Chapter 2

Related Concepts and Literature Review

2.1 Uncertainty and Decision Making

Uncertainty must be taken in a sense radically distinct from the familiar notion of risk, from which it has never been properly separated. The essential fact is that 'risk' means in some cases a quantity susceptible of measurement, while at other times it is something distinctly not of this character; and there are far-reaching and crucial differences in the bearings of the phenomena depending on which of the two is really present and operating. It will appear that a measurable uncertainty, or 'risk' proper, but, so far it’s a different term from an immeasurable one which is not in effect an uncertainty at all.

Uncertainty has been a part of our everyday life. The study of uncertainty can be dated back to as early as sixteenth century when Gerolamo Carnado analyzed ‘Games of Chance’. In mid seventeenth century, Blaise Pascal’s Wager is also one of the earliest instances to use uncertainty and decision theory which is also marked the evolution of probability theory. Statistical models are the first models of uncertainty in the literature. Uncertainty must be taken in a sense radically distinct from the familiar notion of risk, from which it has never been properly separated. The essential fact is that 'risk' means in some cases a quantity susceptible of measurement, while at other times it is something distinctly not of this character; and there are far-reaching and crucial differences in the bearings of the phenomena depending on which of the two is really present and operating. It will appear that a measurable uncertainty, or 'risk' proper, but, so far it’s a different term from an immeasurable one which is not in effect an uncertainty at all. The taxonomy for decision making under uncertainty is shown in Fig. 2.1.
Decision making is a cognitive process to identify or choose a final choice among several alternative possibilities based on the situation, values and preferences of the decision maker. Need of taking decisions in every aspect of life is inabatable. Decision making refers to a single decision maker by default. Whereas, sometimes it is not possible to take a particular decision for a single person. The process of taking a decision by multiple decision makers is called as group decision making. Some decisions are based on multiple criteria. The decision making which spreads to multiple criteria is referred as multiple criteria decision making or multiple criteria analysis. Soft set can play a big role in MCDM because of it’s topological nature. Each criteria can be taken as a parameter of a soft set. There are many algorithms to take decision using soft set.

2.2 Some Basic Models to Handle Uncertainty

In this section, some basic models to handle uncertainty are discussed. There are several models in literature to handle uncertainty and vague data. Some such models are: Probability Theory, Interval Mathematics, Neural Network, Fuzzy Set, Intuitionistic Fuzzy Set, Rough Set, Soft Set and many more hybrid models using these basic uncertainty models.
2.2.1 Probability Theory

The first model introduced to deal with uncertainty and imprecision is probability theory, proposed by Blaise Pascal in the 17th century. In the early parts of the development it was mostly associated with gambling. However, after the introduction of axioms of probability, it has got its rigour and is used in many branches of science as a model of uncertainty. This mathematical tool is used to solve non-deterministic events and randomness. The basic definition of probability of occurrence of an event requires that the experiment (random) is to be repeated sufficiently large number of times under similar conditions. It is used to analyze large sets of data which will be very difficult for the traditional computing to do so. The main objective of this model is that it involves human interaction activity which is needed for solving complex problems.

2.2.2 Artificial Neural Networks

A Neural Network is an information processing paradigm that is inspired by the way biological nervous systems, such as the brain, process information. The main objective of neural networks is to design the artificial brain. These networks are mostly connected with neurons which will be used to transfer data from one to another. The connections are associated with weights. These weights actually contain information about the input signals. There are two phases for a neural network before it can be used. These are the training and the testing phases. In the training phase, a set of input, output pairs are taken and until appropriate outputs are obtained the weights are adjusted. After this phase, in the testing one, another set of input, output pairs are taken and the accuracy of the network is tested. After these two phases the network becomes ready to use. The complexity of the system increases and the efficiency also increases when besides the input and output layers, a set of layers called the hidden layers are added in between. A set of functions, called the activation functions are used to transform the input to the net to generate the outputs.

2.2.3 Fuzzy Set

Zadeh (1965) introduced the notion of fuzzy sets as an extension to the classical notion of sets introduced by Gorge Cantor. Fuzzy set theory can be used in a wide range of domains such as bioinformatics in which information is incomplete or
imprecise. The elements of fuzzy sets are having degrees of membership unlike classical sets. In classical set an element can either be present or not present in a particular set. Whereas, in fuzzy sets an element may have the partial membership. A fuzzy set $A$ is defined through a function $\mu_A$, called its membership function.

Perhaps the most fruitful model of uncertainty is the fuzzy set and its variants. The membership of elements into a set in crisp set theory is dichotomous by nature; that is an element may belong to a set or may not belong to it. However, it was observed that, this convention leads to trouble when modeling many real life situations like; characteristics of age such as very young, young, middle-aged, old and very old. The solution to this problem was provided by the notion of graded membership of elements introduced by Zadeh and the model capturing this concept being called as Fuzzy set.

Fuzzy set theory can be used in a wide range of domains in which information is incomplete or imprecise, such as bioinformatics. The elements of fuzzy sets are having degrees of membership unlike classical sets, where elements are either be present or not present in a particular set. A fuzzy set can be defined mathematically as:

Definition 1: A fuzzy set $A$ over a universe $U$ is defined through a function $\mu_A$, called its membership function $\mu_A : U \to [0,1]$, such that

\[ \mu_A(x) = \alpha, \quad 0 \leq \alpha \leq 1, \quad \forall x \in U; \]

The fuzzy set model has many extensions in the form of intuitionistic fuzzy sets, hesitant fuzzy sets and interval valued fuzzy sets to state only a few. We shall deal with all these models or their hybrid models in the works of this thesis.

2.2.4 Intuitionistic Fuzzy Set

In the context of fuzzy sets, the non-membership values of elements are 1’s complements of their membership values. However, this is not the case in many real life situations. For example, in a voting, some members may vote in favor of a resolution, some may vote against it, whereas some other may abstain from voting. This occurs because of the hesitancy of the voter. In order to model such situations and also to incorporate the hesitation component into a model, intuitionistic fuzzy sets
can be used. In order to capture this idea, Atanassov (1986) introduced the notion of intuitionistic fuzzy set as an extension of the fuzzy set model. The elements in an intuitionistic fuzzy set are having degrees of membership and non-membership.

Definition 2: An intuitionistic fuzzy set \( A \) over \( U \) is associated with a pair of functions \( \mu_A, \nu_A : U \rightarrow [0,1] \) such that for any \( \forall x \in U \), \( 0 \leq \mu_A(x) + \nu_A(x) \leq 1 \).

The hesitation function \( \pi_A \) is defined as \( \pi_A(x) = 1 - \mu_A(x) - \nu_A(x), \forall x \in U \). It is the indeterministic part of \( x \). An intuitionistic fuzzy set reduces to a fuzzy set when \( \nu_A(x) = 1 - \mu_A(x), \forall x \in U \) or \( \pi_A(x) = 0 \).

2.2.5 Rough Set

Rough set as a model of uncertainty was introduced by Pawlak in (Pawlak, 1982). Let \( U \) be a universe and \( R \) be an equivalence relation defined over \( U \). Then \( R \) decomposes \( U \) into disjoint equivalence classes. We denote the equivalence class of an element \( x \) with respect to \( R \) by \([x]_R\), which is defined as \([x]_R = \{y | yRx\}\). Then for any \( X \subseteq U \), we associate two crisp sets \( RX \) and \( \overline{RX} \), respectively called the lower and upper approximations of \( X \) with respect to \( R \) and are defined as,

\[
RX = \{y \in X | [y]_R \subseteq X\} \quad \text{and} \quad \overline{RX} = \{y \in U | [y]_R \cap X \neq \phi\}
\] (2.1)

\( X \) is said to be rough with respect to \( R \) iff \( RX \neq \overline{RX} \). The boundary of \( X \) with respect to \( R \) is denoted by \( BN_R(X) \) and it is given by \( BN_R(X) = \overline{RX} \setminus RX \). So, \( X \) is rough with respect to \( R \) iff \( BN_R(X) \neq \phi \).

A diagrammatic representation of a rough set is shown in Fig. 2.2. Any element \( x \) of the lower approximation of \( X \) is said to be fully included. This means, the element \( x \) belongs to \( X \). The boundary region is the region of uncertainty of \( X \) with respect to \( R \). This convention follows the idea of boundary region of uncertainty introduced by Gotlab Frege, the father of modern logic.
2.2.6 Soft Set

(Molodtsov, 1999) noticed that most of the uncertainty based models proposed till that time do not support enough of parametrization. So, he proposed a new model of uncertainty which he calls as soft set, which is a family of parametrized subsets of the universal set $U$. In the definition of a soft set he took $E$ as a set of parameters defined over $U$. The pair $(U, E)$ is often regarded as a soft universe. Let us denote the power set of $U$ by $P(U)$. A soft set is defined as follows:

Definition 3: A pair $(F, E)$ is called a soft set (over $U$) if and only if $F : E \rightarrow P(U)$.

In other words, a soft set associates every parameter ‘$e$’ in $E$ with a subset of $U$, called the $e$-approximate elements of $U$ and we denote this set by $F(e)$. $F(e)$ may be empty for some elements ‘$e$’ and also the intersection of subsets associated with different parameters may be non-empty.

Soft Set theory which is a fusion of the notions of topology and set theory as a new mathematical tool to deal with uncertainties and has enough of provision for parametrization. Soft Sets can be called as (Binary, Basic, Elementary) neighbourhood systems. It is worth noting that Molodtsov has shown that all fuzzy sets are soft sets.
Example: A soft set \((F, E)\) describes the attractiveness of house Mr. X is going to buy. The set \(U\) is the set of houses and there are 6 houses under consideration. Let \(U = \{h_1, h_2, h_3, h_4, h_5, h_6\}\) and \(E = \{e_1, e_2, e_3, e_4, e_5\}\). \(E\) is the set of parameters describing the condition of all houses. Each parameter is a word or sentence. That is, \(E= \{\text{Expensive}; \text{Beautiful}; \text{Wooden}; \text{Cheap}; \text{In the green surroundings}\}\) which can be represented as \(E= \{e_1, e_2, e_3, e_4, e_5\}\). We consider a mapping \(F: E \rightarrow P(U)\) which is defined as follows.

\[
F(e_1) = \{h_2, h_3\}, \quad F(e_2) = \{h_1, h_5\}, \quad F(e_3) = \{h_1, h_3\}, \quad F(e_4) = \{h_3, h_5\}, \quad \text{and} \quad F(e_5) = \{h_3\}
\]

Thus we have the soft set as follows.

\[
(F, E) = \begin{cases} 
\text{Expensive houses} = \{h_2, h_3\} \\
\text{Beautiful houses} = \{h_1, h_3\} \\
\text{Wooden houses} = \{h_1, h_3\} \\
\text{Cheap houses} = \{h_3, h_5\} \\
\text{In the green surrounding houses} = \{h_3\}
\end{cases}
\]

2.3 Literature Review

In this section, we shall discuss on soft sets and its hybrid models. At the end of it, we shall mention about the gaps identified in the literature and solutions to these problems obtained in the works of this thesis.
2.3.1 Literature Review on Soft Set

In the introductory article of soft set, Molodtsov (1999) discussed some applications of soft sets, which include soft sets in game theory, stability and regularization and soft analysis. In soft sets, parameters can be words, phrases, functions, real numbers etc. Many operations on soft sets were introduced by (Maji et al, 2002). Also, they developed an algorithm to handle the problem of decision making using soft sets. They introduced the tabular representation of soft sets. Like attribute reduction in rough set theory, the concept of parameter reduction for soft sets was discussed by them. The basic operations are used in any type of set theory to generate new ones from the existing ones. The first attempt in this direction is also due to Maji et al. in 2003, where they discussed some of the operations like union, intersection, complement etc. However, it was observed later by many researchers that their definitions are faulty. In one such paper, (Ali et al., 2009) tried to propose new definitions. There has been a sequence of such definitions, one set trying to rectify the previous one till the paper of Tripathy and Arun (2015), where soft sets were defined through their characteristic functions, which are highly authentic and helpful in defining the basic operations of soft sets.

Maji et al. (2002) applied the theory of soft sets to solve decision-making problems by using techniques similar to that of rough sets. To be precise they proposed the notion of reduct soft set of a soft set, which is borrowed from the corresponding notion in rough set theory. The selection of fewer parameters using this approach has made the process faster as the time taken for execution of any decision making algorithm in soft sets is directly proportional to the number of parameters. Similarly Chen et al. (2005) introduced the concept of parameter reduction in soft set theory, which is parallel to the concept of attribute reduction in rough set theory. Their approach is for finding optimal decisions on a general Boolean dataset and the difference among all objects according to the parameter in the parameter set does not influence the final decision. Kong et al. (2008) proposed the concept of Normal Parameter Reduction (NPR) to find optimal decisions as well as suboptimal decision. Zou et al. (2008) proposed algorithms for parameter reduction based on invariability of the optimal choice object, to efficiently decrease the number of parameters used for evaluation and cut down the work load.
Herawan et al. (2009) presented an attribute reduction in multi-valued information systems to decrease the number of attributes in multi-valued information using soft set. Another technique for decision making through parameter reduction to determine maximal supported sets is described by Rose et al. (2010). The main advantage of their approach is to determine the maximal supported sets from Boolean value information systems. Cagman et al. (2010) initiated a novel soft set based decision making scheme, called uni-int decision making. This method could decrease an extensive set of alternatives to its subset of optimal objects based on the criteria given by decision maker. Ma et al. (2011) proposed a method called New Efficient Normal Parameter reduction (NENPR), which makes the earlier proposed notion of normal parameter reduction more understandable and less complex. Ali (2012) discussed the idea of reduction of parameters in case of soft sets. The method of reduction of parameters proposed in this paper is very much similar to reduction of attributes in the case of rough sets. In this approach instead of a subset of parameters as a whole, parameters are reduced one by one. Deng et al. (2012) proposed two algorithms for pseudo parameter reductions and normal parameter reductions of a soft set using which all pseudo parameter reductions and normal parameter reductions can be directly evaluated. Yuan et al. (2013) proposed an alternative approach of parameter reduction, which they used in decision analysis systems.

Maji et al. (2003) realised that most of the parameters in soft sets are of fuzzy behaviour. So, they put forward the concept of fuzzy soft sets, which is a hybrid model of fuzzy sets and soft sets. They introduced many operations on fuzzy soft sets and also provided an application of fuzzy soft sets in decision making problems. Like in fuzzy sets, membership function plays an important role in fuzzy soft set theory. Kong et al. (2013) proved that the decision making algorithm provided by Maji et al. (2003) are not giving appropriate decision with the help of an example. In 2012, Majumdar (2012) introduced generalised fuzzy soft sets and also improved the decision making algorithm using similarity measure. They also discussed an application in medical diagnosis with the help of similarity measure.

Yang et al. (2007) consolidated traditional soft set to fuzzy set to improve its quality, and discussed some immediate outcomes of the fuzzy soft set. A method of object recognition from an imprecise multi-observer data to reduce the problem of uncertainty in decision making using fuzzy soft set are discussed by Roy et al.(2007).
Çağman et al. (2010) introduced fuzzy parameterized fuzzy soft (FPFS) sets and its operations. They also discussed FPFS-decision making and the main advantage of this method is that it reduces parameter by using fuzzy parameterized fuzzy soft set. Jiang et al. (2011) presented an extended fuzzy soft set theory by using approach of fuzzy description logics to serve as the parameters of fuzzy soft sets. Kong et al. (2011) discussed an algorithm for multiple evaluation bases for fuzzy soft set decision problem based on grey relational analysis. They combined multiple evaluation methods into single evaluation to make decision. In 2012, they presented a new parameter reduction for fuzzy soft set to improve the condition of redundant parameters because it is so strict that the number of deleting parameters is very few. Basu et al. (2012) introduced a new approach called mean potentiality approach (MPA) to get a balanced solution of a fuzzy soft set based decision making problem. This method reduces the parameters based on relational algebra using MPA algorithm. Deng et al. (2013) introduced the concept of complete distance between two objects and relative dominance degree between two parameters, based on an object-parameter to predict unknown data in incomplete fuzzy soft sets.

Yang et al. (2013) introduced multi-fuzzy soft set by means of combining the multi-fuzzy set and soft set model to handle imprecision. Li et al. (2015) presented an approach to fuzzy soft set in decision making by combining grey relational analysis and Dempster-Shaper theory of evidence where they used relational algebra and theory of evidence to make decision. Tao et al. (2015) developed a novel concept of uncertain linguistic fuzzy soft set (ULFSS), which uses the external aggregation process that is TOPSIS technique for the collection of decision aggregate information from individual. Tang (2015) employed grey relational analysis and Dempster–Shafer theory of evidence in proposing a novel fuzzy soft set approach in decision making, which is effective and feasible by comparing it with that of the mean potentiality approach. Wang et al. (2015) introduced hybrid hesitant fuzzy set and fuzzy soft sets and they applied it into multi-criteria group decision making problems. Das and Kar (2015) introduced a combination of fuzzy sets and soft sets and presented a matrix representation of the soft sets. This is very useful for computation of the proposed combination methods in decision making involving uncertainties. In 2016, Majumdar introduced hybrid models of fuzzy parameterized soft set and vague soft set and applied it into decision making problems (Majumdar, 2016). He also discussed the
notion of similarity measure between two hybrid soft sets and the use of similarity measures in medical diagnosis.

Yang et al. (2009) introduced an interval valued fuzzy soft sets by combing interval valued fuzzy sets and soft sets. They also applied interval-valued fuzzy soft set in decision making. Khalid et al. (2015) initiated the study of distance measures for interval valued fuzzy soft set (combination of interval valued fuzzy and soft set), and Hausdorff metric-based measures for intuitionistic fuzzy soft set. They used distance measure to combine interval-value fuzzy set and soft set. Feng et al. (2010) introduced the concept of level soft sets of fuzzy soft sets (combination of fuzzy and soft set) to develop an adjustable decision-making approach using fuzzy soft sets. Qin et al. (2011) presented an adjustable approach to interval-valued intuitionistic fuzzy soft sets based on decision making by using reduct intuitionistic. They computed the reduct of intuitionistic fuzzy soft set and interval value intuitionistic fuzzy soft set and converted it into crisp soft set.

Alkhazaleh et al. (2011) presented the generalization of fuzzy soft set to possibility fuzzy soft set (PFSS) and discussed a decision making application. They also introduced a measure of similarity between two PFSSs. The set theoretic approach has been taken in this regard because it is easier for calculation and is a very popular method too. In 2011, they extended this concept to introduce fuzzy parameterized interval-valued fuzzy soft set where they defined a mapping from the fuzzy soft set parameters to interval-valued fuzzy subset of the universal set (Alkhazaleh et al., 2011). Ma et al. (2014) proposed the parameter reduction of the interval-valued fuzzy soft set to deal with uncertainty and imprecision in decision making, where they considered invariable rank of decision choice of parameter reduction. Alkhazaleh (2015) introduced multi-fuzzy soft set by combining multi sets concept and interval-value fuzzy set. They applied the multi-interval-valued fuzzy soft set to decision making. Mukherjee and Sarkar (2014) introduced three types of similarity measures for hybrid interval-valued fuzzy set and soft set. Three types of similarity measures are done based on matching function, distance and set theoretic approach. An application of similarity measure between two interval-valued fuzzy soft sets in a decision making problem is illustrated.

Maji et al. (2001) introduced intuitionistic fuzzy soft sets based on the combination of intuitionistic fuzzy set and soft set model. They applied it in decision making with the help of similarity measurement method. Jiang et al. presented an
adjustable approach to intuitionistic fuzzy soft set based on decision making by using level soft sets of intuitionistic fuzzy soft set. They used decision criteria function as a threshold on membership value and non-member ship value. Mao et al. (2013) presented a group decision making method based on intuitionistic fuzzy soft matrix and uses median and p-quantile to compute threshold vectors. Later in 2015, Das et al. (2015) proposed an algorithmic approach on intuitionistic fuzzy soft matrix (IFSM) to investigate a particular disease that mirroring or reflecting the agreement of all experts. This approach is guided by group decision-making model and uses cardinal IFSS. Das et al. (2014) discussed an approach for multiple attribute group decision making (MAGDM) problems. They assigned a confident weight to each expert based on their opinion before making decision.

Feng et al. (2012) proposed two new concepts of choice value soft sets and k-satisfaction relations and then several new soft decision making schemes are presented. The main advantage of this method is that it improves the classical uni–int decision making approach. Çagman et al. (2012) defined fuzzy parameterized fuzzy soft sets (FPFS) in which the approximate functions are defined from fuzzy parameters to the fuzzy subsets of universal set. They defined the operations of FPFS and soft aggregation operator to form FPFS-decision making method. They also introduced a soft decision making method which selects a set of optimum elements from the alternatives without using the rough sets and fuzzy soft sets. The uni-int decision making method reduces a set to its subsets according to the parameters of the decision makers, which helps the decision makers to work on a small number of attributes instead of large numbers. Çagman and Enginoğlu (2010a) introduced soft matrices and discussed a soft max_min decision making method which can be applied to the problems that contain uncertainties. Later, they extended the soft matrix approach to fuzzy soft matrix, which are the representations of fuzzy soft sets, and discussed few advantages of using fuzzy soft matrices. Deli and Cagman (2015) proposed intuitionistic fuzzy parameterized soft sets for dealing with uncertainties based on intuitionistic fuzzy sets and soft sets where the decision is obtained by using the operations of soft sets and intuitionistic fuzzy sets. The similarity measure of hybrid interval valued intuitionistic fuzzy set and soft set of root type are discussed in (Shanthi and Naidu, 2015). The proposed similarity measure can be used for decision making. Chen (2015) presented an inclusion-based technique for order preference by similarity to ideal solution method with interval-valued intuitionistic fuzzy sets. This
can be used for solving multi-criteria decision making. Tripathy et al. (2016a, b) introduced a hybrid model combining intuitionistic fuzzy set and soft sets, where they introduced membership function and non-membership function for IFSS. Intuitionistic fuzzy set has the defect that cannot dynamically deal with membership degree of investigated subjects, so Jia et al. (2016) introduced the concept of sequence intuitionistic fuzzy soft sets and its operations. They applied it into the multi criteria decision making problems.

Karaaslan and Karataş (2016) proposed $\land$-aggregate decision making based on intuitionistic fuzzy parameterized fuzzy soft sets. Later on Deli and Karataş (2016) combined a hybrid interval valued intuitionistic fuzzy parameterized set and soft set which they applied for parameter reduction and decision making problems containing uncertain data. Wu and Su (2016) introduced the hybrid group generalized interval-valued intuitionistic fuzzy soft set model. This model is more universal than interval-valued intuitionistic fuzzy environment. Feng et al. (2010) presented a hybrid model called rough soft sets to provide framework to consolidate fuzzy sets, rough set and soft sets all together. The key point is that a soft set instead of an equivalence relation is used to granulate the universe of discourse. Hu et al. (2010) developed a new model called soft fuzzy rough sets (combination of soft set, fuzzy set and rough set) to reduce the influence of noise because datasets in a real-world applications are contaminated by noise. Meng et al. (2011) used a fuzzy soft set to granulate the universe of discourse, and obtain a new hybrid model called soft fuzzy rough sets, which can be seen as an extension of soft rough fuzzy sets. They also employed soft fuzzy rough set model to decision making.

Ali (2011) presented the concept of an approximation space associated with each parameter in a soft set which can be used for hybridization and decision making. Zhang (2013) proposed some concepts and conditions for two fuzzy soft sets to generate the same lower soft fuzzy rough approximation operators and the same upper soft fuzzy rough approximation operators. He proposed two concepts of reduct and exclusion which can be used to find the reduct or exclusion of a set of parameters, where they obtained necessary and sufficient conditions for parameter reduction. Zhang and Shu (2015) presented a generalization of a hybrid interval-valued fuzzy set and rough set which is based on constructive and descriptive (axiomatic) approaches. In the constructive approach, by employing an interval-valued fuzzy residual implicator and its dual operator, generalized upper and lower interval-valued fuzzy
rough approximation operators with respect to an arbitrary interval-valued fuzzy approximation space are defined. In the axiomatic approach, generalized interval-valued fuzzy rough approximation operators are defined by axioms. Zhang et al. (2015) discussed a hybrid hesitant fuzzy set and rough set over two universes based on constructive approach. This can be used for practical applications in decision making. Zhan et al. (2016) presented a novel soft rough set: soft rough hemirings, which is useful for multi-criteria group decision making.

Babitha and John (2010) added the concept of soft set relations as a sub soft set of the Cartesian product of the soft sets and many related concepts such as equivalent soft set relation, partition, composition, function etc. are discussed. The concept of generalized intuitionistic fuzzy soft sets was introduced by them in 2011 (Babitha and John, 2011). They also proposed a similarity measure for generalized intuitionistic fuzzy soft set and used it to find out the similarity between synthetic texture and natural texture. Alhazaymeh et al. (2012) introduced generalized vague soft sets and discussed its application in decision making problems. They used degree of preference to find the appropriate decision.

Torra (2010) introduced a new extension of fuzzy sets so-called Hesitant Fuzzy Sets (HFSs), motivated for the common difficulty that often appears when the membership degree of an element must be established and the difficulty is not because of an error margin or due to some possibility distribution, but rather because there are some possible values that make to hesitate about which one would be the right one. This situation is very usual in decision making when an expert might consider different degrees of membership {0.67, 0.72, 0.74} of the element ‘x’ in the set A. In 2013, Babitha and John (2013) introduced hesitant fuzzy soft sets by merging hesitant fuzzy set and soft set. Wang et al. (2014) discussed the complement, “AND”, “OR”, union and intersection operations on hesitant fuzzy soft sets. The basic properties such as De Morgan’s laws and the relevant laws of hesitant fuzzy soft sets are proved. Finally, with the help of level soft set, they applied the hesitant fuzzy soft sets to decision making problem and the effectiveness is proved by a numerical example. Wang et al. (2015) applied HFSS to a multi criteria group decision making approach. Generalized hesitant fuzzy soft sets and some operations on generalized hesitant fuzzy soft sets are defined and some of their properties are studied in (Bin, 2016). Sooraj et al. (2016) introduced membership function for HFSS and they discussed a group decision making application.
By combining the interval-valued hesitant fuzzy set and soft set models, Zhang et al. (2015) introduced the concept of interval-valued hesitant fuzzy soft sets. Then, by means of reduct interval-valued fuzzy soft sets and level hesitant fuzzy soft sets, they discussed an adjustable approach to interval-valued hesitant fuzzy soft sets based on decision making. Finally, the weighted interval-valued hesitant fuzzy soft set is also introduced and its application in decision making problem is shown.

Jiang et al. (2010) defined interval valued intuitionistic fuzzy soft sets which are the extensions of interval valued fuzzy soft sets and intuitionistic fuzzy soft sets. Peng et al. (2015) introduced the average function and proposed the interval-valued intuitionistic hesitant fuzzy soft sets (IVIHFSSs) which are a combination of the interval-valued intuitionistic hesitant fuzzy sets and soft sets. With the help of aggregation operators they discussed an algorithm for decision making problems.

Herawan and Deris (2011) introduced the concept of co-occurrence of parameters in defining support and confidence of association rules. However, it cannot be applied in general transactional data. Vo et al. (2016) found alternative methods which are not obtained by using the methods for regular association rule mining and this method is applicable for mining maximal association rules in text data. Feng et al. (2016) refined several existing concepts from (Herawan and Deris, 2011) to improve the generality and clarity of former definitions and it. This method is useful for mining association rules in transactional data. Cetia (2010) combined a hybrid interval-valued fuzzy set and soft set where they use Sanchez’s approach for medical diagnosis in interval valued fuzzy environment for analyzing patients with fever and malaria. Herawan et al. (2010) discussed a soft set based technique for medical decision making. The technique is used to reduce the number of dispensable symptoms and make an optimal decision. Kumar et al. (2013) introduced a novel approach based on bijective soft sets for the generation of classification rules from the data set. The novel approach showed to be a valuable tool as compared the well-known decision tree classifier algorithm and Naïve Bayes. Lashari et al. (2013) proposed a framework for medical image classification based on soft set and achieved higher performance comparing to baseline techniques. Alcantud et al. (2015) used an automated combination and analysis of information from structural and functional diagnostic techniques in order to obtain an enhanced Glaucoma detection. Alcantud and García (2016) presented two related techniques in analyzing incomplete soft sets.
Mushrif et al. (2006) introduced soft set theory based texture classification algorithm. Their method has better performance as compared to existing texture classification algorithms. Xiao et al. (2009) used combined forecasting approach based on fuzzy soft sets and each individual forecast. This approach improves forecasting accuracy than the combined rough set approach for forecasting. Senan et al. (2010) introduced a Soft set-based feature selection method which generates best feature set for Malaysian musical sound dataset. Qin et al. (2012) discussed a novel soft set approach in selecting clustering attribute which has higher accuracy and lower computational time as compared to rough set based approach. Later on Handaga et al. (2012) proposed a hybrid fuzzy soft set theory for numerical data classification which has better computational results compared to other existing algorithms. Mamat et al. (2013) discussed an alternative soft set approach in selecting clustering attribute which has higher accuracy and lower computational time as compared to soft set-based approach (Qin et al., 2012). Ma et al. (2014) introduced a normal parameter reduction and decision making method of soft sets whose experimental results demonstrate that this algorithm is feasible for dealing with the online shopping. Sutoyo et al. (2016) proposed multi-soft sets for conflict analysis which is efficient in computing support, strength, certainty and coverage of conflict situation.

Up to 2015, there was not a unique way to define the operations on soft sets. So, to generalize the operations on soft sets, Tripathy and Arun (2015) defined soft sets through their characteristic functions. They also reframed the fundamental notions and operations on soft sets by associating it with characteristic functions. This approach has been highly authentic and helpful in defining the basic operations on soft sets. These definitions remove the ambiguities in the earlier definitions and make the proofs of properties authentic and attractive. So, in the next section we discuss some of the authentic definitions given by Tripathy and Arun (2015).

2.3.2 Operations on Soft Set

The following definitions for the intersection and union operation were proposed by Maji et al (2003)
Definition 4 (AND operation): If \((F, A)\) and \((G, B)\) are two soft sets over the universe \(U\), then \((F, A)\) AND \((G, B)\) is denoted by \((F, A) \land (G, B) = (H, A \times B)\), where \(H(a, b) = F(a) \cap G(b), \forall (a, b) \in A \times B\).

Definition 5 (OR operation): If \((F, A)\) and \((G, B)\) are two soft sets over the universe \(U\), then \((F, A)\) OR \((G, B)\) is denoted by \((F, A) \lor (G, B) = (I, A \times B)\), where \(I(a, b) = F(a) \cup G(b), \forall (a, b) \in A \times B\).

Note 1: The two basic definitions AND and OR are not the operations applied to two or more soft sets on \(U\) to realize new soft sets on \(U\). In fact, it is worth noting that these two operations realize soft sets over the generalized soft universe \((U, E \times E)\).

In fact, it was noted by Molodtsov that his aim was to extend the notion of soft sets to get soft sets over a wider set of parameters and sometimes it may be useful. However, the union operation for soft sets was proposed by Maji et al (2003) as given below.

Definition 6: If \((F, A)\) and \((G, B)\) are two soft sets, then union of both \((F, A)\) and \((G, B)\) is \((H, C)\) where \(C = A \cup B\), and \(\forall e \in C, H(e) =\)

\[
H(e) = \begin{cases} 
F(e), & \text{if } e \in A \setminus B; \\
G(e), & \text{if } e \in B \setminus A; \\
F(e) \cup G(e), & \text{if } e \in A \cap B.
\end{cases}
\]  

(2.3)

Note 2: Ali et al (2009) had shown that, this definition of union is not correct and they proposed a modified definition. Also, some of the results basing upon their definition of union like the De Morgan’s laws have been shown to be incorrect. With his modified definition, Ali et al have shown that de Morgan’s laws hold good. The intersection operation was proposed by Maji et al as follows:

Definition 7: If \((F, A)\) and \((G, B)\) are two soft sets, then intersection of both \((F, A)\) and \((G, B)\) is \((H, C)\) where \(C = A \cap B\), and \(\forall e \in C, H(e) = F(e)\) or \(G(e)\), (As both are same set).
Also, the complement of a soft set was introduced by Maji et al as given below:

Definition 8 (Complement operation): Given a set of parameters $A$, by $\neg A$ we mean the set $\{-a \mid a \in A\}$, where $\neg a$ means ‘not a’. The complement of a soft set $(F, A)$ is represented by $(F, A)^c$ and is defined by the soft set $(F^c, \neg A)$, where $(F^c : \neg A \to P(U))$ is given by $F^c(a) = U - F(\neg a), \forall a \in \neg A$.

Note 3: Tripathy and Arun (2015) noted that the above notion is not the complement of $(F, A)$. However, they call it the negation of $(F, A)$. The complement of a soft set $(F, A)$ is also defined by them, which is correct notion of complement in the usual sense.

Tripathy and Arun (2015) introduced the notion of characteristic function for soft sets in order to make the above definitions of union and intersection more uniform.

Let $(F, A)$ be a soft set over $U$. Then we define the characteristic function $\varphi_{(F, A)} = \{\varphi_{(F, A)}^a \mid a \in A\}$ of $(F, A)$ as follows:

Definition 9 (Characteristic function): For any $a \in A$, we define the characteristic function $\varphi_{(F, A)}^a : U \to \{0, 1\}$ such that

$$\varphi_{(F, A)}^a(x) = \begin{cases} 1, & \text{if } x \in F(a); \\ 0, & \text{otherwise.} \end{cases} \quad (2.4)$$

Note 4: Without loss of generality we can assume that for any $b \notin A$, $\varphi_{(F, A)}^b(x) = 0, \forall x \in U$. So, in this sense the characteristic function of a soft set can be considered as being defined over the whole set of parameters $E$.

We are making this modification in order to make the definition of the union of two soft sets in terms of the characteristic function feasible.

Definition 10 (Absolute set): A soft set $(F, A)$ over a soft space $U$ is said to be absolute soft set denoted by $\tilde{A}$, if $\forall e \in A, F(e) = U$. 
Note 5: This definition does not seem to be correct as it is dependent upon $A$, which is a subset of $E$, the parameter set for the universe $U$. The correct definition which is independent upon $A$ can be as follows:

A soft set $(F, E)$ over the soft universe $(U, E)$ is said to be the absolute soft set over the soft universe if $F(e) = U$, $\forall \ e \in E$. We denote it by $\bar{U}$. So, $\forall e \in E$ and $\forall x \in X$, we have $\chi_{\bar{U}}^e(x) = 1$.

Definition 11 (Null set): A soft set $(F, A)$ over $U$ is said to be null soft set denoted by $\phi$, if $\forall e \in A$, $F(e) = \phi$.

Note 6: This definition does not seem to be correct as it is dependent upon $A$, which is a subset of $E$, the parameter set for the soft universe $U$. The correct definition which is independent upon $A$ can be as follows:

A soft set $(F, E)$ over $U$ is said to be the null soft set over the universe $U$, where $F(e) = \phi$, $\forall \ e \in E$. We denote it by $\phi$. So, $\forall e \in E$ and $\forall x \in X$, we have $\chi_{\phi}^e(x) = 0$.

Now, we mention the operations on soft sets using the characteristic function as follows:

Definition 12 (Modified Union operation): For any two soft sets $(F, A)$ and $(G, B)$ over a common universe $U$, then the union of $(F, A)$ and $(G, B)$ is the soft set $(H, C)$ where $C = A \cup B$, and $\forall a \in C$ and $x \in U$, we have

$$
\chi_{(H,C)}^a(x) = \max\{\chi_{(F,A)}^a(x), \chi_{(G,B)}^a(x)\}.
$$

Note 7: Tripathy and Arun (2015) proved here that with the changed definition of the absolute soft set and the null soft set we have, for any soft set $(F, A)$ defined over the soft space.

$$(F, A) \cup \bar{U} = \bar{U}$$

$$(F, A) \cup \phi = (F, A)$$
Definition 13 (Modified Intersection operation): For any two soft sets \((F, A)\) and \((G, B)\) over a common universe \(U\), then the intersection of \((F, A)\) and \((G, B)\) is the soft set \((H, C)\) where \(C = A \cap B\), and \(\forall a \in C\) and \(\forall x \in U\),

\[
\chi^a_{(H,C)}(x) = \min\{\chi^a_{(F,A)}(x), \chi^a_{(G,B)}(x)\}.
\] (2.8)

Note 8: Tripathy and Arun (2015) proved that with the changed definition of the absolute soft set and null soft set we have, for any soft set \((F, A)\) defined over the soft space \(U\).

\[
(F,A) \cap \tilde{U} = (F,A).
\] (2.9)

\[
(F,A) \cap \tilde{\phi} = \tilde{\phi}.
\] (2.10)

Definition 14 (soft subset): Given two soft sets \((F, A)\) and \((G, B)\) over a common universe \(U\), then \((F, A)\) is said to be a soft subset of \((G, B)\), written as \((F, A) \subseteq (G, B)\) if \(A \subseteq B\) and \(\forall a \in A, \ x \in U, \ \chi^a_{(F,A)}(x) \leq \chi^a_{(G,B)}(x)\).

Definition 15 (Equal soft set): For any two soft sets \((F, A)\) and \((G, B)\) over a common universe \(U\), we say that \((F, A)\) is equal to \((G, B)\), written as \((F, A) = (G, B)\) if \(A = B\) and \(\forall a \in E, \ A = B\) and \(\forall x \in U\), \(\chi^a_{(F,A)}(x) = \chi^a_{(G,B)}(x)\).

Definition 16 (Complement operation): For any two soft sets \((F, A)\) and \((G, B)\) over a common universe \(U\), we define the complement \((H, C)\) of \((G, B)\) in \((F, A)\) as,

\[
\chi^a_{(H,C)}(x) = \max\{0, \chi^a_{(F,A)}(x) - \chi^a_{(G,B)}(x)\}, \ \forall a \in A \text{ and } \forall x \in U.
\]

Definition 17: The complement of a soft set over the universe \(U\) can be derived from the above definition 16 by taking \((F, A)\) as \(\tilde{U}\) and \((G, B)\) as \((F, A)\).

We denote it by \((F,A)^c\) and clearly, \(\chi^a_{(F,A)^c}(x) = \max\{0, \chi^a_{(F,A)}(x) - \chi^a_{(F,A)}(x)\} \ \forall x \in U\) and \(\forall e \in E\). It can be seen easily that \(\chi^a_{(F,A)^c}(x) = 1 - \chi^a_{(F,A)}(x)\).

Note 9: We note that it was pointed out that with the earlier definition of complement of a soft set, the axioms of contradiction and exclusion, that is for any soft set \((F, A)\) defined over a universe \(U\) the following hold true.

\[
(F,A) \cup (F,A)^c = U.
\] (2.11)
\[(F,A) \bigcap (F,A)^C = \phi. \]  

(2.12)

Definition 18 (Symmetric difference): For any two soft sets \((F, A)\) and \((G, B)\) over a common universe \(U\), we define the symmetric difference \((H, C)\) of \((F, A)\) and \((G, B)\) where \(C = A \cup B\), as

\[
\chi_{(H,C)}^a(x) = \max\{0, \chi_{(F,A)}^a(x) - \chi_{(G,B)}^a(x), \chi_{(G,B)}^a(x) - \chi_{(F,B)}^a(x)\}
\]  

(2.13)

2.4 Problem Identification

After a thorough literature review, we have found certain gaps and tried to find solutions to them. In this section we present the problems identified.

Problem-1
There was a necessity to redefine the fuzzy soft sets model and interval valued fuzzy soft set (IVFSS) model. Also, the basic operations are needed to be defined suitably. We propose to develop algorithms based on IVFSS for handling decision making problems and also illustrate their applicability in handling real life problems.

Problem-2
The definition of Interval valued intuitionistic fuzzy soft sets (IVIFSS) proposed by Jiang (2010) is needed to be changed following the membership function approach and define operations on these models. We propose to develop a decision making algorithm basing upon IVIFSS, and show its utility to handle real life problems.

Problem-3
Sunil et al. (2013) introduced the notion of hesitant fuzzy soft set and Peng et al. (2015) introduced interval valued hesitant fuzzy soft set (IVHFSS) by combining interval valued hesitant fuzzy sets and soft sets. We propose to redefine these two generalised hybrid models using the membership function approach and modify the operations on them. As usual, we propose to develop decision making algorithms based upon these models and show their utility by illustrating the applications in handling real life problems.

Problem-4
Extending the IVHFSS model introduced by them, Peng et al. (2015) proposed the interval valued intuitionistic hesitant fuzzy soft set (IVHIFSS) model. As proposed in
all the above three problems, we propose to redefine this model, develop a decision making algorithm based on it and show its applications.

2.5 Solutions Obtained

- Defined IVFSS model using the membership function approach and operations on them. Proposed two algorithms to handle decision making problems; one based upon IVFSS for individual decision making and the other based upon IVFSS for group decision making. The seed selection problem was solved by using the first algorithm and the selection of candidates in interviews problem was solved by using the second algorithm.

- Defined IVIFSS model using the membership function approach and operations on it. Proposed two decision making algorithms (one each for individual decision making and the other for group decision making) and illustrated their applications by solving the car selection and candidate selection in interviews real life problems respectively.

- Defined IVHFSS model using the membership function approach and operations on it. Proposed a decision making algorithm for individual decision making and illustrated its application by solving the stock market real life problem.

- Defined IVIHFSS model using the membership function approach and operations on it. Proposed a decision making algorithm for individual decision making and illustrated its application by solving the selection of players into a national team.

Significant contribution in the above solutions is the introduction of the concepts of positive and negative parameters and fixing user priority of the parameters.

In this chapter some of the related concepts and the existing papers are reviewed and analyzed thoroughly to get the gaps and problems behind the concepts. Based on the problems identified, there is a need to uniquely define the hybrid models of soft sets. All these problems are clearly mentioned in this chapter. The detailed expansion of the problems will be discussing in the next chapters.