CHAPTER 3

Dichromatic polynomial of product digraphs

3.1 Introduction

An interesting notion is that the dichromatic polynomial of a digraph. For an arc $a = uv$ of a digraph $D$, $D.a$ is obtained from $D$ by identifying the vertices $u$ and $v$, is called the contraction of $a$. For a digraph $D$ and positive integer $\lambda$, the function $P(D; \lambda)$ is defined to be the number of ways of colouring $D$ using the $\lambda$ colours. We recall the definition of Cartesian and tensor product of digraphs. We write $u \rightarrow v$ to denote that there is an arc from $u$ to $v$ in a digraph. The Cartesian product of the digraphs $D_1$ and $D_2$, denoted by $D_1 \square D_2$, has the vertex set $V(D_1 \square D_2) = V_1 \times V_2$ and $(u_1, v_1) \rightarrow (u_2, v_2)$ in $D_1 \square D_2$ if and only if, either $u_1 = u_2$ in $D_1$ and $v_1 \rightarrow v_2$ in $D_2$ or $u_1 \rightarrow u_2$ in $D_1$ and $v_1 = v_2$ in $D_2$. The tensor product of the digraphs $D_1$ and $D_2$, denoted by $D_1 \times D_2$, has the vertex set $V(D_1 \times D_2) = V_1 \times V_2$ and $(u_1, v_1) \rightarrow (u_2, v_2)$ in $D_1 \times D_2$ if and only if, $u_1 \rightarrow u_2$ in $D_1$ and $v_1 \rightarrow v_2$ in $D_2$. In this chapter, we derive exact dichromatic polynomial for some digraphs. We prove that the dichromaticity of the cartesian product of any two digraphs is the maximum of the dichromaticity of two digraphs and the dichromaticity of the tensor product of
any two digraphs is the minimum of the dichromaticity of two digraphs. Contents of
this chapter have been accepted for publication in [32].

3.2 Dichromatic polynomial

In this section, we obtain some results on dichromatic polynomial of digraphs.

Theorem 3.2.1. A digraph $D$ on $n$ vertices is acyclic if and only if $P(D; \lambda) = \lambda^n$.

Proof. Since $D$ is acyclic, by Theorem 2.2.5, $\chi_d(D) = 1$. Therefore, each vertex has $\lambda$ choices. Hence $P(D; \lambda) = \lambda^n$. Conversely, let $P(D; \lambda) = \lambda^n$. This implies that all the vertices of $D$ are assigned the same colour. Therefore $\chi_d(D) = 1$. By Theorem 2.2.5, $D$ is acyclic.

Corollary 3.2.2. $P(P_n; \lambda) = \lambda^n$.

Proof. Since $\chi_d(P_n) = 1$, there are $\lambda$ choices for each vertex. Therefore

$P(P_n; \lambda) = \lambda^n$.

Corollary 3.2.3. $P(S_n; \lambda) = \lambda^n$.

Proof. Since $\chi_d(S_n) = 1$, there are $\lambda$ choices for each vertex. Therefore

$P(S_n; \lambda) = \lambda^n$.

Corollary 3.2.4. $P(C_n; \lambda) = \lambda^n - \lambda$. 

3.2 Dichromatic polynomial

Proof. Since $\chi_d(C_n) = 2$, all the vertices are not assigned the same colour, at least one of the vertices must be assigned distinct colour. Therefore, we eliminate $\lambda$ choices from the total number of choices. Hence $P(C_n; \lambda) = \lambda^n - \lambda$.

Corollary 3.2.5. $P(K^*_n; \lambda) = \lambda(\lambda - 1)(\lambda - 2) \cdots (\lambda - (n - 1))$.

Proof. Since each vertex is assigned distinct colours, $P(K^*_n; \lambda) = \lambda(\lambda - 1)(\lambda - 2) \cdots (\lambda - (n - 1))$.

Ararat Harutyunyan has proved the following theorem as a recurrence formula in his dissertation [82].

Theorem 3.2.6. Let $C$ be a directed cycle in a digraph $D$ such that no edges of $C$ appear in any other cycle of $D$. Then

$$P(D; x) = P(D - E(C); x) - P(D/C; x),$$

where $D/C$ is the digraph obtained from $D$ by deleting $E(C)$ and identifying all vertices of $C$.

Here we derive exact dichromatic polynomial of unicyclic digraph.

Theorem 3.2.7. If $D$ is an unicyclic digraph of length $r$, then $P(D; \lambda) = \lambda^n - \lambda^{n-r+1}$.

Proof. Let $D$ be a unicyclic digraph with the directed cycle $C$ of length $r$ (see Figure 3.1). As there are $\lambda$ colours, the number of ways to colour the cycle $C$ is $\lambda^r$. All the vertices of the cycle $C$ cannot be assigned the same colour, therefore we can eliminate $\lambda$ choices from the total number of choices. So, the number of colourings for the...
directed cycle is $\lambda^r - \lambda$. Now, the remaining vertices of the digraph which is not in $C$ can be coloured in $\lambda^{n-r}$ ways. Hence the total number of different colourings of $D$ is equal to $\lambda^n - \lambda^{n-r+1}$. \hfill \Box

**Theorem 3.2.8.** If $D$ is a unicyclic digraph, $a$ is an arc of $D$ and $C$ be the unique directed cycle in $D$, then

$$P(D - a; \lambda) = \begin{cases} P(D; \lambda), & \text{if } a \notin A(C) \\ \lambda^n, & \text{if } a \in A(C) \end{cases}$$

**Proof.** Suppose $a \notin A(C)$. Every colouring of $D - a$ is a colouring of $D$ whether the end vertices of $a$ are assigned the same colour or different colours. Hence $P(D - a; \lambda) = P(D; \lambda)$. Let $a$ be an arc in $C$. $D - a$ becomes acyclic, then by Theorem 2.2.5, $\chi_d(D - a) = 1$. Clearly, $P(D - a; \lambda) = \lambda^n$. \hfill \Box

![Figure 3.1 Unicyclic digraph](image)

**Corollary 3.2.9.** Let $D$ be an unicyclic digraph and $C$ be the unique cycle. If $a$ is an arc of $C$, then the number of colourings which are colourings of $D - a$ but not for $D$ is $\lambda^{n-r+1}$.

**Proof.** By Theorem 3.2.8, $P(D - a; \lambda) = \lambda^n$, when $a$ is in $A(C)$. Therefore, the number of colourings which are colourings of $D - a$ but not for $D$ is equal to $\lambda^n - \{\lambda^n - \lambda^{n-r+1}\}$.
3.2 Dichromatic polynomial

$$= \lambda^{n-r+1}. \square$$

**Theorem 3.2.10.** Let $D$ be a digraph and $a=uv$ be an arc in $D$. Then

$$P(D; \lambda) = \begin{cases} P(D-a; \lambda), & \text{if } a \notin A(D) \\ P(D-a; \lambda) + P(D.a; \lambda), & \text{if } a \in A(D) \text{ and } u, v \text{ must be assigned distinct colours in } D-a. \end{cases}$$

*Proof.* Let $a=uv$ be an arc which is not in any directed cycle. If $u$ and $v$ are assigned the different colours in $D-a$, then $P(D; \lambda) = P(D-a; \lambda)$. Suppose $u$ and $v$ are assigned the same colour say $c_1$ in $D-a$. Let $V_1 = \{ v \in V / v \text{ has the colour } c_1 \}$. Let $H$ be the subdigraph induced by $V_1$. By Theorem 2.2.5, $\chi_d(H) = 1$. In this case, $P(D; \lambda) = P(D-a; \lambda)$. Let $a=uv$ be an arc in directed cycle. We partition all the colourings of $D$ as two sets as follows:

(i) Colourings where the end vertices $u$ and $v$ are assigned the same colour. (ii) Colourings where the end vertices $u$ and $v$ are assigned different colours. The number of colourings where $u$ and $v$ are assigned the same colour is equal to $P(D.a; \lambda)$. The number of colourings where $u$ and $v$ are assigned different colours is equal to $P(D-a; \lambda)$ where the end vertices $u$ and $v$ in $D-a$ are assigned distinct colours. Hence $P(D; \lambda) = P(D-a; \lambda) + P(D.a; \lambda)$, where the end vertices in $D-a$ are assigned distinct colours. \(\square\)

**Theorem 3.2.11.** Let $D$ be a digraph obtained from $k$ directed cycles $C_1, C_2, \ldots, C_k$ of length $r_1, r_2, \ldots, r_k$ respectively and has a vertex $u$ common to all $C_i$, $1 \leq i \leq k$. Then

$$P(D; \lambda) \equiv \frac{(\lambda^{r_1} - \lambda)(\lambda^{r_2} - \lambda)\ldots(\lambda^{r_k} - \lambda)}{\lambda^{k-1}}$$

*Proof.* Without loss of generality, let us choose a directed cycle $C_1$ of length $r_1$. The number of colourings of $C_1$ is $\lambda^{r_1} - \lambda$. Then choose another cycle $C_2$ of length $r_2$, the
number of colourings of $C_2$ is $\lambda^{r_2} - \lambda$. Since $u$ is common to $C_1$ and $C_2$, the number of colourings of $C_1$ and $C_2$ is $\left(\frac{\lambda^{r_1} - \lambda}{\lambda}\right) \text{ and } \left(\frac{\lambda^{r_2} - \lambda}{\lambda}\right)$ (see Figure 3.2).

![Figure 3.2 A vertex is common to two directed cycles](image)

As there are $\lambda$ - choices, the number of colourings of $C_1$ and $C_2$ is $\lambda\left(\frac{\lambda^{r_1} - \lambda}{\lambda}\right)\left(\frac{\lambda^{r_2} - \lambda}{\lambda}\right)$. Inductively, the number of colourings of $C_1, C_2, C_3, C_4, ..., C_k$ is $\lambda\left(\frac{\lambda^{r_1} - \lambda}{\lambda}\right) \left(\frac{\lambda^{r_2} - \lambda}{\lambda}\right) ... \left(\frac{\lambda^{r_k} - \lambda}{\lambda}\right)$. Hence $P(D; \lambda) = \frac{(\lambda^{r_1} - \lambda)(\lambda^{r_2} - \lambda)...(\lambda^{r_k} - \lambda)}{\lambda^{k-1}}$.

**Theorem 3.2.12.** Let $D$ be a digraph obtained from $k$ directed cycles $C_1, C_2, ..., C_k$ of length $r_1, r_2, ..., r_k$ respectively and have $P_l$ as a directed path on $l$ vertices common to all $C_i$, $1 \leq i \leq k$. Then $P(D; \lambda) = (\lambda^l - \lambda)(\lambda^{r_1-l}) \left(\lambda^{r_2-l} \right) ... \left(\lambda^{r_k-l} \right) + \lambda(\lambda^{r_1-l} - 1)(\lambda^{r_2-l} - 1) ... (\lambda^{r_k-l} - 1)$.

**Proof.** Without loss of generality, let us choose one directed cycle $C_1$ of length $r_1$ with common directed path $P_l$ common to all $C_i$, $1 \leq i \leq k$ (refer Figure 3.3). If all the vertices of $P_l$ are assigned the same colour, then there are $(\lambda^{r_1-l} - 1)$ choices for $C_1$. Similarly, there are $(\lambda^{r_2-l} - 1)$ colourings for $C_2$. As there are $\lambda$ choices, the number of colourings for all $C_i$ in which all the vertices of $P_l$ are assigned the same colour is
3.2 Dichromatic polynomial

\[ \lambda(\lambda^{r_1-1} - 1)(\lambda^{r_2-1} - 1)\ldots(\lambda^{r_k-1} - 1). \]

Figure 3.3 Directed path on \( l \) vertices common to all directed cycles

Now, assume that all the vertices of \( P \) are not assigned the same colour. Then there are \((\lambda^l - \lambda)\) ways to colour the directed path \( P \). The number of possible colouring of \( C_1, C_2, \ldots, C_k \) are \((\lambda^{r_1-1})(\lambda^{r_2-1})\ldots(\lambda^{r_k-1})\). Therefore, the number of colourings of \( D \) is \((\lambda^l - \lambda)(\lambda^{r_1-1})(\lambda^{r_2-1})\ldots(\lambda^{r_k-1})\). Hence the total number of colourings of \( D \) is \((\lambda^l - \lambda)(\lambda^{r_1-1})(\lambda^{r_2-1})\ldots(\lambda^{r_k-1}) + \lambda(\lambda^{r_1-1} - 1)(\lambda^{r_2-1} - 1)\ldots(\lambda^{r_k-1} - 1)\). \( \square \)

**Corollary 3.2.13.** Let \( D \) be a digraph obtained from \( k \) directed cycles \( C_1, C_2, \ldots, C_k \) of length \( r_1, r_2, \ldots, r_k \) respectively and \( a=uv \) be a common arc to all \( C_i, \ 1 \leq i \leq k \) (see Figure 3.4). Then \( P(D; \lambda) = \lambda(\lambda - 1)(\lambda^{r_1+r_2+\ldots+r_k-2k}) + \lambda(\lambda^{r_1-2} - 1)(\lambda^{r_2-2} - 1)\ldots(\lambda^{r_k-2} - 1). \)

**Proof.** By Theorem 3.2.12, if both \( u \) and \( v \) are assigned the same colour, then the number of colouring is \( \lambda(\lambda^{r_1-2} - 1)(\lambda^{r_2-2} - 1)\ldots(\lambda^{r_k-2} - 1) \). If two vertices \( u \) and \( v \) are assigned the distinct colours, then the number of colouring of
3.3 Dichromatic number of Cartesian product of digraphs

In this section, we prove results on dichromaticity of Cartesian products of digraphs.

**Theorem 3.3.1.** Let $D_1$ and $D_2$ be any two digraphs. Then

$G(D_1 \square D_2) = G(D_1) \square G(D_2).$

**Proof.** Let $D_1$ and $D_2$ be two digraphs. Obviously, $V(G(D_1 \square D_2)) = V(G(D_1) \square G(D_2))$. We prove that every edge of $G(D_1 \square D_2)$ is an edge of $G(D_1) \square G(D_2)$. Suppose $(u_i, v_i)$ and $(u_j, v_j)$ are adjacent in $G(D_1 \square D_2)$. Here $u_i, u_j \in V(G(D_1))$ and $v_i, v_j \in V(G(D_2))$. Then either $(u_i, v_i) \rightarrow (u_j, v_j)$ or $(u_j, v_j) \rightarrow (u_i, v_i)$ in $D_1 \square D_2$. Without loss of generality, let us assume that $(u_i, v_i) \rightarrow (u_j, v_j)$. This implies that $u_i = u_j$ and $v_i \rightarrow v_j$ or $u_i \rightarrow u_j$ and $v_i = v_j$. So, either $v_i$ and $v_j$ are
adjacent in $G(D_2)$ or $u_i$ and $u_j$ are adjacent in $G(D_1)$. Hence $u_i = u_j$ and $v_i$ is adjacent to $v_j$ or $u_i$ is adjacent to $u_j$ and $v_i = v_j$ in $G(D_1) \square G(D_2)$. Therefore $G(D_1 \square D_2) \subseteq G(D_1) \square G(D_2)$. Next we can prove that every edge of $G(D_1) \square G(D_2)$ is an edge of $G(D_1 \square D_2)$. Let $(u_iv_j, u_kv_l) \in E(G(D_1) \square G(D_2))$. Then either $u_i = u_k$ and $v_j$ is adjacent to $v_l$ or $u_i$ is adjacent to $u_k$ and $v_j = v_l$. This implies $v_j \to v_l$ in $D_2$ and $u_i \to u_k$ in $D_1$. These two arcs in $D_1 \square D_2$ are the corresponding edges in $G(D_1) \square G(D_2)$. Therefore $G(D_1) \square G(D_2) \subseteq G(D_1 \square D_2)$. Hence $G(D_1 \square D_2) = G(D_1) \square G(D_2)$.

Lemma 3.3.2. If any two vertices in the directed cycle $C$ of an unicyclic digraph are assigned two different colours, then a colouring is possible in which all the remaining vertices are assigned with any one of the colours.

Proof. Let $D$ be the unicyclic digraph in which any two vertices in the unique directed cycle are assigned two different colours say $c_1$ and $c_2$. Let $v_i$ and $v_j$, $i < j$ be two vertices in the directed cycle $C$ of $D$. Now, we construct a colour sequence for $D$, where $v_j$ alone is assigned colour $c_2$ and all other vertices are assigned the colour $c_1$. Let the directed cycle $C$ be $v_1v_2...v_rv_1$. Removing all the arcs in the directed cycle $C$, we get $r$ subdigraphs in which each is acyclic. Moreover, each subdigraph contains only one vertex of $C$. By Theorem 2.2.5, all the vertices in all subdigraphs are assigned the colour $c_1$. Let $s_k$ be the colour sequence corresponding to the subdigraph that contains $v_k$, $k = 1, 2, \ldots, r$. Form a sequence $s_is_{i-1}...s_1s_1s_{r-1}...s_js_{j-1}...s_{i+1}$. This is a colour sequence for $D$ where $v_j$ alone is assigned the colour $c_2$ and all the other vertices are assigned the colour $c_1$. 

\[\square\]
3.3 Dichromatic number of Cartesian product of digraphs

**Theorem 3.3.3.** Let $D_1$ and $D_2$ be two digraphs. Then

$$\chi_d(D_1 \square D_2) = \max \{ \chi_d(D_1), \chi_d(D_2) \}.$$  

**Proof.** The Cartesian product $D_1 \square D_2$ contains copies of $D_1$ and $D_2$ as subdigraphs. So $\chi_d(D_1 \square D_2) \geq \max \{ \chi_d(D_1), \chi_d(D_2) \}$. Let $k = \max \{ \chi_d(D_1), \chi_d(D_2) \}$. To prove the upper bound, we produce a $k$-colouring of $D_1 \square D_2$ using optimal colourings of $D_1$ and $D_2$. Let $g$ be a $\chi_d(D_1)$-colouring of $D_1$ and $u_1, u_2, ..., u_{n_1}$ be the corresponding colour sequence of $D_1$. Let $h$ be a $\chi_d(D_2)$-colouring of $D_2$ and $v_1, v_2, ..., v_{n_2}$ be the corresponding colour sequence of $D_2$. Define a colouring $f$ of $D_1 \square D_2$ by letting $f(u, v) \equiv [g(u) + h(v)](\text{mod } k)$. Thus $f$ assigns colours to $V(D_1 \square D_2)$ from a set of size $k$.

Now, we prove that $f$ properly colours $D_1 \square D_2$ corresponding to the sequence

$$(u_1, v_1)(u_2, v_2)(u_3, v_3)...(u_{n_1}, v_{n_1})(u_2, v_1)(u_2, v_1)...(u_2, v_{n_2})...(u_{n_1}, v_1)...(u_{n_1}, v_{n_2}) \quad (3.1)$$

Suppose $(u, v)$ and $(u', v')$ are assigned the same colour and $(u, v) \rightarrow (u', v')$. Then we prove that $(u', v')$ precedes $(u, v)$ in the above sequence (3.1). $(u, v) \rightarrow (u', v')$ implies that either $u = u'$ and $v \rightarrow v'$ or $v = v'$ and $u \rightarrow u'$. Without loss of generality, let $u = u'$ and $v \rightarrow v'$. Now $f(u, v) = f(u', v')$. This implies that $g(u) + h(v) \equiv [g(u') + h(v')](\text{mod } k)$, implies that $h(v) = h(v')$, since $u = u'$ and $h(v), h(v') \leq k$. Therefore $v$ and $v'$ are assigned the same colour. Now $v$ and $v'$ are assigned the same colour and $v \rightarrow v'$. So, $v'$ must precede $v$ in the sequence $v_1, v_2, ..., v_{n_2}$. Hence $(u', v')$ precede $(u, v)$ in (3.1). Similarly, we can discuss the case for $v = v'$ and $u \rightarrow u'$. 

**Corollary 3.3.4.** $\chi_d(P_n \square P_m) = 1$. 
3.4 Dichromatic number of tensor product of digraphs

Proof. By Theorems 2.2.5 and 3.3.3, we get the required result.

Corollary 3.3.5. $\chi_d(P_n \square C_m) = 2$.

Proof. The dichromatic number of $P_n$ is one and dichromatic number of $C_n$ is two.

By Theorem 3.3.3, $\chi_d(P_n \square C_m) = 2$.

Corollary 3.3.6. $\chi_d(C_n \square C_m) = 2$.

Proof. Since $\chi_d(C_n) = 2$, this corollary is obvious by Theorem 3.3.3.

Corollary 3.3.7. $\chi_d(D_1 \square D_2) = 2$ if and only if either $\chi_d(D_1) = 2$ and $\chi_d(D_2) \leq 1$ or $\chi_d(D_2) = 2$ and $\chi_d(D_1) \leq 1$.

Proof. Let us assume that $\chi_d(D_1 \square D_2) = 2$. Then either $D_1$ or $D_2$ must contain a directed cycle, therefore either $\chi_d(D_1) = 2$ and $\chi_d(D_2) \leq 1$ or $\chi_d(D_2) = 2$ and $\chi_d(D_1) \leq 1$. Converse is obvious.

Corollary 3.3.8. $P((P_n \square P_m); \lambda) = \lambda^n$.

Proof. By Corollary 3.3.4 and by Theorem 3.2.1, we get the required result.

Corollary 3.3.9. $P((P_n \square C_m); \lambda) = P((C_n \square C_m); \lambda) = \lambda^n - \lambda$.

Proof. By Corollaries 3.3.5, 3.3.6 and 3.2.4, we get the required result.
3.4 Dichromatic number of tensor product of digraphs

We begin this section with few observations and prove results on dichromatic number of tensor product of digraphs.

**Observation 3.4.1.** Let us take two directed cycles of length $C_l$ and $C_m$ such that $l \leq m$. Let $C_l = v_1v_2\ldots v_l$ and $C_m = u_1u_2\ldots u_m$. In tensor product $C_n \times C_m$, there is a unique directed cycle of length $r = \text{lcm}(l, m)$, denoted by $[l, m]$. The cycle is as follows. $(u_1, v_1) \rightarrow (u_2, v_2) \rightarrow (u_3, v_3) \rightarrow \cdots \rightarrow (u_l, v_l) \rightarrow (u_{l+1}, v_1) \rightarrow (u_{l+2}, v_2) \rightarrow \cdots \rightarrow (u_m, v_1)$. For example, in $C_3 \times C_4$, $u_1v_1, u_2v_2, u_3v_3, u_1v_4, u_2v_1, u_3v_2, u_1v_3, u_2v_4, u_3v_1, u_1v_2, u_2v_3, u_3v_4, u_1v_1$ is a directed cycle (see Figure 3.5).

**Observation 3.4.2.** If $l$ and $m$ are odd positive integer, then $r = [l, m]$ is odd positive integer. If $l$ and $m$ are even positive integer, then $r = [l, m]$ is even positive integer. If
3.4 Dichromatic number of tensor product of digraphs

$l$ is even positive integer and $m$ is odd positive integer, then $r = [l, m]$ is even positive integer.

**Lemma 3.4.3.** $D_1 \times D_2$ has a directed cycle if and only if both $D_1$ and $D_2$ have directed cycle.

*Proof.* Let $D_1 \times D_2$ has a directed cycle. Let $u_1v_1, u_2v_2, \ldots, u_tv_t$ be a directed cycle. Let us arbitrarily choose any arc from $D_1 \times D_2$. Suppose $u_i v_j \rightarrow u_k v_l$, then $u_i \rightarrow u_k$ in $D_1$ and $v_j \rightarrow v_l$ in $D_2$. This is true for all arcs in the directed cycle. This implies that $D_1$ and $D_2$ have a directed cycle.

Conversely, let $D_1$ and $D_2$ have a directed cycle. Let $u_1u_2 \cdots u_l$ be a directed cycle of length $l$ in $D_1$ and $v_1v_2 \cdots v_m$ be a directed cycle of length $m$ in $D_2$. Without loss of generality, let $l \leq m$, by Observation 3.4.1, we get a directed cycle in $D_1 \times D_2$. \hfill \Box

**Corollary 3.4.4.** $D_1 \times D_2$ has an odd symmetric cycle if and only if both $D_1$ and $D_2$ have odd symmetric cycles.

*Proof.* Let $D_1 \times D_2$ has an odd symmetric cycle. Let $u_1v_1, u_2v_2, \ldots, u_tv_t$ be the odd symmetric cycle. Let us arbitrarily choose any symmetric arc from $D_1 \times D_2$. Suppose $u_iv_j$ is symmetric to $u_kv_l$, then $u_i$ is symmetric to $u_k$ in $D_1$ and $v_j$ is symmetric to $v_l$ in $D_2$. This is true for all the symmetric arcs in the odd symmetric cycle. This implies that $D_1$ and $D_2$ have odd symmetric cycles.

Conversely, let $D_1$ and $D_2$ have odd symmetric cycles. Let $u_1u_2 \cdots u_l$ be the odd symmetric cycle of length $l$ in $D_1$ and $v_1v_2 \cdots v_m$ be the odd symmetric cycle
of length $m$ in $D_2$. Without loss of generality, let $l \leq m$, by Observation 3.4.2, we get an odd symmetric cycle in $D_1 \times D_2$. 

Theorem 3.4.5. Let $D_1$ and $D_2$ be two digraphs. Then $\chi_d(D_1 \times D_2) \leq \min \{\chi_d(D_1), \chi_d(D_2)\}$.

Proof. Let $V(D_1) = \{u_1, u_2, ..., u_{n_1}\}$, $V(D_2) = \{v_1, v_2, ..., v_{n_2}\}$ and $\chi_d(D_1) = k$. Let $g$ be a $k$-colouring of $D_1$. Without loss of generality, let us assume that $u_1, u_2, ..., u_{n_1}$ be the corresponding colour sequence for $g$. Define a colouring $f$ for $D_1 \times D_2$ such that $f(u, v) = g(u)$. We claim that $f$ is a proper colouring of $D_1 \times D_2$. Consider the sequence

$$(u_1, v_1)(u_1, v_2) \cdots (u_1, v_{n_2})(u_2, v_1)(u_2, v_2) \cdots (u_2, v_{n_2}) \cdots (u_{n_1}, v_1) \cdots (u_{n_1}, v_{n_2}) \quad (3.2)$$

Suppose $(u, v)$ and $(u', v')$ are assigned the same colour and $(u, v) \rightarrow (u', v')$. Clearly $u$ and $u'$ are assigned the same colour, $u \rightarrow u'$ and $v \rightarrow v'$. Here $u'$ precedes $u$ in the colour sequence $u_1, u_2, ..., u_{n_1}$. Hence $(u', v')$ precedes $(u, v)$ in the sequence (3.2). So $f$ is a proper colouring of $D_1 \times D_2$ implying that $\chi_d(D_1 \times D_2) \leq \chi_d(D_1)$. Similarly, $\chi_d(D_1 \times D_2) \leq \chi_d(D_2)$. Hence $\chi_d(D_1 \times D_2) \leq \min\{\chi_d(D_1), \chi_d(D_2)\}$.

Corollary 3.4.6. $\chi_d(D_1 \times D_2) = 1$ if and only if either $D_1$ or $D_2$ is acyclic.

Proof. Let us assume that $\chi_d(D_1 \times D_2) = 1$. Then by Theorem 2.2.5, either $D_1$ or $D_2$ must be acyclic. Converse is obvious.

Theorem 3.4.7. $\chi_d(D_1 \times D_2) = 2$ if and only if either $\chi_d(D_1) = 2$ and $\chi_d(D_2) \geq 2$ or $\chi_d(D_1) \geq 2$ and $\chi_d(D_2) = 2$. 


Proof. Let \( \chi_d(D_1) = 2 \) and \( \chi_d(D_2) \geq 2 \). Then by Theorem 3.4.5, \( \chi_d(D_1 \times D_2) \leq 2 \). We claim that \( \chi_d(D_1 \times D_2) = 2 \). Suppose \( \chi_d(D_1 \times D_2) = 1 \). By Corollary 3.4.6, then there is no directed cycle in \( D_1 \times D_2 \). Since \( \chi_d(D_1) = 2 \) and \( \chi_d(D_2) \geq 2 \), \( D_1 \) and \( D_2 \) have directed cycles. By Lemma 3.4.3, \( D_1 \times D_2 \) also has a directed cycle, a contradiction. Hence \( \chi_d(D_1 \times D_2) = 2 \).

Conversely, let \( \chi_d(D_1 \times D_2) = 2 \). Obviously \( \chi_d(D_1) \) and \( \chi_d(D_2) \geq 2 \). We claim that either \( \chi_d(D_1) \) or \( \chi_d(D_2) = 2 \). Since \( \chi_d(D_1 \times D_2) = 2 \), any one of the conditions stated in the theorem is true. Let \( D_1 \times D_2 \) have directed cycle but no symmetric cycle. By Lemma 3.4.3, both \( D_1 \) and \( D_2 \) have directed cycle and none has symmetric cycle. So, by Theorem 2.2.6, \( \chi_d(D_1) = \chi_d(D_2) = 2 \). Let us assume that \( D_1 \times D_2 \) has only even symmetric cycles (one such symmetric cycle exists) and the set of vertices in the odd or even positions in each symmetric cycle do not form a directed cycle. In case, both \( D_1 \) and \( D_2 \) have odd symmetric cycles, then by Lemma 3.4.4, \( D_1 \times D_2 \) will have an odd symmetric cycle, so that \( \chi_d(D_1 \times D_2) \geq 3 \), a contradiction. So, either \( D_1 \) or \( D_2 \) have only even symmetric cycles. Without loss of generality, let \( D_1 \) have only even symmetric cycles. In case, in all the even symmetric cycles, the set of vertices in the odd or even positions in each symmetric cycles do not form a directed cycle, then \( \chi_d(D_1) = 2 \). Otherwise \( \chi_d(D_1) \geq 3 \). In this case, \( \chi_d(D_2) = 2 \), since \( 2 = \chi_d(D_1 \times D_2) = \min \{ \chi_d(D_1), \chi_d(D_2) \} \).

Corollary 3.4.8. \( \chi_d(P_n \times P_m) = 1 \).

Proof. The dichromatic number of path on \( n \) vertices is one. By Theorem 3.4.5,
3.4 Dichromatic number of tensor product of digraphs

corollary is true.

**Corollary 3.4.9.** $\chi_d(P_n \times C_m) = 1$.

*Proof.* The dichromatic number of $P_n$ is one and dichromatic number of $C_n$ is two. Therefore, by Theorem 3.4.5, $\chi_d(P_n \square C_m) = 1$. 

**Corollary 3.4.10.** $\chi_d(C_n \times C_m) = 2$.

*Proof.* The dichromatic number of $C_n$ is two. Hence, by Theorem 3.4.5, corollary is true.

**Corollary 3.4.11.** $P((P_n \times P_m); \lambda) = P((P_n \times C_m); \lambda) = \lambda^n$.

*Proof.* By Corollary 3.4.8 and by Theorem 3.2.1, corollary is true.

**Corollary 3.4.12.** $P((C_n \times C_m); \lambda) = \lambda^n - \lambda$.

*Proof.* By Corollary 3.4.10 and 3.3.5, corollary is true.

Here we propose the following two open problems based on dichromatic polynomial of digraphs.

**Problem 3.4.13.** Find dichromatic number of Strong and Lexicographic product of any two digraphs.

**Problem 3.4.14.** $P(D; \lambda)$ is a monic polynomial of degree $n$ in $\lambda$ with integer coefficients and constant term zero, in addition, its coefficients alternate in sign and the coefficient of $\lambda^{n-1}$ is $-m$. 