Chapter 7

A Genetic Algorithm Approach for Fuzzy Goal Programming Formulation of Chance Constrained Problems Using Stochastic Simulation

7.1 Introduction

The two types of prominent approaches for solving decision making problems in an uncertain environment are the stochastic programming (SP) and fuzzy programming (FP). In the case of SP, inexactness of the parameter values is inherent to problem description in the decision making situations.

On the other hand, imprecision in data description is inherent due to inexactness of human judgments.

The field of SP based on the probability theory, initially introduced by Charnes and Cooper [73] as the chance constrained programming (CCP), has been studied [76, 128, 332] extensively in the past and applied to several real-life problems in [57, 103].

The study on single objective SP problems made in the past has been surveyed by Luhandjula [308] in 2006.

Now considering the multiobjective in nature of most of the real-life decision problems, various approaches including the Goal Programming (GP) approach [206] as a prominent tool to solve MODM problems have been studied [226] to solve SP problems with multiplicity of objectives in the past.

However, in most of the studies [88, 197], the CCP problems with known distribution of the stochastic parameters are converted into their deterministic equivalents in using the well established crisp approaches to solve the problems. But, with the realization of the fact that there are different types of distribution functions involved with the real-world CCP problems in which deterministic equivalent become computationally expensive and computational complexity often arises [299] in actual practice.

To overcome the above difficulty, the stochastic simulation [243, 392] based approaches to various versions of CCP problems have been studied [290, 296] in the past.

Now, another type of prominent approach for solving decision problems in an imprecise environment is the FP approach based on the theory of fuzzy sets introduced by Zadeh [515] in 1965.
Several FP approaches [527] including FGP approach [484] within the framework of conventional GP [206, 279] have been studied extensively in the past and applied to different real-life problems in [47, 352].

Now, in the real-life decision situations, it has been realized that both the probabilistic and fuzzy data would have to be simultaneously taken into account to make more realistic decision in an inexact decision making environment. The study on the area of CCP has been made by the pioneer researchers [297, 303, 336] in the past. The use of GAs to stochastic decision problems by using stochastic simulation [289] as well as consideration of deterministic equivalent of them in [229] has been studied [230] in the recent past.

Here, it may be mentioned that the consideration of both the aspects discussed previously to MODM problems is a great challenge to the researchers in this field and the methodological development of it is at an early stage from the viewpoint of their implementation to the real-world complex decision problems.

In this article, achievement of fuzzily described objectives of MODM problems under a system of chance constraints is considered. In the model formulation of the problem, instead of converting the constraints set to the deterministic one, a stochastic simulation to the defined goals and with certain probability distributions of the chance parameters is introduced in the solution search process.

In the solution process, an GA scheme is employed to reach a satisfactory decision by minimizing the under-deviational variables of the defined membership goals of the fuzzy objective goals to the highest degree (unity) to the extent possible on the basis of the weight of importance of achieving the goals is taken into consideration in the decision making environment.

A numerical example is solved and the model solution of proposed FGP framework is compared with the solution of the conventional FP approach to expound the potential use of the method presented here in the decision making context.
7.2 Problem Formulation

The generic form of a chance constrained MODM problem can be presented as:

Find \( X(x_1, x_2, x_3) \) so as to

Maximize \( Z_k(X), \quad k \in K_1 \)

and Minimize \( Z_k(X), \quad k \in K_2 \)

subject to

\[
X \in S = \left\{ X \in \mathbb{R}^n \mid \text{Prob}[A X \leq b] \geq \beta, X \geq 0, b \in \mathbb{R}^m \right\}
\]

\[
L \leq X \leq U,
\]

where \( X \) is a vector of deterministically defined decision variables, \( L \) and \( U \) denote upper- and lower- bounds, respectively, of the vector \( X \), and dimension of them depends on the dimension of \( X \), ‘Prob’ designates the probabilistically defined constraints, \( A \) is an \((m \times n)\) real coefficient matrix, \( b \) is a resource vector, and where \( \beta (0 < \beta < 1) \) represents the vector of confidence level of probabilities for satisficing the defined constraints.

It is assumed that the feasible region \( S(\neq \Phi) \) is bounded, and where \( K_1 \cup K_2 = \{1, 2, \ldots, K\} \) with \( K_1 \cap K_2 = \emptyset \).

Now, in the present problem formulation, it is assumed that the coefficient vectors as well as the resource vector are independently random in nature and follow different continuous probability distributions in the decision making context.

Then, the stochastic simulation approach to search the solutions subject to the defined chance constraints is presented in the following Section 7.2.1.

7.2.1 Stochastic Simulation for Solution Estimation

The generation of random numbers for simulation run and thereby estimation of probability for feasible solutions have been well documented by Rubenstein [417], and widely used in [296] in the context of making decision under chance constraints.

Now, the constraints in (7.1) can be explicitly stated as
Then, the simulation process can be defined as follows:

Let, \( \text{Prob}\left[\sum_{j=i}^{n} a_j x_j \leq b_i \right] \geq p_{i}, \quad i=1,2,\ldots,m. \quad (7.2) \)

where, \( \mathbf{r} = (r_1, r_2, \ldots, r_{n+1}) \) is the \((n+1)\) dimensional random vector, and where \( r_j = (a_{j1}, a_{j2}, \ldots, a_{jm}), j = 1,2,\ldots,n \) and \( r_{n+1} = (b_1, b_2, \ldots, b_n) \).

Then, for a given \( X \), \( P \) independent random vectors are generated in such a way that

\[
\mathbf{r}^{(p)} = (r_{1}^{(p)}, r_{2}^{(p)}, \ldots, r_{n+1}^{(p)}), \quad p=1,2,\ldots,P.
\]

for the given distributions of the defined vectors of random variables.

Then, let \( P' \) be the number of occasions on which

\[
g_i(x, \mathbf{r}^{(p)}) \leq 0, \quad i=1,2,\ldots,m; \quad p=1,2,\ldots,P.
\]

are satisfied.

Then, the probability of occurrence of constraint satisfaction is defined by

\[
V = \frac{P'}{P} \quad (7.3)
\]

The simulation process is summarized in the following steps:

Step 1. Initialize \( P' = 0 \).

Step 2. Generate the vectors of random numbers according to their defined distribution functions.

Step 3. If the constraints \( g_i(x, \mathbf{r}) \leq 0, \ i = 1, 2, \ldots, m \)

then set \( P' = P' + 1 \)

Step 4. Repeat the steps 2 and 3 \( P \) times

Step 5. Compute \( V = \frac{P'}{P} \).

Now, how the proposed simulation process works for feasibility verification of a candidate solution (a chromosome) in a genetic search process is briefly described as follows [289]:

\[
\text{Prob}\left[\sum_{j=i}^{n} a_j x_j \leq b_i \right] \geq p_{i}, \quad i=1,2,\ldots,m. \quad (7.2)
\]
7.2.2 Use of Stochastic Simulation for GA Scheme

- Determine the initial candidate solutions (the chromosome)
- Verify the feasibility criteria using the stochastic simulation and determine the fitness scores (the objective function values)
- Use the sampling mechanism in GA for parent selections and update the chromosomes (offspring) using the genetic operators.
- Repeat the process through the feasibility testing until the stopping criteria (termination condition for GA search process) is reached.

Now, the GA scheme adopted in process of FP formulation of the problem in (7.1) and thereby solving the executable GP model of the problem is presented in the following Section 7.3.

7.3 Design of the GA Scheme

The algorithm steps in the genetic search process adopted are presented as follows:

Step 1: Representation and initialization

Let $V_L$ denotes the binary coded representation of chromosome in a population as $V_L = \{x_1, x_2, ..., x_n\}_L$, $L=1, 2, ..., \text{pop}_\text{size}$, the population size \text{pop}_\text{size} number of chromosomes are randomly initialized in its search domain.

Step 2: Feasibility Check

The proposed stochastic simulation approach presented in the Section 7.2.2 is used to check the feasibility of the candidate solution.

Step 3: Fitness function

The fitness value of each chromosome is determined by the value of an objective function. The fitness function is defined as

$$\text{eval}(V_L) = (Z_K)_k, \quad k \in K_1$$

(maximization)

or

$$\text{eval}(V_L) = (Z_K)_k, \quad k \in K_2$$

(minimization) \hspace{1cm} (7.4)

where $Z_K$ represents the $k$-th objective function value.

The best and least values objective function values of the fittest chromosome can be presented as:
\[ V^* = \max \{ \text{eval} (V_L) \mid L = 1, 2, \ldots, \text{pop}_\text{size} \}, \]
or \[ V^* = \min \{ \text{eval}(V_L) \mid L = 1, 2, \ldots, \text{pop}_\text{size} \}, \]
which depends on the needs and desires of the DM in the decision making situation.

Step 4: Selection

The simple roulette-wheel scheme [164] is used for selecting two parents for mating purposes in the genetic search process.

Step 5. Crossover

The parameter \( P_c \) is defined as the probability of crossover. The arithmetic crossover operator (1-point crossover) of a genetic system is applied here in the sense that the resulting offspring always satisfy the linear constraints set \( S \). Here a chromosome is selected as a parent, if for a defined random number \( r \in [0, 1] \), \( r < P_c \) is satisfied.

For example, arithmetic crossover for two parents \( V_1, V_2 \in S \) yields two offspring

\[ E_1 = \alpha_1 V_1 + \alpha_2 V_2, \quad E_2 = \alpha_2 V_1 + \alpha_1 V_2, \]

where \( \alpha_1, \alpha_2 \geq 0 \) with \( \alpha_1 + \alpha_2 = 1 \), always belong to \( S \) and where \( S \) is a convex set.

Step 6. Mutation

As in the conventional scheme, a parameter \( P_m \) of the genetic system is defined as the probability of mutation. The mutation operation is performed on a bit-by-bit basis, where for a Random Number \( r \in [0, 1] \), a chromosome is selected for mutation provided that \( r < P_m \).

Step 6. Termination

The execution of whole process terminates when the number of iterations is reached to the generation number specified in the genetic process. The best-generated chromosome is reported finally in the genetic search process.

Now, representation of each type of the objectives in (7.1) in fuzzy versions of them and thereby FGP formulation of the problem by using the proposed GA scheme is presented in the following Section 7.4.
7.4 FGP Formulation

In the field of FP, an imprecise aspiration level is assigned to each of the objectives and certain tolerance limit for achievement of aspired level of each of them is taken into consideration.

Let $(X_{Bk}', Z_{Bk}')$ and $(X_{Wk}', Z_{Wk}')$ be the best and worst solutions for maximization of the $k$-th objective $Z_k$, $k \in K_1$, subjects to the constraints in (7.1) where B and W are used to indicate the best and worst solutions, respectively. Again, let $(X_{Bk}', Z_{Bk}')$ and $(X_{Wk}', Z_{Wk}')$ be the best and worst solutions respectively for minimization of the $k$-th objective $Z_k$, $k \in K_2$.

Then, the fuzzy goals of the problem appear as:

$$Z_k \geq Z_{Bk}', \quad k \in K_1,$$

and

$$Z_k \leq Z_{Bk}', \quad k \in K_2$$

with the lower and upper tolerance limits $Z_{ma}(k \in K_1)$, and $Z_{wa}(k \in K_1)$, respectively, where $\geq$ and $\leq$ indicates the fuzzy version of $\geq$ and $\leq$ respectively.

Now, in the field of FP, the fuzzy goals are characterized by their respective membership functions.

7.4.1 Characterization of Membership Function

Let $\mu_k(X)$ be the membership function representation of the $k$-th fuzzy goal.

Then, for $\geq$ type of restriction, $\mu_k(X)$ takes the form

$$\mu_k(X) = \begin{cases} 
1 & \text{if } Z_k(X) \geq Z_{Bk}' \\
\frac{Z_k(X) - Z_{wk}'}{t_{ik}} & \text{if } Z_{wk}' \leq Z_k(X) < Z_{Bk}' \\
0 & \text{if } Z_k(X) < Z_{wk}'
\end{cases}$$

(7.6)

Where, $t_{ik} = (Z_{Bk}' - Z_{Wk}')$ is the tolerance range for achievement of the $k$-th fuzzy goal, $k \in K_1$. 

Similarly, for $\leq$ type of restriction, $\mu_k(X)$ appear as

$$
\mu_k(X) = \begin{cases} 
1 & \text{if } Z_k(X) \leq Z_{bk}^*, \\
\frac{Z_{wk}^* - Z_k(X)}{t_{2k}} & \text{if } Z_{bk}^* < Z_k(X) \leq Z_{wk}^* \\
0 & \text{if } Z_k(X) > Z_{wk}^*,
\end{cases}
$$

(7.7)

Where, $t_{2k} = (Z_{wk}^* - Z_{bk}^*)$ is the tolerance range for achievement of the $k$-th fuzzy goal, $k \in K_1$.

The FGP model of the problem is presented in the following Section 7.4.2.

### 7.4.2 FGP Model Formulation

In FGP model formulation, the defined membership functions are converted into the membership goals by introducing the under- and over- deviational variables and assigning unity as the aspiration level to each of them. In the 'goal achievement function', under-deviational variables on the basis of weights of importance of achieving the goals are taken into consideration. Such a version of FGP called the minsum FGP is the simplest and widely used approach [376] to solve fuzzy MODM problems.

Then, the standard minsum FGP formulation appears as:

Find $X$ so as to:

Minimize 

$$
Z = \sum_{k=1}^{K} w_k^- d_k^- 
$$

and satisfy

$$
\frac{Z_k(X) - Z_{wk}^*}{t_{1k}} + d_k^- - d_k^+ = 1, \quad k = 1,2,...,k_1
$$

$$
\frac{Z_{wk}^* - Z_k(X)}{t_{2k}} + d_k^- - d_k^+ = 1, \quad k = (k_1+1, k_1+2,...,K)
$$

(7.8)

subject to the system constraints in (7.1) and (7.2), where $Z$ represents the fuzzy goal achievement function, $d_k^- (\geq 0)$ and $d_k^+ (\geq 0)$ represents the under- and over- deviational variables associated with the $k$-th membership goal, $w_k (\geq 0), k = 1,2,..., K$, represent the numerical weights associated with the respective under deviational variables, and where $w_k^-, k = 1,2,..., K$, are determined as [376]:
Now, the fitness function in using the GA scheme to the problem (7.8) is defined in the following Section.

### 7.4.3 Definition of Fitness Function

The goal achievement function $Z$ in (7.8) appears as the fitness function in the evaluation process of using the GA scheme. The evaluation function for measuring the fitness of a chromosome can be presented as:

\[
eval(V_L) = (Z)_L = (\sum_{k=1}^{K} w_k^* d_k^*)_L,
\]

where, $k = 1, 2, \ldots, K$ and $L = 1, 2, \ldots, \text{pop\_size}$.

(7.9)

Here, the chromosome $V^*$ with the best fitness value at each generation is defined as:

\[
V^* = \min \{ \eval(V_L) | L = 1, 2, \ldots, \text{pop\_size} \},
\]

in the genetic search process.

To illustrate the proposed approach, a numerical example is solved.

### 7.5 Numerical Illustration

To illustrate the use of the proposed procedure, the following MOCCP problem having two objective functions are considered.

Find $X(x_1, x_2, x_3)$ so as to

- Maximize $Z_1 = 4x_1 + 3x_2 + 5x_3$ (7.10)
- Minimize $Z_2 = 5x_1 + 4x_2 + x_3$ (7.11)

subject to,

- $\Prob[a_{11}x_1 + a_{12}x_2 + a_{13}x_3 \leq b_1] \geq 0.95$ (7.12)
- $\Prob[a_{21}x_1 + a_{22}x_2 + a_{23}x_3 \leq b_2] \geq 0.8$ (7.13)
- $\Prob[a_{31}x_1 + a_{32}x_2 + a_{33}x_3 \leq b_3] \geq 0.9$ (7.14)
- $1 \leq x_i \leq 2$, $i = 1, 2, 3$. (7.15)
In the decision situation, the distribution of parameters of the problem and their values are defined as follows:

(i) \( a_{11}, a_{12}, a_{13}, b_1 \) follow beta distribution with parameters (1,50), (2,100), (4,90), (8, 5), respectively.

(ii) \( a_{21}, a_{22}, a_{23}, b_2 \) follow gamma distribution with parameters (2, 0.01), (2,1), (3, 0.1), (3, 5) respectively.

(iii) \( a_{31}, a_{32}, a_{33}, b_3 \) follow Weibull distribution with parameters (2,3), (3, 2), (3, 1), (5, 30), respectively.

Now, to formulate the fuzzy goals of the objectives of the problem and their respective tolerance limits, the proposed GA is coded in Programming Language C, and the values \( \text{pop\_size} = 100, \ p_c = 0.8, \ p_m = 0.08 \) and the generation number = 300 are adopted in the genetic search process.

The obtained best and least values of the objectives are presented in the Table 7.1.

<table>
<thead>
<tr>
<th>Best value</th>
<th>((x_1, x_2, x_3))</th>
<th>Least value</th>
<th>((x_1, x_2, x_3))</th>
</tr>
</thead>
<tbody>
<tr>
<td>(Z_1^B = 36)</td>
<td>(3.0, 3.0, 3.0)</td>
<td>(Z_1^W = 12)</td>
<td>(1.0, 1.0, 1.0)</td>
</tr>
<tr>
<td>(Z_2^B = 10)</td>
<td>(1.0, 1.0, 1.0)</td>
<td>(Z_2^W = 30)</td>
<td>(3.0, 3.0, 3.0)</td>
</tr>
</tbody>
</table>

Now, based on the numerical values presented in the Table 1, the fuzzy goals can be constructed as

\[ Z_1 \geq 36, \quad \text{and} \quad Z_2 \leq 10. \]

The membership functions are then found as:

\[ \mu_1(x) = \frac{4x_1 + 3x_2 + 5x_3 - 12}{24}, \quad (7.16) \]

\[ \mu_2(x) = \frac{30 - (5x_1 + 4x_2 + x_3)}{20}, \quad (7.17) \]
Now, following the procedure, the resulting FGP model of the problem appears as:

\[ \text{Find } \mathbf{X}(x_1, x_2, x_3) \text{ so as to} \]

\[ \text{Minimize } Z = 0.04167 d_1^- + 0.05d_2^- \]

and satisfy

\[ 0.04167 (4x_1 + 3x_2 + 5x_3) - 0.5 + d_1^- - d_1^+ = 1 \]
\[ 1.5 - 0.05(5x_1 + 4x_2 + x_3) + d_2^- - d_2^+ = 1. \]  \hspace{1cm} (7.18)

subject to the constraints in (7.12) - (7.15).

Now, employing the proposed stochastic simulation based GA method by addressing the fitness function defined in (7.18) and using the same genetic parameters set discussed previously, the model solution is obtained as

\[ (x_1, x_2, x_3) = (1.0, 1.0, 3.0) \text{ with } (Z_1, Z_2) = (22.0, 12.0). \]

The achieved membership values are

\[ (\mu_1, \mu_2) = (0.41, 0.90). \]

The result shows that a satisfactory decision is achieved here in the decision environment.

Note 7.1: In the decision making context, if the max-min fuzzy operator \([523]\) is used to solve the problem of optimizing the defined membership functions of the fuzzy goals, where the objective is to maximize \(\lambda\) (say), subject to \(\lambda\) 'less than or equal to' each of the defined membership functions with \((0 \leq \lambda \leq 1)\) and also the given system constraints then in the same decision environment the solution of the problem is obtained as:

\[ (x_1, x_2, x_3) = (1.8529, 1.5285, 1.6345) \text{ with } (Z_1, Z_2) = (20.1696, 17.013). \]

A diagrammatic representation of the results obtained under the above two approaches is shown in the following Figure 7.1.
The results show that a better decision is obtained here under the proposed GA approach from the viewpoint of optimizing both the objective functions $Z_1$ and $Z_2$ in the decision making environment.

**Note 7.2:** It is to be noted that when the random parameters of the chance constraints follow normal distribution, then deterministic equivalent of the constraints can easily be obtained by using the means and variances of the of the parameter values which also appear with the characteristic of normal distribution. In such a case, the conventional SP can be efficiently used to solve the problem.

But, in a decision situation, when the parameters follow any other distribution, the corresponding means and variances may not always agree with the distribution pattern followed by the random parameters. As a matter of fact, computational complexity arises there to convert the chance constraints into their deterministic equivalent in the decision making process. Here, proposed stochastic simulation as the ‘trial and error’ approach is the most fruitful to employ to any type of probability distribution followed by the random parameters of the problem.
7.6 Conclusion

In this paper, how the stochastic simulation based GA can be efficiently used for solving fuzzily described MODM problems under chance constraints with different continuously distributed random parameters are presented.

The main advantage of the approach is that the computational complexity in converting the chance constraints with different distribution of the parameters to deterministic equivalent does not involve here due to the use of stochastic simulation in the solution search process. Again, since the objective of a MODM problem often conflict each other for goal achievement, the use of the GA method as GA is a goal satisficer rather than objective optimizer, to the FGP formulation of the problem always provides a satisfactory decision and the proposed approach may be claimed as a superior one over the conventional crisp / fuzzy approaches for solving CCP problems.

Further, the proposed approach can easily be extended to solve non-linear including fractional MODM problems without involving linearization of the objectives as employed in the conventional approaches [112, 384] to the decision problems in both the crisp and uncertain environment.

An extension of the approach to different chance constrained hierarchical decision making problems may be a problem for future research. Finally, it is expected that the approach presented here may lead to future research for its implementation to real-life MODM problems in the current complex uncertain decision making environment.