Chapter 5

Function projective synchronization in chaotic and hyperchaotic systems through OPCL coupling

5.1 Introduction

Open-plus-closed-loop (OPCL) control method is a more general and physically realizable coupling scheme that can provide stable CS (complete synchronization) and AS (anti-synchronization) in identical and mismatched oscillators. Grosu et al.,[111, 112] recently reported projective synchronization in chaotic systems through open-plus-closed-loop (OPCL) coupling. Hyperchaotic systems possessing at least two positive Lyapunov exponents have more complex behaviour and abundant dynamics than chaotic systems and are more suitable for some engineering applications such as secure communication. Hence how to realize synchronization of hyperchaotic systems through OPCL method is an interesting question.

In this chapter, we propose a physically realizable coupling function through unidirectional OPCL coupling for the FPS of chaotic as well as more complex hyperchaotic systems.

5.2 Unidirectional OPCL coupling for FPS

In chapter 1 we considered the OPCL method for complete synchronization. Here, we briefly discuss the OPCL method for the FPS of two mismatched chaotic or
Function projective synchronization in chaotic and hyperchaotic systems through OPCL coupling

hyperchaotic oscillators. A chaotic or hyperchaotic driver is defined by

\[ \dot{y} = f(y) + \Delta f(y), \quad y \in \mathbb{R}^n, \]  \hspace{1cm} (5.1)

where \( \Delta f(y) \) contains the mismatched terms. It drives another chaotic or hyperchaotic oscillator \( \dot{x} = f(x), \quad x \in \mathbb{R}^n \) to achieve a goal dynamics \( g(t) = \alpha(t)y(t) \), where \( \alpha(t) \) is an arbitrary scaling function.

After coupling, the response system is given by

\[ \dot{x} = f(x) + D(x, g), \]  \hspace{1cm} (5.2)

where the coupling function is defined as

\[ D(x, g) = \dot{g} - f(g) + \left( H - \frac{\partial f(g)}{\partial g} \right)(x - g), \]  \hspace{1cm} (5.3)

\( \frac{\partial f(g)}{\partial g} \) is the Jacobian of the dynamical system and \( H \) is an arbitrary constant Hurwitz matrix \( (n \times n) \) whose eigenvalues all have negative real parts.

The error signal of the coupled system is defined by \( e = x - g \) and \( f(x) \) can be written using Taylor series expansion, as

\[ f(x) = f(g) + \frac{\partial f(g)}{\partial g}(x - g) + \cdots. \]  \hspace{1cm} (5.4)

Keeping the first-order terms in (5.4) and substituting in (5.2), the error dynamics is obtained as

\[ \dot{e} = He \]  \hspace{1cm} (5.5)

Since \( H \) is a Hurwitz matrix with all of its eigenvalues having negative real parts, \( e \rightarrow 0 \) as \( t \rightarrow \infty \) and we obtain asymptotic FPS.

### 5.3 FPS in chaotic oscillators

#### 5.3.1 Numerical simulation: Identical oscillators

We consider FPS in two identical oscillators using Rössler oscillator systems[116]. The Rössler system arises from work in chemical kinetics and it is given by the
following differential equations.

\[
\begin{align*}
\dot{y}_1 &= -\omega y_2 - y_3 \\
\dot{y}_2 &= y_1 + by_2 \\
\dot{y}_3 &= c + y_3(y_1 - d)
\end{align*}
\] (5.6)

which has a chaotic attractor when the system’s parameters are chosen as $\omega = 1, b = 0.15, c = 0.2$ and $d = 10$

The Jacobian of the model is

\[
\frac{\partial f}{\partial y} = \begin{bmatrix}
0 & -\omega & -1 \\
1 & b & 0 \\
y_3 & 0 & y_1 - d
\end{bmatrix}
\] (5.7)

System (5.6) is considered as the drive system. For simplicity, in all simulations the arbitrary Hurwitz matrix is chosen as $-I$.

The response system after coupling is then given by

\[
\begin{align*}
\dot{x}_1 &= -\omega x_2 - x_3 + \dot{\alpha} y_1 - e_1 + \omega e_2 + e_3 \\
\dot{x}_2 &= x_1 + bx_2 + \dot{\alpha} y_2 - e_1 - (1 + b)e_2 \\
\dot{x}_3 &= c + x_3(x_1 - d) + \dot{\alpha} y_3 + (\alpha - 1)c + \alpha(1 - \alpha)y_3 y_1 \\
&\quad -\alpha y_3 e_1 - (1 + (\alpha y_1 - d))e_3
\end{align*}
\] (5.8)

For numerical simulations, the arbitrary scaling function is chosen as $\alpha(t) = 3 + 1.5 \sin(2\pi t/10)$. Results of the numerical simulations are shown in figures 5.1 and 5.2. Figure 5.1(a) shows the time series of $x_1$ and $y_1$ under OPCL coupling. Figure 5.1(b) depicts the evolution of FPS errors, which shows errors tending to zero asymptotically. The ratio $\|x\| / \|y\|$ plotted against time tends to the scaling function as shown in figure 5.2, indicating FPS.

5.3.2 Numerical simulation : Mismatched oscillators

FPS in mismatched case can be illustrated with the example of Lorenz oscillators[6]. The nonlinear differential equations that describe the Lorenz system considered as drive system is given by

\[
\begin{align*}
\dot{y}_1 &= \sigma(y_2 - y_1) + \Delta \sigma(y_2 - y_1) \\
\dot{y}_2 &= r y_1 - y_2 - y_1 y_3 + \Delta r y_1 \\
\dot{y}_3 &= -b y_3 + y_1 y_2 - \Delta b y_3
\end{align*}
\] (5.9)
Figure 5.1: Identical Rössler system. (a) time series of $y_1$ (solid line) and $x_1$ (dashed line). (b) the evolution of FPS errors.
Figure 5.2: Identical Rössler system. $\|x\|/\|y\|$ plotted against time tends to the scaling function

which has a chaotic attractor when the parameters are respectively chosen as $\sigma = 10, b = \frac{8}{3}, r = 28$.

The Jacobian of the system is

$$\frac{\partial f}{\partial y} = \begin{bmatrix} -\sigma & \sigma & 0 \\ (r - y_3) & -1 & -y_1 \\ y_2 & y_1 & -b \end{bmatrix} \quad (5.10)$$

The response system after the OPCL coupling is given by

$$\begin{align*}
\dot{x}_1 &= \sigma(x_2 - x_1) + \alpha\Delta\sigma(y_2 - y_1) \\
&\quad + \dot{\alpha}y_1 - (1 - \sigma)e_1 - \sigma e_2 \\
\dot{x}_2 &= rx_1 - x_2 - x_1x_3 + \alpha(\alpha - 1)y_1y_3 \\
&\quad + \alpha\Delta y_1 + \dot{\alpha}y_2 - (r - \alpha y_3)e_1 + \alpha y_1 e_3 \\
\dot{x}_3 &= -bx_3 + x_1x_2 + \alpha(1 - \alpha)y_1y_2 - \alpha\Delta by_3 \\
&\quad + \dot{\alpha}y_3 - \alpha y_2 e_1 - \alpha y_1 e_2 - (1 - b)e_3 \quad (5.11)
\end{align*}$$

For numerical simulations, the driver is chosen identical to the response except
that \( \Delta r = 5 \) and the arbitrary scaling function is chosen as \( \alpha(t) = 5 + 2 \cos(2\pi t/15) \).

Results of the numerical simulations are shown in figures 5.3-5.4.

5.4 FPS in Hyperchaotic oscillators

5.4.1 Numerical simulation: Identical oscillators

Hyperchaotic systems possessing more than one positive Lyapunov exponents exhibit rich and complex dynamics than chaotic systems and are more suitable for some applications like secure communications. In this section we apply OPCL control to obtain FPS in two identical hyperchaotic Lorenz systems. The hyperchaotic Lorenz system[120] is described by following nonlinear differential equations

\[
\begin{align*}
\dot{y}_1 &= a(y_2 - y_1) \\
\dot{y}_2 &= \beta y_1 + y_2 - y_1 y_3 - y_4 \\
\dot{y}_3 &= y_1 y_2 - \gamma y_3 \\
\dot{y}_4 &= \theta y_2 y_3
\end{align*}
\] (5.12)

The system exhibits hyperchaotic behaviour when \( a = 10, \beta = 28, \gamma = 8/3 \) and \( \theta = 0.1 \).

The Jacobian is given by

\[
\frac{\partial f}{\partial y} = \begin{bmatrix}
-a & a & 0 & 0 \\
(\beta - y_3) & 1 & -y_1 & -1 \\
y_2 & y_1 & -\gamma & 0 \\
0 & \theta y_3 & \theta y_2 & 0
\end{bmatrix}
\] (5.13)

An identical response system after coupling is then given by

\[
\begin{align*}
\dot{x}_1 &= a(x_2 - x_1) + \dot{\alpha} y_1 - (1 - \alpha) e_1 - \alpha e_2 \\
\dot{x}_2 &= \beta x_1 + x_2 - x_1 x_3 - x_4 + \alpha(\alpha - 1) y_1 y_3 + \dot{\alpha} y_2 - (\beta - \alpha y_3) e_1 - 2 e_2 + \alpha y_1 e_3 + e_4 \\
\dot{x}_3 &= x_1 x_2 - \gamma x_3 + \alpha(1 - \alpha) y_1 y_2 + \dot{\alpha} y_3 - \alpha y_2 e_1 - \alpha y_1 e_2 - (1 - r) e_3 \\
\dot{x}_4 &= \theta x_2 x_3 + \alpha(1 - \alpha) y_2 y_3 + \dot{\alpha} y_4 - \theta \alpha y_3 e_2 - \theta \alpha y_3 e_3 - e_4
\end{align*}
\] (5.14)

Simulation results when the arbitrary scaling function matrix is chosen as \( \alpha(t) = \)}
Figure 5.3: Mismatched Lorenz system. Driver identical to response except $\Delta r = 5$(a) time series of $y_1$ (solid line) and $x_1$ (dashed line)(b) the evolution of FPS errors
5.4.2 Numerical simulation: Mismatched oscillators

For the mismatched case we choose hyperchaotic Chen \cite{121} oscillators. The hyperchaotic Chen system with mismatch which is chosen as driver is given by

\[
\begin{align*}
\dot{y}_1 &= a(y_2 - y_1) + y_4 + \Delta a(y_2 - y_1) \\
\dot{y}_2 &= d y_1 - y_1 y_3 + cy_2 + \Delta d y_1 + \Delta c y_2 \\
\dot{y}_3 &= y_1 y_2 - b y_3 - \Delta b y_3 \\
\dot{y}_4 &= y_2 y_3 + r y_4 + \Delta r y_4
\end{align*}
\]  

(5.15)

When \( a = 35, b = 3, c = 12, d = 7, 0.085 \leq r \leq 0.798 \), system is hyperchaotic.

The Jacobian is

\[
\frac{\partial f}{\partial y} = \begin{bmatrix}
-a & a & 0 & 1 \\
(d - y_3) & c & -y_1 & 0 \\
y_2 & y_1 & -b & 0 \\
0 & y_3 & y_2 & r
\end{bmatrix}
\]

(5.16)
Figure 5.5: Identical hyperchaotic Lorenz system. (a) time series of $y_1$ (solid line) and $x_1$ (dashed line). (b) the evolution of FPS errors.
Function projective synchronization in chaotic and hyperchaotic systems through OPCL coupling

The response system after coupling is obtained as

\[
\begin{align*}
\dot{x}_1 &= a(x_2 - x_1) + x_4 + \dot{\alpha}y_1 \\
&\quad - (1 - a)e_1 - ae_2 - e_4 \\
\dot{x}_2 &= dx_1 - x_1x_3 + cx_2 + a(\alpha - 1)y_1y_3 + \dot{\alpha}y_2 + \alpha\Delta dy_1 \\
&\quad + \alpha\Delta e_2 - (d - \alpha y_3)e_1 - (1 + c)e_2 + \alpha y_1 e_3 \\
\dot{x}_3 &= x_1x_2 - bx_3 + \alpha(1 - \alpha)y_1y_2 + \dot{\alpha}y_3 \\
&\quad - \alpha y_2 e_1 - \alpha y_1 e_2 - (1 - b)e_3 \\
\dot{x}_4 &= x_2x_3 + rx_4 + \alpha(1 - \alpha)y_2y_3 + \alpha\Delta y_4 \\
&\quad + \dot{\alpha}y_4 - \alpha y_3 e_2 - \alpha y_2 e_3 - (1 + r)e_4 \\
\end{align*}
\] (5.17)

For numerical simulations, the parameters of the system are selected as \(a = 35, b = 3, c = 12, d = 7\) and \(r = 0.2\). The mismatch in parameters are chosen as \(\Delta a = 0, \Delta b = 0, \Delta c = 0, \Delta d = 0\) and \(\Delta r = 0.5\) so that both the drive and response Chen systems exhibits hyperchaotic behaviour. The arbitrary scaling function is chosen as \(\alpha(t) = 4 + 2\sin(2\pi t/10 + 20)\). The simulation results indicated in Fig 5.7-5.8 show that the two mismatched systems are in FPS.
5.5 Conclusions

FPS is a more general definition of synchronization. In FPS, the master and slave system synchronize up to a scaling function which can be used to get more secure communication in application to secure communication because it is obvious that the unpredictability of the scaling function can additionally enhance the security of communication. The OPCL coupling method is a physically realizable method which is suited for practical applications. We design controls through OPCL coupling for the FPS of identical as well as mismatched chaotic and hyperchaotic systems. Numerical simulations verify robust FPS.
Figure 5.7: Mismatched hyperchaotic Chen system. Driver identical to response except $\Delta r = 0.5$ (a) time series of $y_1$ (solid line) and $x_1$ (dashed line) (b) the evolution of FPS errors
Figure 5.8: Mismatched hyperchaotic Chen system. $\|x\| / \|y\|$ plotted against time tends to the scaling function