V(\lambda | x) = \left( \frac{n - 2c_1 + 2}{T} \right)^2 \left( \frac{n - 2c_1 + 2}{n - 2c_1} \right)^{n-2c_1+1/2} e^{-2} \exp \left( \frac{n - 2c_1 + 1}{n - 2c_1} \right)^{e-2c_1+1/2} e^{-1} \].

Under the Pareto1 prior \( g(\lambda) = be^{-\lambda(b+1)} \), \( c, b > 0; \lambda > 0 \) where \( b \) and \( c \) are the known hyper parameters, the posterior distribution for \( \lambda \) is given in (2.2.8)

\[ -nh(\lambda) = (n - b - 1) \log \lambda - \lambda T \quad \Rightarrow \quad \lambda^* = \frac{(n - b - 1)}{\lambda} - T \Rightarrow \hat{\lambda} = \frac{(n - b - 1)}{T} \].

Thus, using (2.3.1) we have

\[ E(\lambda | x) = \left( \frac{n - b - 1}{T} \right) \frac{(n - b)^{1/2} \exp \{ (n - b) \ln \hat{\lambda} - \hat{\lambda} T \}}{(n - b - 1)^{1/2} \exp \{ (n - b - 1) \ln \hat{\lambda} - \hat{\lambda} T \}} \]

\[ \Rightarrow E(\lambda | x) = \left( \frac{n - b - 1}{T} \right) \left( \frac{n - b}{n - b - 1} \right)^{n-b+1/2} e^{-1}. \]  

Note that the relative error is \( \left( \frac{n - b}{n - b - 1} \right)^{n-b+1/2} e^{-1} \).

Similarly, \( E(\lambda^2 | x) \approx \frac{\hat{\phi}^* \exp \{ -nh^*(\lambda^*) \}}{\hat{\phi} \exp \{ -nh(\hat{\lambda}) \}} \).

Here, \( -nh^*(\lambda^*) = \ln(\lambda^*) - nh(\lambda) = (n - b + 1) \ln(\lambda^*) - \hat{\lambda} T \)

\[ E(\lambda^2 | x) = \frac{(n - b - 1)(n - b + 1)}{T^2} \left( \frac{n - b + 1}{n - b - 1} \right)^{n-b+1/2} e^{-2}, \]  

\[ \Rightarrow V(\lambda | x) = \left( \frac{(n - b - 1)(n - b + 1)}{T^2} \right) \left( \frac{n - b + 1}{n - b - 1} \right)^{n-b+1/2} e^{-2} \exp \left( \frac{n - b - 1}{n - b - 1} \right)^{e-2c_1+1/2} e^{-1} \].
Under the Inverse Levy prior \( g(\lambda) = \sqrt{\frac{a}{2\pi}} \lambda^{-\frac{1}{2}} e^{-\frac{a}{2} \lambda} \); \( a > 0; \lambda > 0 \), where \( a \) is the known hyper parameter, thus the posterior distribution for \( \hat{\lambda} \) is given in (2.2.11)

\[
-nh(\lambda) = \left( n - \frac{1}{2} \right) \log \lambda - \lambda \left( T + \frac{a}{2} \right) \;

\Rightarrow

-nh'(\lambda) = \left( n - \frac{1}{2} \right) \frac{1}{\lambda} - \left( T + \frac{a}{2} \right) \;

\Rightarrow \hat{\lambda} = \frac{(n - \frac{1}{2})}{(T + \frac{a}{2})}.
\]

Also, \( -nh''(\hat{\lambda}) = -\frac{(T + \frac{a}{2})^2}{(n - \frac{1}{2})} \).

\[
\therefore \quad \hat{\phi}^2 = \frac{-[nh''(\hat{\lambda})]}{[n(\hat{\lambda})]} = \frac{(n - \frac{1}{2})}{(T + \frac{a}{2})^2} \text{ or } \hat{\phi} = \frac{\sqrt{(n - \frac{1}{2})}}{(T + \frac{a}{2})}.
\]

Now, \( -nh''(\hat{\lambda}^*) = -nh(\hat{\lambda}) + \ln h(\hat{\lambda}) = (n + \frac{1}{2}) \ln(\hat{\lambda}^*) - \hat{\lambda}^* \left( T + \frac{a}{2} \right) \)

\[
\Rightarrow

\hat{\lambda}^* = \frac{(n + \frac{1}{2})}{(T + \frac{a}{2})},
\]

\[
-nh''(\hat{\lambda}^*) = -\frac{(T + \frac{a}{2})^2}{(n + \frac{1}{2})},
\]

\[
\hat{\phi}^2 = \frac{-[nh''(\hat{\lambda}^*)]}{[n(\hat{\lambda}^*)]} = \frac{(n + \frac{1}{2})}{(T + \frac{a}{2})^2} \text{ or } \hat{\phi}^* = \frac{\sqrt{(n + \frac{1}{2})}}{(T + \frac{a}{2})}.
\]

Thus, using (2.3.1) we have

\[
E(\lambda | x) = \frac{(n + \frac{1}{2})}{(T + \frac{a}{2})} \exp \left\{ (n + \frac{1}{2}) \ln \hat{\lambda} - \hat{\lambda} \left( T + \frac{a}{2} \right) \right\}
\]

\[
\Rightarrow \quad E(\lambda | x) = \left( \frac{\sqrt{(n + \frac{1}{2})(n - \frac{1}{2})}}{T + \frac{a}{2}} \right) \left( \frac{n + \frac{1}{2}}{n - \frac{1}{2}} \right)^{n + \frac{1}{2}} e^{-1}.
\]

Note that the relative error is \( \left( \frac{n + \frac{1}{2}}{n - \frac{1}{2}} \right)^{n + \frac{1}{2}} e^{-1} \).

Similarly, \( E(\lambda^2 | x) \equiv \frac{\hat{\phi}^2 \exp \{-nh''(\hat{\lambda}^*)\}}{\hat{\phi} \exp \{-nh'(\hat{\lambda})\}}. \)

Here, \( -nh''(\hat{\lambda}^*) = \ln(\hat{\lambda}^*) - nh(\hat{\lambda}) = (n + \frac{3}{2}) \ln(\hat{\lambda}^*) - \hat{\lambda}^* \left( T + \frac{a}{2} \right) \)

\[
E(\lambda^2 | x) = \frac{\sqrt{(n + \frac{3}{2})(n - \frac{1}{2})}}{T + \frac{a}{2}} \left( \frac{n + \frac{3}{2}}{n - \frac{1}{2}} \right)^{n + \frac{1}{2}} e^{-2},
\]

\[
\Rightarrow \quad V(\lambda | x) = \frac{\sqrt{(n + \frac{3}{2})(n - \frac{1}{2})}}{T + \frac{a}{2}} \left( \frac{n + \frac{3}{2}}{n - \frac{1}{2}} \right)^{n + \frac{1}{2}} e^{-2} \left[ \frac{\sqrt{(n + \frac{1}{2})(n - \frac{1}{2})}}{T + \frac{a}{2}} \left( \frac{n + \frac{1}{2}}{n - \frac{1}{2}} \right)^{n + \frac{1}{2}} e^{-1} \right]^2.
\]
2.4 Exponentiated Inverse Exponential Distribution

Several authors have developed and generalized many standard distributions based on the Exponentiated distributions. Such as, Gupta and Kundu (2001) firstly proposed Exponentiated Exponential distribution. The Exponentiated Gamma, Exponentiated Weibull, Exponentiated Gumbel and Exponentiated Frechet distributions developed and considered by Nadarajah and Kotz (2006). A generalization of the Inverse Exponential distribution called the Exponentiated Inverse Exponential distribution has been introduced by Kawsar and Ahmad (2017).

We first provide general definitions of the Exponentiated Inverse Exponential distribution which will subsequently reveal its pdf and cdf.

**Definition 1:** If \( X \) follows a lifetime distribution with cdf \( G(x) \) and then the cdf of Exponentiated distribution of \( X \) can be defined as:

\[
F(x) = \left[G(x)\right]^\alpha.
\]

**Definition 2:** If \( X \) follows a lifetime distribution with pdf \( g(x) \) and then the pdf of Exponentiated distribution of \( X \) can be defined as:

\[
f(x) = \alpha g(x) \left[G(x)\right]^{\alpha - 1}.
\]

**Theorem 2.4.1:** Let \( X \) be a random variable of an IE distribution with pdf \( g(x) \). Then, pdf of \( X \) is given by:

\[
f(x) = \frac{\alpha \lambda}{x^2} \left\{ \left[ \frac{\lambda}{x} \right]^{-\frac{1}{\alpha}} \right\}^\alpha \quad ; x > 0, \alpha, \lambda > 0.
\]

Where \( \lambda \) and \( \alpha \) are scale and shape parameters respectively.

**Proof:** By definition 2, substitute (2.1.1) and (2.1.2) into (2.4.2), then the pdf for the EIE distribution can be obtained by:

\[
f(x) = \frac{\alpha \lambda}{x^2} e^{-\frac{\lambda}{x}} \left\{ \left[ \frac{\lambda}{x} \right]^{-\frac{1}{\alpha}} \right\}^{\alpha - 1}.
\]

\[
f(x) = \frac{\alpha \lambda}{x^2} \left\{ e^{-\frac{\lambda}{x}} \right\}^\alpha \quad ; x > 0, \alpha, \lambda > 0.
\]

**Theorem 2.4.2:** Let \( X \) be a random variable of an IE distribution with cdf \( G(x) \), then the distribution function of the EIE distribution takes the following form:

\[
F(x) = \left\{ \left[ \frac{\lambda}{x} \right]^{-\frac{1}{\alpha}} \right\}^\alpha \quad ; x > 0, \alpha, \lambda > 0
\]
**Proof:** By definition 1, substitute (2.1.2) into (2.4.1), then the cdf for the EIE distribution can be obtained by:

\[
F(x) = \left\{ \frac{-1}{x} \right\}^\alpha
\]

The graphs of density function and cumulative distribution function are plotted for different values of parameters \( \alpha \) and \( \lambda \) are given in Figure 2.1 and Figure 2.2 respectively.

Figure 2.1 shows that the density function of EIE distribution is positively skewed and the shape is unimodal, for fixed \( \lambda \), it becomes more and more peaked as the value of \( \alpha \) is decreased.

**Theorem 2.4.3:** Let \( X \) be a random variable of the EIE distribution with parameters \( \alpha \) and \( \lambda \), then the survival function of the EIE distribution can be written as:

\[
S(x) = 1 - \left\{ e^{-\frac{x}{\lambda}} \right\}^\alpha \quad ; x > 0, \alpha, \lambda > 0.
\] (2.4.5)

**Proof:** By definition, the survival function of the random variable \( X \) is given by:

\[
S(x) = 1 - F(x)
\]

Using (2.4.4), the survival function of the EIE distribution can be expressed by:

\[
S(x) = 1 - \left\{ e^{-\frac{x}{\lambda}} \right\}^\alpha \quad ; x > 0, \alpha, \lambda > 0
\]
Theorem 2.4.4: Let \( X \) be a random variable of the EIE distribution with parameters \( \alpha \) and \( \lambda \), then the hazard rate of the EIE distribution takes the form:

\[
h(x) = \frac{\alpha \lambda x^{-2} e^{-\frac{x}{\alpha}}}{1 - \left( e^{-\frac{x}{\alpha}} \right)^{\alpha}} \quad ; x > 0, \alpha, \lambda > 0. \tag{2.4.6}
\]

Proof: Let \( X \) be a continuous random variable with pdf and survival function, \( f(x) \) and \( S(x) \), respectively, then the hazard rate is defined by:

\[
h(x) = \frac{f(x)}{S(x)}. \tag{2.4.7}
\]

Substituting (2.4.3) and (2.4.5) into (2.4.7), we obtain:

\[
h(x) = \frac{\alpha \lambda x^{-2} e^{-\frac{x}{\alpha}}}{1 - \left( e^{-\frac{x}{\alpha}} \right)^{\alpha}} \quad ; x > 0, \alpha, \lambda > 0.
\]

The plots for the survival and hazard functions are shown in Figure 2.3 and Figure 2.4 respectively.

We can infer from Figure 2.4 that the shape of the hazard rate is unimodal, it increases at the initial stage and later decreases. Also, we observed that the survival curves show the decreasing rate.
2.5 Relationship with other distributions
Some well-known theoretical distributions can be derived from the proposed EIE distribution. For example;

1) For $\alpha = 1$, Equation (2.4.3) reduces to give the one-parameter Inverse Exponential distribution with probability density function as:

$$f(x) = \frac{\lambda}{x^2} e^{-\frac{\lambda}{x}}; \; x > 0, \lambda > 0.$$  \hspace{1cm} (2.5.1)

2) For $\lambda = 1$, Equation (2.4.3) reduces to give the one-parameter Exponentiated standard inverted exponential distribution with probability density function as:

$$f(x) = \frac{\alpha}{x^2} \left(e^{-\frac{1}{x}}\right)^\alpha; \; x > 0, \alpha > 0.$$  \hspace{1cm} (2.5.2)

3) For $\lambda$ and $\alpha = 1$, Equation (2.4.3) reduces to give the standard inverted exponential distribution with probability density function as:

$$f(x) = \frac{1}{x^2} e^{-\frac{1}{x}}; \; x > 0.$$  \hspace{1cm} (2.5.3)

4) If a random variable is such that $Y = 1/X$ and $\alpha = 1$ in equation (2.4.3) reduces to give the Exponential distribution with pdf as:

$$f(x) = \lambda e^{-\lambda x}; \; x > 0, \lambda > 0.$$  \hspace{1cm} (2.5.4)

5) If a random variable is such that $Y = 1/X$ and $\lambda = 1, \alpha = 1$ in Equation (2.4.3) reduces to give the standard Exponential distribution with pdf as:

$$f(x) = e^{-x}; \; x > 0.$$  \hspace{1cm} (2.5.5)

2.6 Statistical Properties of the EIE Distribution
This section provides some basic statistical properties of the EIE Distribution.

2.6.1 The $r^{th}$ Moment of the EIE Distribution

**Theorem 2.6.1:** If $X \sim \text{EIE}(\alpha, \lambda)$, then $r^{th}$ moment of a continuous random variable $X$ is given as follow:

$$\mu_r' = E(X^r) = \alpha \lambda \Gamma(1-r).$$

**Proof:** Let $X$ is an absolutely continuous non-negative random variable with pdf $f(x)$, then $r^{th}$ moment of $X$ can be obtained by:

$$\mu_r' = E(X^r) = \int_0^\infty x^r f(x) dx.$$
From the pdf of the EIE distribution in (2.4.3), then show that $E(X^r)$ can be written as:

$$E(X^r) = \int_0^\infty x^{r-2} \alpha \lambda \left( e^{-\frac{x}{\alpha \lambda}} \right)^r \, dx,$$

$$\Rightarrow E(X^r) = \alpha \lambda \int_0^\infty \frac{1}{x^r} \left( e^{-\frac{x}{\alpha \lambda}} \right) \, dx.$$

After some calculations,

$$\mu_r = E(X^r) = \alpha' \lambda' \Gamma(1-r). \quad (2.6.1)$$

We observe that Equation (2.6.1) only exists when $r < 1$. The implication is that the first moment, second moment and other higher-order moments does not exist.

### 2.6.2 Harmonic mean of EIE distribution

The harmonic mean (H) is given as:

$$\frac{1}{H} = E\left( \frac{1}{X} \right) = \int_0^\infty \frac{1}{x} f(x) \, dx,$$

$$= \int_0^\infty \frac{\alpha \lambda}{x^{3+1}} e^{-\frac{x}{\alpha \lambda}} \, dx.$$

$$\Rightarrow \frac{1}{H} = \frac{1}{\alpha \lambda}. \Rightarrow H = \alpha \lambda. \quad (2.6.2)$$

### 2.6.3 Mode

Consider the density of the EIE distribution given in (2.4.3), we take the logarithm of (2.4.3) as follows:

$$\log f(x) = \log(\alpha) + \log(\lambda) - 2 \log(x) - \frac{\alpha \lambda}{x}. \quad (2.6.3)$$

Now, \[ \frac{\partial}{\partial x} \log f(x) = -\frac{2}{x} + \frac{\alpha \lambda}{x^2}. \quad (2.6.4) \]

Now, set equation (2.6.4) equal to 0 and solve for $x$, to get

$$x_0 = \frac{\alpha \lambda}{2}. \quad (2.6.5)$$

### 2.6.4 Quantile Function and Median

The Quantile function is given by $Q(u) = F^{-1}(u)$.

Therefore, the corresponding quantile function for the proposed model is given by:
Chapter 2. Classical and Exponentiated Inverse Exponential Distribution

\[ Q(u) = -\frac{\alpha \lambda}{\log(u)}, \quad (2.6.6) \]

where U has the uniform U (0,1) distribution. We obtain the median by substituting \( u=0.5 \). Hence, the median of the proposed model is given by:

\[ F^{-1} = -\frac{\alpha \lambda}{\log(0.5)}. \quad (2.6.7) \]

### 2.6.5 Moment generating function

In this sub section, we derived the moment generating function of EIE distribution.

**Theorem 2.6.2:** Let \( X \) have an EIE distribution. Then the moment generating function of \( X \) denoted by \( M_X(t) \) is given by:

\[ M_X(t) = \sum_{r=0}^{\infty} \frac{t^r}{r!} \alpha^r \lambda^r \Gamma(1-r). \quad (2.6.8) \]

**Proof:** By definition

\[ M_X(t) = E(e^{tx}) = \int_0^{\infty} e^{tx} f(x)dx, \]

using Taylor series expansion

\[ M_X(t) = \sum_{r=0}^{\infty} \frac{t^r}{r!} \int_0^{\infty} x^r f(x)dx \]

\[ \Rightarrow M_X(t) = \sum_{r=0}^{\infty} \frac{t^r}{r!} \int_0^{\infty} x^r f(x)dx \]

\[ \Rightarrow M_X(t) = \sum_{r=0}^{\infty} \frac{t^r}{r!} E(X^r) \]

\[ \Rightarrow M_X(t) = \sum_{r=0}^{\infty} \frac{t^r}{r!} \alpha^r \lambda^r \Gamma(1-r). \]

This completes the proof.

### 2.6.6 Characteristic function

In this sub section, we derived the Characteristic function of EIE distribution.

**Theorem 2.6.3:** Let \( X \) have an EIE distribution. Then the characteristic function of \( X \) denoted by \( \phi_X(t) \) is given by:

\[ \phi_X(t) = \sum_{r=0}^{\infty} \frac{(it)^r}{r!} \alpha^r \lambda^r \Gamma(1-r). \quad (2.6.9) \]
**Proof:** By definition

\[ \phi_X(t) = E(e^{itx}) = \int_0^\infty e^{itx} f(x) dx, \]

Using Taylor series expansion,

\[ \phi_X(t) = \int_0^\infty \left(1 + itx + \frac{(itx)^2}{2!} + \cdots\right) f(x) dx \]

\[ \Rightarrow \quad \phi_X(t) = \sum_{r=0}^\infty \frac{(it)^r}{r!} \int_0^\infty x^r f(x) dx \]

\[ \Rightarrow \quad \phi_X(t) = \sum_{r=0}^\infty \frac{(it)^r}{r!} E(X^r) \]

\[ \Rightarrow \quad \phi_X(t) = \sum_{r=0}^\infty \frac{(it)^r}{r!} \alpha^r \lambda^r \Gamma(1-r). \]

This completes the proof.

### 2.6.7 Order Statistics

In this sub section, we derive closed form expressions for the pdf of the kth order statistic of the EIE distribution. In statistics, the kth order statistic of a statistical sample is equal to its kth smallest value. Together with rank statistics, order statistics are among the most fundamental tools in non-parametric statistics and inference. We know that if \( X_{(1)}, X_{(2)}, \ldots, X_{(n)} \) denotes the order statistics of a random sample \( X_1, X_2, \ldots, X_n \) from a continuous population with cdf \( F_X(x) \) and pdf \( f_X(x) \), then the pdf of kth order statistics \( X_{(k)} \) is given by

\[
f_{X_{(k)}}(x) = \frac{n!}{(k-1)!(n-k)!} f_X(x) (F_X(x))^{k-1} (1 - F_X(x))^{n-k}. \quad (2.6.10)
\]

Substituting equation (2.4.3) and equation (2.4.4) in equation (2.6.10), we get pdf of kth order statistic given as

\[
f_{X_{(k)}}(x) = \frac{n!}{(k-1)!(n-k)!} \frac{\alpha \lambda x^k e^{-\alpha \lambda x}}{x^2} \left(1 - e^{-\alpha \lambda x}\right)^{n-k}. \quad (2.6.11)
\]

Note that at \( \alpha = 1 \), (2.6.11) yields the pdf of the kth order statistic of IE distribution. Therefore, the pdf of the first (smallest) order statistic \( X_{(1)} \) is given by

\[
f_{X_{(1)}}(x) = \frac{n \alpha \lambda x^k e^{-\alpha \lambda x}}{x^2} \left(1 - e^{-\alpha \lambda x}\right)^{n-1}, \quad (2.6.12)
\]
and the pdf of the largest order statistic $X_{(n)}$ is given by

$$f_{X_{(n)}}(x) = \frac{n\alpha\lambda}{x^2} \left( e^{-\alpha\lambda x} \right)^n.$$  \hspace{1cm} (2.6.13)

### 2.7 Estimation of Parameters by using MLE Method

Let $x = (x_1, x_2, \ldots, x_n)$ be a random sample having probability density function (2.4.3), and then the likelihood function is given by

$$L(\alpha, \lambda) = \alpha^n \lambda^n \prod_{i=1}^{n} \frac{1}{x_i^2} \left( e^{-\alpha\lambda x_i} \right)^\alpha.$$  \hspace{1cm} (2.7.1)

$$\ln L(\alpha, \lambda) = n \ln \alpha + n \ln \lambda - 2 \sum_{i=1}^{n} \ln x_i - \sum_{i=1}^{n} \left( \frac{\alpha\lambda}{x_i} \right).$$  \hspace{1cm} (2.7.2)

$$\therefore \frac{\partial}{\partial \alpha} \ln L(\alpha, \lambda) = 0 \quad \Rightarrow \quad \frac{n}{\alpha} - \lambda \sum_{i=1}^{n} x_i^{-1} = 0,$$  \hspace{1cm} (2.7.3)

$$\Rightarrow \quad \hat{\alpha} = \frac{n}{\lambda \sum_{i=1}^{n} x_i^{-1}}.$$  \hspace{1cm} (2.7.4)

$$\therefore \frac{\partial}{\partial \lambda} \ln L(\alpha, \lambda) = 0 \quad \Rightarrow \quad \frac{n}{\lambda} - \alpha \sum_{i=1}^{n} x_i^{-1} = 0,$$  \hspace{1cm} (2.7.5)

$$\Rightarrow \quad \hat{\lambda} = \frac{n}{\alpha \sum_{i=1}^{n} x_i^{-1}}.$$  \hspace{1cm} (2.7.6)

### 2.8 Bayesian Method of Estimation

#### 2.8.1 Posterior Distribution of Exponentiated Inverted Exponential Distribution under Chi-square prior

It is assumed that the prior distribution of $\alpha$ is the Chi-square distribution with known hyper parameter $d$ is given as:

$$g(\alpha) \propto \alpha^{d-1} e^{-\frac{\alpha}{2}}, \quad \alpha > 0, d > 0.$$  \hspace{1cm} (2.8.1)

Combing the prior (2.8.1) and the likelihood function (2.7.1), then the posterior distribution of $\alpha$ is defined by

$$p(\alpha | x) \propto g(\alpha) L(\alpha)$$

$$\Rightarrow \quad p(\alpha | x) = K \alpha^{(n+d/2-1)} e^{-\alpha \sum_{i=1}^{n} x_i^{-1} + (1/2)},$$

$$\Rightarrow \quad \alpha_0 = \frac{1}{\lambda_0} \sum_{i=1}^{n} x_i^{-1}$$

$$\Rightarrow \quad \lambda_0 = \frac{n}{\alpha_0}.$$
The normalizing constant $K$ is determined by the relation
\[
\int_0^\infty p(\alpha \mid x) d\alpha = 1
\]
\[
\Rightarrow K \int_0^\infty \alpha^{(n+d/2-1)} e^{-\alpha(T+1/2)} d\alpha = 1, \text{ where } T = \sum_{i=1}^n x_i^{-1}
\]
Thus, the value of $K$ is given by
\[
K = \frac{(T + 1/2)^{(n+d/2)}}{\Gamma(n + d/2)}.
\]
Hence the posterior distribution of $\alpha$ is given as
\[
p(\alpha \mid x) = \frac{(T + 1/2)^{(n+d/2)}}{\Gamma(n + d/2)} \alpha^{(n+d/2-1)} e^{-\alpha(T+1/2)}.
\] (2.8.2)
$(n + d/2)$ and $(T + 1/2)$ are the parameters of the posterior distribution similar to the gamma distribution $G(T + 1/2, n + d/2)$.

**2.8.2 Baye’s estimators and risk functions under Chi-square prior using KLF, RQLF and ELF**

Under KLF the risk function is given by
\[
R(\hat{\alpha}, \alpha) = \int_0^\infty \frac{(\hat{\alpha} - \alpha)^2}{\hat{\alpha}} \frac{(T + 1/2)^{(n+d/2)}}{\Gamma(n + d/2)} \alpha^{(n+d/2-1)} e^{-\alpha(T+1/2)} d\alpha,
\] (2.8.3)
\[
= \frac{\hat{\alpha}(T + 1/2)}{(n + d/2 - 1)} + \left(\frac{n + d/2}{\hat{\alpha}}\right)^2 - 2.
\] (2.8.4)
\[
\therefore \frac{\partial}{\partial \hat{\alpha}} R(\hat{\alpha}, \alpha) = 0. \Rightarrow \hat{\alpha}_{KLF} = \frac{\sqrt{(n + d/2 - 1)(n + d/2)}}{(T + 1/2)}
\] (2.8.5)

Under RQLF the risk function is given by
\[
R(\hat{\alpha}, \alpha) = \int_0^\infty \left(\frac{\hat{\alpha} - \alpha}{\alpha}\right)^2 \frac{(T + 1/2)^{(n+d/2)}}{\Gamma(n + d/2)} \alpha^{(n+d/2-1)} e^{-\alpha(T+1/2)} d\alpha,
\] (2.8.6)
\[
= \frac{\hat{\alpha}^2(T + 1/2)^2}{(n + d/2 - 1)(n + d/2 - 2)} + 1 - 2 \frac{\hat{\alpha}(T + 1/2)}{(n + d/2 - 1)}.
\] (2.8.7)
\[
\therefore \frac{\partial}{\partial \hat{\alpha}} R(\hat{\alpha}, \alpha) = 0. \Rightarrow \hat{\alpha}_{RQLF} = \frac{(n + d/2 - 2)}{(T + 1/2)}.
\] (2.8.8)

Under ELF the risk function is given by
\[
R(\hat{\alpha}, \alpha) = \int_0^\infty \left[\frac{\hat{\alpha}}{\alpha} - \log\left(\frac{\hat{\alpha}}{\alpha}\right) - 1\right] \frac{(T + 1/2)^{(n+d/2)}}{\Gamma(n + d/2)} \alpha^{(n+d/2-1)} e^{-\alpha(T+1/2)} d\alpha.
\] (2.8.9)
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\[
R(\hat{\alpha}, \alpha) = \frac{1}{\hat{\alpha}} \left( \frac{(\hat{\alpha} - \alpha)^2}{\hat{\alpha}} \right) \frac{(T + 1/a)^{a+b}}{\Gamma(n+b)} \alpha^{(a+b-1)} e^{-\alpha(T+1/a)} d\alpha,
\]

(2.8.13)

\[
\hat{\alpha}_{KLF} = \frac{\sqrt{(n+b)(n+b-1)}}{(T+1/a)}.
\]

(2.8.15)

Under RQLF the risk function is given by

\[
R(\hat{\alpha}, \alpha) = \frac{1}{\hat{\alpha}} \left( \frac{(\hat{\alpha} - \alpha)^2}{\alpha} \right) \frac{(T + 1/a)^{a+b}}{\Gamma(n+b)} \alpha^{(a+b-1)} e^{-\alpha(T+1/a)} d\alpha.
\]

(2.8.16)

2.8.3 Posterior Distribution of EIE Distribution under Erlang Prior

It is assumed that the prior distribution of \( \alpha \) is the Erlang distribution with known hyper parameters \( a \) and \( b \) is given as:

\[ g(\alpha) \propto \alpha^{b-1} e^{-\alpha/a}, \quad \alpha, a, b > 0. \]

The posterior distribution is obtained in a similar way as in case of Chi-square distribution and is given by

\[ p(\alpha \mid x) = K \alpha^{(n+b+1)} e^{-\alpha(T+1/a)}, \]

\[ \Rightarrow K = \frac{(T + 1/a)^{(n+b+1)}}{\Gamma(n+b + 1)}, \text{ where } T = \lambda \sum_{i=1}^{n} x_i^{-1}. \]

Thus, the posterior distribution of \( \alpha \) is given as

\[ p(\alpha \mid x) = \frac{(T + 1/a)^{(n+b)}}{\Gamma(n+b)} \alpha^{(n+b-1)} e^{-\alpha(T+1/a)}. \]

(2.8.12)

\( (n+b) \) and \( (T + 1/a) \) are the parameters of the posterior distribution similar to the gamma distribution \( G(T + 1/a, n + b) \)

2.8.4 Baye’s estimators and risk functions under Erlang prior using KLF, RQLF and ELF

Under KLF the risk function is given by

\[
R(\hat{\alpha}, \alpha) = \frac{1}{\hat{\alpha}} \left( \frac{(\hat{\alpha} - \alpha)^2}{\hat{\alpha}} \right) \frac{(T + 1/a)^{(a+b)}}{\Gamma(n+b)} \alpha^{(a+b-1)} e^{-\alpha(T+1/a)} d\alpha,
\]

(2.8.13)

\[
\hat{\alpha}_{KLF} = \frac{\sqrt{(n+b)(n+b-1)}}{(T+1/a)}.
\]

(2.8.15)
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\[ \hat{\alpha}^2(T + 1/a)^2 + 1 - \frac{2\hat{\alpha}(T + 1/a)}{(n + b - 1)(n + b - 2)} = 0, \]  
\[ \Rightarrow \hat{\alpha}_{RQLF} = \frac{(n + b - 2)}{(T + 1/a)}. \]  

(2.8.17)

Under ELF the risk function is given by

\[ R(\hat{\alpha}, \alpha) = \int_0^{\infty} \left[ \frac{\hat{\alpha} - \alpha}{\alpha} \log \left( \frac{\hat{\alpha}}{\alpha} \right) - 1 \right] \frac{(T + 1/a)^{\alpha(n+b-1)}}{(n+b-1)!} e^{-a(T+1/a)} d\alpha, \]

(2.8.19)

\[ = e \left[ \frac{\hat{\alpha}^2(T + 1/a)}{(n + b - 1)} - \log(\hat{\alpha}) + \frac{\Gamma'(n + b)}{\Gamma(n + b)} - 1 \right]. \]

(2.8.20)

\[ \Rightarrow \hat{\alpha}_{ELF} = \frac{(n + b - 1)}{(T + 1/a)}. \]

(2.8.21)

2.8.5 Posterior Distribution of EIE Distribution under Gamma prior

The gamma prior is defined as \( \alpha \sim \alpha^{-1} e^{-a\alpha} \); where \( l, r > 0 \) are hyperparameters, then the posterior distribution of \( \alpha \) using the gamma prior distribution is given by

\[ p(\alpha | \chi) = \frac{(T + l)^{(n+r)}}{\Gamma(n+r)} \alpha^{(n+r-1)} e^{-a(T+l)}. \]

(2.8.22)

2.8.6 Baye’s estimators and risk functions under Gamma prior using KLF, RQLF and ELF

Under KLF the risk function is given by

\[ R(\hat{\alpha}, \alpha) = \int_0^{\infty} \left( \frac{\hat{\alpha} - \alpha}{\hat{\alpha}} \right)^2 \frac{(T + l)^{(n+r)}}{\Gamma(n+r)} \alpha^{(n+r-1)} e^{-a(T+l)} d\alpha, \]

(2.8.23)

\[ = \hat{\alpha}(T + l) + \frac{(n + r)}{\hat{\alpha}(T + l)} - 2. \]

(2.8.24)

\[ \Rightarrow \hat{\alpha}_{KLF} = \frac{\sqrt{(n + r)(n + r - 1)}}{(T + l)}. \]

(2.8.25)

Under RQLF the risk function is given by

\[ R(\hat{\alpha}, \alpha) = \int_0^{\infty} \left( \frac{\hat{\alpha} - \alpha}{\alpha} \right)^2 \frac{(T + l)^{(n+r)}}{\Gamma(n+r)} \alpha^{(n+r-1)} e^{-a(T+l)} d\alpha, \]

(2.8.26)

\[ = \frac{\hat{\alpha}^2(T + l)^2}{(n + r - 1)(n + r - 2)} + 1 - \frac{2\hat{\alpha}(T + l)}{(n + r - 1)(n + r - 2)}. \]

(2.8.27)

\[ \Rightarrow \hat{\alpha}_{RQLF} = \frac{(n + r - 2)}{(T + l)}. \]

(2.8.28)
Under ELF the risk function is given by
\[
R(\hat{\alpha}, \alpha) = \int_{0}^{\infty} \left[ \frac{\hat{\alpha}}{\alpha} - \log\left( \frac{\hat{\alpha}}{\alpha} \right) - 1 \right] \frac{(T + l)^{(n+r)}}{\Gamma(n + r)} \alpha^{(n+r-1)} e^{-a(T+l)} d\alpha,
\]
(2.8.29)

\[
= c \left[ \frac{\hat{\alpha}^2(T + l)}{(n + r - 1)} - \log(\hat{\alpha}) + \frac{\Gamma(n + r)}{\Gamma(n + r) - 1} \right].
\]
(2.8.30)

\[
\therefore \frac{\partial}{\partial \hat{\alpha}} R(\hat{\alpha}, \alpha) = 0. \quad \Rightarrow \quad \hat{\alpha}_{ELF} = \frac{(n + r - 1)}{(T + l)}.
\]
(2.8.31)

### 2.8.7 Prior Predictive Distribution under Chi-square, Erlang and Gamma priors

The prior predictive distribution is defined as
\[
g(y) = \int_{0}^{\infty} f(y | \alpha) g(\alpha) d\alpha
\]

Under Chi-square prior predictive distribution is
\[
g(y) = \frac{\alpha \lambda}{x^2} e^{-\frac{\lambda}{x}} \frac{1}{2^{d/2} \Gamma(d/2)} \alpha^{d/2-1} e^{-\alpha} d\alpha.
\]
\[
\Rightarrow \quad g(y) = \frac{(d/2)\lambda}{2^{d/2} \chi^2 \left( \frac{\lambda}{x} + \frac{1}{2} \right)}.
\]
(2.8.32)

Under Erlang prior, prior predictive distribution is as
\[
g(y) = \frac{\alpha \lambda}{x^2} e^{-\frac{\lambda}{x}} \frac{1}{a^b \Gamma(b)} \alpha^{b-1} e^{-a} d\alpha.
\]
\[
\Rightarrow \quad g(y) = \frac{b\lambda}{a^b \chi^2 \left( \frac{\lambda}{x} + \frac{1}{a} \right)^{b+1}}.
\]
(2.8.33)

Under gamma prior, prior predictive distribution is obtained as
\[
g(y) = \frac{\alpha \lambda}{x^2} e^{-\frac{\lambda}{x}} \frac{1'}{\Gamma(r)} \alpha^{r-1} e^{-a} d\alpha.
\]
\[
\Rightarrow \quad g(y) = \frac{l'r\lambda}{x^2 \left( \frac{\lambda}{x} + l \right)^{r+1}}.
\]
(2.8.34)

### 2.9 Simulation Study and Real Data Analysis for Inverse exponential distribution

We have generated a sample of size 25, 50 and 100 from Inverse exponential distribution by using R software. To examine the performance of Bayesian estimates for parameter of inverse exponential distribution under different approximation
techniques, estimates are presented along with posterior variances given in parenthesis in the tables 2.1 to 2.2.

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<td>0.32143</td>
</tr>
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<td>(0.00147)</td>
<td>(0.00145)</td>
<td>(0.00146)</td>
<td>(0.00142)</td>
</tr>
</tbody>
</table>
Real life Application: Here, we analyze the real life data set is given by Pavur et al (1992). The results recorded as the following which are the number of revolutions (in the millions) to failure of 23 ball bearings in a life test study.


Table 2.3: Posterior Mean and posterior variance (in parenthesis) under normal approximation based on real data set

<table>
<thead>
<tr>
<th>λ</th>
<th>Quasi Prior</th>
<th>Extension of Jeffrey’s Prior</th>
<th>Pareto 1 Prior</th>
<th>Inverse Levy Prior</th>
</tr>
</thead>
<tbody>
<tr>
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<td>d = 0.2</td>
<td>d = 1.2</td>
<td>b = 0.5</td>
<td>b = 1.5</td>
</tr>
<tr>
<td>0.5</td>
<td>1.83498</td>
<td>1.75450</td>
<td>1.78669</td>
<td>1.62573</td>
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<td>(0.35697)</td>
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<td>(0.34758)</td>
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<td>(0.01964)</td>
<td>(0.01999)</td>
<td>(0.01819)</td>
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<tr>
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<td>0.56449</td>
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<td>0.54964</td>
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<tr>
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<td>(0.00654)</td>
<td>(0.00625)</td>
<td>(0.00637)</td>
<td>(0.00579)</td>
</tr>
</tbody>
</table>

Table 2.4: Posterior Mean and posterior variance (in parenthesis) under T-K approximation based on real data set

<table>
<thead>
<tr>
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<th>Quasi Prior</th>
<th>Extension of Jeffrey’s Prior</th>
<th>Pareto 1 Prior</th>
<th>Inverse Levy Prior</th>
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</thead>
<tbody>
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<td>d = 0.2</td>
<td>d = 1.2</td>
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<tr>
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<tr>
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<td>(0.42191)</td>
<td>(0.40418)</td>
<td>(0.41127)</td>
<td>(0.37582)</td>
</tr>
<tr>
<td>1.5</td>
<td>0.57897</td>
<td>0.55465</td>
<td>0.56438</td>
<td>0.51574</td>
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<tr>
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<td>(0.01837)</td>
<td>(0.01760)</td>
<td>(0.01791)</td>
<td>(0.01636)</td>
</tr>
<tr>
<td>2.5</td>
<td>0.32019</td>
<td>0.30674</td>
<td>0.31212</td>
<td>0.28522</td>
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<td>(0.01108)</td>
<td>(0.01061)</td>
<td>(0.01080)</td>
<td>(0.00987)</td>
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</table>

Results and Discussion for Inverse Exponential distribution

Here our main focus was to examine the importance of Bayesian approximation techniques. We presented approximate to Bayesian integrals of Inverse Exponential distribution depending upon numerical integration and simulation study and showed how to study posterior distribution by means of simulation study. We observe that under informative as well as non-informative priors, the normal
approximation behaves well than T-K approximation, although the posterior variances
in case of T-K approximation are very close to that of normal approximation.

From the tables (2.1, 2.2, 2.3 and 2.4) it can be observed that the large sample
distribution could be improved when prior is taken into account. In both cases normal
approximation as well as T-K approximation, posterior variance under Extension of
Jeffrey’s prior are less as compared to other assumed priors especially the Extension
of Jeffrey’s prior \( c_1 = 1.4 \). Further we conclude that the posterior variance based on
different priors tends to decrease with the increase in sample size. It implies that the
estimators obtained are consistent.

It is observed that the real-life data also confirms to the simulated data results.
Therefore, we conclude that the extension of Jeffery’s prior performs well in the
Inverse Exponential distribution.

2.10 Application and Simulation Study for Exponentiated Inverse exponential
distribution

In this section, we use a real data set to show that the Exponentiated Inverse
Exponential distribution can be a better model than the Exponentiated Inverse
Rayleigh distribution. We consider a data set corresponding to remission times (in
months) of a random sample of 128 bladder cancer patients given in Lee and Wang
(2003). The data set is given as follows: 0.08, 2.09, 2.73, 3.48, 4.87, 6.94, 8.66,
13.11, 23.63, 0.20, 2.22, 3.52, 4.98, 6.99, 9.02, 13.29, 0.40, 2.26, 3.57, 5.06, 7.09,
9.22, 13.80, 25.74, 0.50, 2.46, 3.64, 5.09, 7.26, 9.47, 14.24, 25.82, 0.51, 2.54, 3.70,
5.17, 7.28, 9.74, 14.76, 26.31, 0.81, 2.62, 3.82, 5.32, 7.32, 10.06, 14.77, 32.15, 2.64,
3.88, 5.32, 7.39, 10.34, 14.83, 34.26, 0.90, 2.69, 4.18, 5.34, 7.59, 10.66, 15.96, 36.66,
1.05, 2.69, 4.23, 5.41, 7.62, 10.75, 15.62, 43.01, 1.19, 2.75, 4.26, 5.41, 7.63, 17.12,
46.12, 1.26, 2.83, 4.33, 5.49, 7.66, 11.25, 17.14, 79.05, 1.35, 2.87, 5.62, 7.87, 11.64,
17.36, 1.40, 3.02, 4.34, 5.71, 7.93, 11.79, 18.10, 1.46, 4.40, 5.85, 8.26, 11.98, 19.13,
1.76, 3.25, 4.50, 6.25, 8.37, 12.02, 2.02, 3.31, 4.51, 6.54, 8.53, 12.03, 20.28, 2.02,
3.36, 6.93, 8.65, 12.63 and 22.69. These data are used here only for illustrative
purposes. The required numerical evaluations are carried out using R software.

We have fitted Exponentiated Inverse Rayleigh (EIR) and Exponentiated
Inverse Exponential (EIE) models to this data. These two distributions are fitted to the
subject data using maximum likelihood estimation. The MLEs of the parameters with
Chapter 2. Classical and Exponentiated Inverse Exponential Distribution

standard errors in parentheses and the corresponding log-likelihood values, AIC, AICC and BIC are displayed in Table 2.5.

Table 2.5: MLEs (S.E in parentheses) and Criteria for Comparison

<table>
<thead>
<tr>
<th>Distribution</th>
<th>Parameter Estimates</th>
<th>-2LogL</th>
<th>AIC</th>
<th>AICC</th>
<th>BIC</th>
</tr>
</thead>
<tbody>
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<td></td>
<td>( \alpha )</td>
<td>( \lambda )</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>EIE</td>
<td>1.66680 (117.611)</td>
<td>1.51083 (106.605)</td>
<td>883.417</td>
<td>887.4167</td>
<td>893.2168</td>
</tr>
<tr>
<td>EIR</td>
<td>0.81073 (28.8669)</td>
<td>0.76225 (27.1406)</td>
<td>1497.54</td>
<td>1501.54</td>
<td>1507.34</td>
</tr>
</tbody>
</table>

Plot of the fitted densities for the Cancer Patients Data

2.10.2 Simulation Study for Exponentiated Inverse Exponential distribution

In the simulation study, three data sets of size 25, 50 and 100 have been generated from R software to examine the performance of Classical and Bayesian estimates for the shape parameter \( \alpha \) of EIE distribution under different priors using different loss functions. The value of the shape parameter \( \alpha \) is 0.5, 1.0 and 1.5 and the value of the scale parameter \( \lambda \) is 0.5. The data sets are obtained by using the inverse
cdf method as discussed in section (2.6.4) and the summary of results is presented in the tables 2.6 to 2.8 given below:

Table 2.6: Baye’s estimators and MSE (in parenthesis) of $\hat{\alpha}$ under Chi-square prior

<table>
<thead>
<tr>
<th>n</th>
<th>$\lambda$</th>
<th>$\alpha$</th>
<th>$m$</th>
<th>$\hat{\alpha}_{ML}$</th>
<th>$\hat{\alpha}_{KLF}$</th>
<th>$\hat{\alpha}_{RQLF}$</th>
<th>$\hat{\alpha}_{ELF}$</th>
</tr>
</thead>
<tbody>
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<td>0.4</td>
<td></td>
<td>0.59522 (0.02324)</td>
<td>0.58104 (0.02052)</td>
<td>0.54586 (0.01605)</td>
<td>0.56939 (0.01876)</td>
</tr>
<tr>
<td></td>
<td></td>
<td>1.4</td>
<td></td>
<td>0.59522 (0.02324)</td>
<td>0.59280 (0.02284)</td>
<td>0.55763 (0.01755)</td>
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<tr>
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<td>1.14856 (0.07658)</td>
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</tr>
<tr>
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<td>1.10228 (0.06605)</td>
<td>1.14879 (0.07773)</td>
</tr>
<tr>
<td></td>
<td>1.5</td>
<td>0.4</td>
<td></td>
<td>1.78565 (0.20914)</td>
<td>1.70304 (0.16107)</td>
<td>1.59994 (0.12984)</td>
<td>1.66891 (0.14837)</td>
</tr>
<tr>
<td></td>
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<td>1.73753 (0.17864)</td>
<td>1.63443 (0.14029)</td>
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<tr>
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<td>0.54647 (0.00861)</td>
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<tr>
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### Table 2.7: Baye’s estimators and MSE (in parenthesis) of $\hat{\alpha}$ under Erlang prior

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<th>$\hat{\alpha}_{RQLF}$</th>
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<td>1.70706 (0.07201)</td>
<td>1.69839 (0.06811)</td>
<td>1.67314 (0.05873)</td>
<td>1.68998 (0.06485)</td>
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</table>
Table 2.8: Baye’s estimators and MSE (in parenthesis) of $\hat{\alpha}$ under Gamma prior

<table>
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<tr>
<th>$n$</th>
<th>$\lambda$</th>
<th>$\alpha$</th>
<th>$l = r$</th>
<th>$\hat{\alpha}_{ML}$</th>
<th>$\hat{\alpha}_{KLF}$</th>
<th>$\hat{\alpha}_{RQLF}$</th>
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<td>0.2</td>
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<td>0.58517 (0.02140)</td>
<td>0.54974 (0.01662)</td>
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<td>0.59477 (0.02302)</td>
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<td>0.2</td>
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<td>1.07135 (0.06115)</td>
<td>1.00649 (0.05611)</td>
<td>1.04987 (0.05855)</td>
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<td>1.06839 (0.05783)</td>
<td>1.00622 (0.05319)</td>
<td>1.04780 (0.05544)</td>
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<td>0.2</td>
<td>1.78565 (0.20914)</td>
<td>1.73902 (0.18209)</td>
<td>1.63374 (0.14285)</td>
<td>1.70416 (0.16665)</td>
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<td>0.56064 (0.00691)</td>
<td>0.56629 (0.00763)</td>
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<td>0.2</td>
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<td>1.13203 (0.03035)</td>
<td>1.11501 (0.02615)</td>
<td>1.12637 (0.02888)</td>
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<td>1.67063 (0.05811)</td>
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<td>1.68448 (0.06235)</td>
<td>1.65941 (0.05373)</td>
<td>1.67614 (0.05934)</td>
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</tbody>
</table>
2.10.3 Results and Discussion for Exponentiated Inverse Exponential Distribution

On examining the summary statistics of the data set which are fitted to the Exponentiated inverse exponential distribution and Exponentiated inverse Rayleigh distribution using the Akaike information criterion (AIC), corrected Akaike information criterion (AICC) and the Bayesian information criterion (BIC). The best distribution corresponds to lower values of \(-2\log L, \text{AIC}, \text{BIC}, \text{AICC}\) value.

The results presented in table 2.5 shows that Exponentiated inverse exponential distribution provides adequate fit to the data among the models considered.

We consider the Bayesian analysis of the Exponentiated inverse exponential distribution using different informative priors. After analysis we conclude that Bayesian method of estimation is better than classical method of estimation. By comparing the results of our study, we observe that the Relative quadratic loss function has the least MSE under all the priors in the simulation study. Thus we can say that Relative quadratic loss function is better than other loss functions. It is also observed that among all the priors, the Erlang prior provides the Bayes estimators with least MSE.