Chapter 2

Virus Spread in the Network Through an Intuitionistic Fuzzy Graph

2.1 INTRODUCTION

The first definition of intuitionistic fuzzy relations and intuitionistic fuzzy graphs were proposed by Atanassov (1999). Vinoth Kumar and Geetha Ramani (2011) introduced the concept of product intuitionistic fuzzy graphs. Karunambigai et al. (2011) proposed constant intuitionistic fuzzy graphs and totally constant intuitionistic fuzzy graphs. Akram and Davvaz (2012) described the idea of strong intuitionistic fuzzy graphs and examined some of their properties and propositions. The energy of the graph is extended to the energy of the fuzzy graph by Anjali and Mathew (2013).

In computer networks, we frequently come across the link structure of a website that can be demonstrated by the directed graph in terms of their links and the way interfacing between the links. In many situations, the complete topology for the alive network might not have been constantly accessible to the communication systems in a state of time because of the reason that a few or many of its links (edges or arcs) might be incidentally disabled owing to harm or attack.
or blockage upon them. Obviously, they are under repair by then of time. Such cases are so frequent nowadays that it calls for rigorous and good consideration of the researchers, in especially the individuals who are worried with great quality of service while in a network. Indeed, even in at most of the cases, the parameters relating to its links are not crisp numbers, rather intuitionistic fuzzy numbers or fuzzy numbers.

Many of the real life environments can be represented by graph. However, when the system or network is large and complex, it is difficult to extract the exact information about the system using the classical graph theory. In such cases, the fuzzy graph or intuitionistic fuzzy graph is used to analyze the system. In our proposed work, the concept of energy of the fuzzy graph is extended to the energy of the intuitionistic fuzzy graph.

In this chapter, we consider an intuitionistic fuzzy graph. The energy of such an intuitionistic fuzzy graph is determined and the lower and the upper bound for the energy of an intuitionistic fuzzy graph are derived. There are no exact methods to analyze the virus spread in the user flow of the website. Hence we made an attempt to find the virus spread in a network in terms of energy of an intuitionistic fuzzy graph. These concepts are explained and illustrated by taking the website network of the web navigation of http://www.pantechsolutions.net/.

2.2 ENERGY OF AN INTUITIONISTIC FUZZY GRAPH

**Definition 2.1.** An intuitionistic fuzzy graph is defined as $G = (V, E, \mu, \gamma)$ where $V$ is the set of vertices and $E$ is the set of edges. $\mu$ is an intuitionistic fuzzy membership value defined on $V \times V$ and $\gamma$ is an intuitionistic fuzzy non-membership value defined on $V \times V$. We denote $\mu(v_i, v_j)$ by $\mu_{ij}$ and $\gamma(v_i, v_j)$ by $\gamma_{ij}$ such that
(i) $0 \leq \mu_{ij} + \gamma_{ij} \leq 1$ (ii) $0 \leq \mu_{ij}, \gamma_{ij}, \pi_{ij} \leq 1$, where $\pi_{ij} = 1 - \mu_{ij} - \gamma_{ij}$. Also we denote $\mu_{ij}$ represents the strength of the relationship between $v_i$ and $v_j$ and $\gamma_{ij}$ represents the strength of the non-relationship between $v_i$ and $v_j$.

**Definition 2.2.** An intuitionistic fuzzy adjacency matrix of an intuitionistic fuzzy graph $G = (V, E, \mu, \gamma)$ is defined as $A(G) = [a_{ij}]$ where $a_{ij} = (\mu_{ij}, \gamma_{ij})$.

**Definition 2.3.** An intuitionistic fuzzy adjacency matrix of an intuitionistic fuzzy graph can be written as two matrices one containing the entries as membership value and the other containing the entries as non-membership value. i.e. $A(G) = [(\mu_{ij}, \gamma_{ij})]$. We denote $A_\mu(G) = [\mu_{ij}]$ is an intuitionistic fuzzy adjacency matrix of membership value and $A_\gamma(G) = [\gamma_{ij}]$ is an intuitionistic fuzzy adjacency matrix of non-membership value.

**Definition 2.4.** The spectrum of an intuitionistic fuzzy adjacency matrix $A(G)$ is defined as $(X, Y)$ where $X$ is the set eigenvalues of $A_\mu(G)$ and $Y$ is the set of eigenvalues of $A_\gamma(G)$. It is denoted by $\text{spec}(A(G))$.

**Definition 2.5.** Let $\lambda_1, \lambda_2, ..., \lambda_n$ be the eigen values of $A_\mu(G)$ and let $\delta_1, \delta_2, ..., \delta_n$ be the eigen values of $A_\gamma(G)$. Then the energy of an intuitionistic fuzzy graph $G = (V, E, \mu, \gamma)$ is defined as

$$E(A(G)) = \left( \sum_{i=1}^{n} |\lambda_i|, \sum_{i=1}^{n} |\delta_i| \right)$$

where

$$\sum_{i=1}^{n} |\lambda_i|$$

is the energy of membership value denoted by $E(A_\mu(G))$ and

$$\sum_{i=1}^{n} |\delta_i|$$

is the energy of non-membership value denoted by $E(A_\gamma(G))$. 

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Theorem 2.1. Let $G = (V, E, \mu, \gamma)$ be an intuitionistic fuzzy graph with $n$ vertices. If $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the eigen values (real or complex) of $A_\mu(G)$ and $\delta_1, \delta_2, \ldots, \delta_n$ are the eigen values (real or complex) of $A_\gamma(G)$, then

\[
\begin{align*}
(i) \quad & \sum_{i=1}^{n} \lambda_i = 0 \\
(ii) \quad & \sum_{i=1}^{n} \lambda_i^2 = 2 \sum_{1 \leq i < j \leq n} \mu_{ij} \mu_{ji} \\
(iii) \quad & \sum_{i=1}^{n} \delta_i = 0 \\
(iv) \quad & \sum_{i=1}^{n} \delta_i^2 = 2 \sum_{1 \leq i < j \leq n} \gamma_{ij} \gamma_{ji}.
\end{align*}
\]

Proof. (i) We know that the sum of the eigen values of $A_\mu(G)$ is equal to the trace of $A_\mu(G)$. That is,

\[
\sum_{i=1}^{n} \lambda_i = \text{trace}(A_\mu(G)) = \sum_{i=1}^{n} \mu_{ii} = 0.
\]

(ii) Also the sum of the squares of the eigen values of $A_\mu(G)$ is equal to the trace of $(A_\mu(G))^2$, that is

\[
\sum_{i=1}^{n} \lambda_i^2 = \text{trace}(A_\mu(G))^2 = 0 + \mu_{12} \mu_{21} + \ldots + \mu_{1n} \mu_{n1} + \\
\mu_{21} \mu_{12} + 0 + \ldots + \mu_{2n} \mu_{n2} + \\
\mu_{n1} \mu_{1n} + \mu_{n2} \mu_{2n} + \ldots + 0.
\]

This implies that,

\[
\sum_{i=1}^{n} \lambda_i^2 = 2 \sum_{1 \leq i < j \leq n} \mu_{ij} \mu_{ji}.
\]

Similarly we can prove (iii) and (iv). \hfill \square

Theorem 2.2. Let $G$ be an intuitionistic fuzzy directed graph (without loops) with $n$ vertices. If $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the real eigen values of $A_\mu(G)$ and $\delta_1, \delta_2, \ldots, \delta_n$ are the real eigen values of $A_\gamma(G)$, then
(i) \( \sqrt{2 \sum_{1 \leq i < j \leq n} \mu_{ij}\mu_{ji} + n(n - 1)|A|^2} \leq E(A_\mu(G)) \leq \sqrt{2n \sum_{1 \leq i < j \leq n} \mu_{ij}\mu_{ji}} \)

(ii) \( \sqrt{2 \sum_{1 \leq i < j \leq n} \gamma_{ij}\gamma_{ji} + n(n - 1)|B|^2} \leq E(A_\gamma(G)) \leq \sqrt{2n \sum_{1 \leq i < j \leq n} \gamma_{ij}\gamma_{ji}} \)

where \(|A|\) is the determinant of \(A_\mu(G)\) and \(|B|\) is the determinant of \(A_\gamma(G)\).

Proof. (i) Upper bound: In Cauchy-Schwarz inequality
\[
\left( \sum_{i=1}^{n} a_i b_i \right)^2 \leq \left( \sum_{i=1}^{n} a_i^2 \right) \left( \sum_{i=1}^{n} b_i^2 \right)
\]

put \(a_i = 1\) and \(b_i = |\lambda_i|\), we get
\[
\sum_{i=1}^{n} |\lambda_i| \leq \sqrt{n} \left( \sum_{i=1}^{n} |\lambda_i|^2 \right)^{1/2}.
\]

We know that
\[
\left( \sum_{i=1}^{n} \lambda_i \right)^2 = \sum_{i=1}^{n} |\lambda_i|^2 + 2 \sum_{1 \leq i < j \leq n} \lambda_i \lambda_j.
\]

By comparing the coefficients of \(\lambda^{n-2}\) in the characteristic polynomial
\[
\prod_{i=1}^{n} (\lambda - \lambda_i) = |A - \lambda I|,
\]

we get
\[
\sum_{1 \leq i < j \leq n} \lambda_i \lambda_j = - \sum_{1 \leq i < j \leq n} \mu_{ij}\mu_{ji}.
\]

Substitute equation (2.4) in equation (2.2) and by Theorem 2.1, we get
\[
\sum_{i=1}^{n} |\lambda_i|^2 = 2 \sum_{1 \leq i < j \leq n} \mu_{ij}\mu_{ji}.
\]

Substitute equation (2.5) in equation (2.1), we get
\[
\sum_{i=1}^{n} |\lambda_i| \leq \sqrt{n} \left( \sum_{1 \leq i < j \leq n} \mu_{ij}\mu_{ji} \right)^{1/2}
\]

\[
\Rightarrow E(A_\mu(G)) \leq \sqrt{2n \sum_{1 \leq i < j \leq n} \mu_{ij}\mu_{ji}}.
\]
Lower bound:

\[
(E (A_\mu (G)))^2 = \left( \sum_{i=1}^{n} |\lambda_i| \right)^2
\]

\[
= \sum_{i=1}^{n} |\lambda_i|^2 + 2 \sum_{1 \leq i < j \leq n} |\lambda_i\lambda_j|
\]

\[
= 2 \sum_{1 \leq i < j \leq n} \mu_{ij}\mu_{ji} + \frac{2n(n-1)}{2} AM \{|\lambda_i\lambda_j|\}.
\]

Since

\[
AM \{|\lambda_i\lambda_j|\} \geq GM \{|\lambda_i\lambda_j|\}, 1 \leq i < j \leq n,
\]

where AM and GM are the arithmetic and geometric mean respectively, we get

\[
E (A_\mu (G)) \geq \sqrt{2 \sum_{1 \leq i < j \leq n} \mu_{ij}\mu_{ji} + n(n-1) GM \{|\lambda_i\lambda_j|\}}. \tag{2.7}
\]

But

\[
GM \{|\lambda_i\lambda_j|\} = \left( \prod_{1 \leq i < j \leq n} |\lambda_i\lambda_j| \right)^{\frac{2}{n(n-1)}}
\]

\[
= \left( \prod_{i=1}^{n} |\lambda_i|^{n-1} \right)^{\frac{2}{n(n-1)}}
\]

\[
= \left( \prod_{i=1}^{n} |\lambda_i| \right)^{\frac{2}{n}} = |A|^{\frac{2}{n}}.
\]

Therefore equation (2.7) becomes

\[
E (A_\mu (G)) \geq \sqrt{2 \sum_{1 \leq i < j \leq n} \mu_{ij}\mu_{ji} + n(n-1) |A|^\frac{2}{n}}. \tag{2.8}
\]

From the above equations (2.6) and (2.8) we conclude that the upper and lower bound of \(E(A_\mu(G))\) as follows

\[
\sqrt{2 \sum_{1 \leq i < j \leq n} \mu_{ij}\mu_{ji} + n(n-1) |A|^\frac{2}{n}} \leq E (A_\mu (G)) \leq \sqrt{2n \sum_{1 \leq i < j \leq n} \mu_{ij}\mu_{ji}}.
\]

Similarly, we can prove the upper and lower bound of \(E(A_\gamma(G))\).
Corollary 2.1. If $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the real eigen values of $A_{\mu}(G)$ and $\delta_1, \delta_2, \ldots, \delta_n$ are the real eigen values of $A_{\gamma}(G)$ and if $E(A_{\mu}(G)) \geq E(A_{\gamma}(G))$, then

$$
2n \sum_{1 \leq i < j \leq n} \mu_{ij} \mu_{ji} \geq 2 \sum_{1 \leq i < j \leq n} \gamma_{ij} \gamma_{ji} + n(n-1)|B|^2.
$$

Proof. From Theorem 2.2, we have

$$
\sqrt{2n \sum_{1 \leq i < j \leq n} \mu_{ij} \mu_{ji}} \geq E(A_{\mu}(G))
$$

$$
\Rightarrow \sqrt{2n \sum_{1 \leq i < j \leq n} \mu_{ij} \mu_{ji}} \geq E(A_{\gamma}(G))
$$

$$
\Rightarrow 2n \sum_{1 \leq i < j \leq n} \mu_{ij} \mu_{ji} \geq (E(A_{\gamma}(G)))^2
$$

$$
\Rightarrow 2n \sum_{1 \leq i < j \leq n} \mu_{ij} \mu_{ji} \geq \left( \sum_{i=1}^{n} |\delta_i| \right)^2
$$

$$
\Rightarrow 2n \sum_{1 \leq i < j \leq n} \mu_{ij} \mu_{ji} \geq \sum_{i=1}^{n} |\delta_i|^2 + 2 \sum_{1 \leq i < j \leq n} |\delta_i \delta_j|
$$

$$
\Rightarrow 2n \sum_{1 \leq i < j \leq n} \mu_{ij} \mu_{ji} \geq 2 \sum_{1 \leq i < j \leq n} \gamma_{ij} \gamma_{ji} + 2 \sum_{1 \leq i < j \leq n} |\delta_i \delta_j|
$$

$$
\Rightarrow 2n \sum_{1 \leq i < j \leq n} \mu_{ij} \mu_{ji} \geq 2 \sum_{1 \leq i < j \leq n} \gamma_{ij} \gamma_{ji} + n(n-1)|B|^2.
$$

Alternative proof. From Theorem 2.2, we have

$$
\sqrt{2 \sum_{1 \leq i < j \leq n} \gamma_{ij} \gamma_{ji} + n(n-1)|B|^2} \leq E(A_{\gamma}(G))
$$

$$
\Rightarrow \sqrt{2 \sum_{1 \leq i < j \leq n} \gamma_{ij} \gamma_{ji} + n(n-1)|B|^2} \leq E(A_{\mu}(G))
$$
\[
\Rightarrow \sqrt{2 \sum_{1 \leq i < j \leq n} \gamma_{ij} \gamma_{ji} + n(n-1)|B|^2} \leq \sqrt{2n \sum_{1 \leq i < j \leq n} \mu_{ij} \mu_{ji}}
\]

\[
\Rightarrow 2 \sum_{1 \leq i < j \leq n} \gamma_{ij} \gamma_{ji} + n(n-1)|B|^2 \leq 2n \sum_{1 \leq i < j \leq n} \mu_{ij} \mu_{ji}
\]

Hence the proof. \(\Box\)

**Corollary 2.2.** If \(\lambda_1, \lambda_2, ..., \lambda_n\) are the real eigen values of \(A_\mu(G)\) and \(\delta_1, \delta_2, ..., \delta_n\) are the real eigen values of \(A_\gamma(G)\) and if \(E(A_\gamma(G)) \geq E(A_\mu(G))\), then

\[
2n \sum_{1 \leq i < j \leq n} \gamma_{ij} \gamma_{ji} \geq 2 \sum_{1 \leq i < j \leq n} \mu_{ij} \mu_{ji} + n(n-1)|A|^2.
\]

**Proof.** From Theorem 2.2, we have

\[
\sqrt{2n \sum_{1 \leq i < j \leq n} \gamma_{ij} \gamma_{ji}} \geq E(A_\gamma(G))
\]

\[
\Rightarrow \sqrt{2n \sum_{1 \leq i < j \leq n} \gamma_{ij} \gamma_{ji}} \geq E(A_\mu(G))
\]

\[
\Rightarrow 2n \sum_{1 \leq i < j \leq n} \gamma_{ij} \gamma_{ji} \geq (E(A_\mu(G)))^2
\]

\[
\Rightarrow 2n \sum_{1 \leq i < j \leq n} \gamma_{ij} \gamma_{ji} \geq \left( \sum_{i=1}^{n} |\lambda_i| \right)^2
\]

\[
\Rightarrow 2n \sum_{1 \leq i < j \leq n} \gamma_{ij} \gamma_{ji} \geq \sum_{i=1}^{n} |\lambda_i|^2 + 2 \sum_{1 \leq i < j \leq n} |\lambda_i \lambda_j|
\]

\[
\Rightarrow 2n \sum_{1 \leq i < j \leq n} \gamma_{ij} \gamma_{ji} \geq 2 \sum_{1 \leq i < j \leq n} \mu_{ij} \mu_{ji} + 2 \sum_{1 \leq i < j \leq n} |\lambda_i \lambda_j|
\]

\[
\Rightarrow 2n \sum_{1 \leq i < j \leq n} \gamma_{ij} \gamma_{ji} \geq 2 \sum_{1 \leq i < j \leq n} \mu_{ij} \mu_{ji} + n(n-1)|A|^2.
\]
Alternative proof. From Theorem 2.2, we have

\[
\sqrt{2 \sum_{1 \leq i < j \leq n} \mu_{ij} \mu_{ji} + n(n-1)|A|^2} \leq E(A_\mu(G))
\]

\[
\Rightarrow \sqrt{2 \sum_{1 \leq i < j \leq n} \mu_{ij} \mu_{ji} + n(n-1)|A|^2} \leq E(A_\gamma(G))
\]

\[
\Rightarrow \sqrt{2 \sum_{1 \leq i < j \leq n} \mu_{ij} \mu_{ji} + n(n-1)|A|^2} \leq \sqrt{2n \sum_{1 \leq i < j \leq n} \gamma_{ij} \gamma_{ji}}
\]

\[
\Rightarrow 2 \sum_{1 \leq i < j \leq n} \mu_{ij} \mu_{ji} + n(n-1)|A|^2 \leq 2n \sum_{1 \leq i < j \leq n} \gamma_{ij} \gamma_{ji}.
\]

Hence the proof. \(\square\)

**Theorem 2.3.** Let \(G\) be an intuitionistic fuzzy directed graph (without loops) with \(n\) vertices and if \(\lambda_1, \lambda_2, \ldots, \lambda_n\) are the complex eigen values of \(A_\mu(G)\) and \(\delta_1, \delta_2, \ldots, \delta_n\) are the complex eigen values of \(A_\gamma(G)\), then

\[
(i) \quad \sqrt{\sum_{i=1}^{n} |\lambda_i|^2 + n(n-1)|A|^2} \leq E(A_\mu(G)) \leq \sqrt{n \left[ \left( \sum_{i=1}^{n} |\lambda_i| \right)^2 - 2 \sum_{1 \leq i < j \leq n} |\lambda_i \lambda_j| \right]}
\]

where \(|A|\) is the determinant of \(A_\mu(G)\) and

\[
(ii) \quad \sqrt{\sum_{i=1}^{n} |\delta_i|^2 + n(n-1)|B|^2} \leq E(A_\gamma(G)) \leq \sqrt{n \left[ \left( \sum_{i=1}^{n} |\delta_i| \right)^2 - 2 \sum_{1 \leq i < j \leq n} |\delta_i \delta_j| \right]}
\]

where \(|B|\) is the determinant of \(A_\gamma(G)\).

Proof. (i) Upper bound: In Cauchy-Schwarz inequality

\[
\left( \sum_{i=1}^{n} a_i b_i \right)^2 \leq \left( \sum_{i=1}^{n} a_i^2 \right) \left( \sum_{i=1}^{n} b_i^2 \right)
\]

put \(a_i = 1\) and \(b_i = |\lambda_i|\), we get

\[
\sum_{i=1}^{n} |\lambda_i| \leq \sqrt{n \sum_{i=1}^{n} |\lambda_i|^2}.
\]
We know that

\[
\left( \sum_{i=1}^{n} |\lambda_i| \right)^2 = \sum_{i=1}^{n} |\lambda_i|^2 + 2 \sum_{1 \leq i < j \leq n} |\lambda_i \lambda_j| \tag{2.10}
\]

\[
\Rightarrow \sum_{i=1}^{n} |\lambda_i|^2 = \left( \sum_{i=1}^{n} |\lambda_i| \right)^2 - 2 \sum_{1 \leq i < j \leq n} |\lambda_i \lambda_j|. \tag{2.11}
\]

Substitute equation (2.11) in equation (2.9), we get

\[
\sum_{i=1}^{n} |\lambda_i| \leq \sqrt{n} \sqrt{\left( \sum_{i=1}^{n} |\lambda_i| \right)^2 - 2 \sum_{1 \leq i < j \leq n} |\lambda_i \lambda_j|}
\]

\[
\Rightarrow E(A_\mu(G)) \leq \sqrt{n \left[ \left( \sum_{i=1}^{n} |\lambda_i| \right)^2 - 2 \sum_{1 \leq i < j \leq n} |\lambda_i \lambda_j| \right]}. \tag{2.12}
\]

Lower bound:

\[
(E(A_\mu(G)))^2 = \left( \sum_{i=1}^{n} |\lambda_i| \right)^2
\]

\[
= \sum_{i=1}^{n} |\lambda_i|^2 + 2 \sum_{1 \leq i < j \leq n} |\lambda_i \lambda_j|
\]

\[
= \sum_{i=1}^{n} |\lambda_i|^2 + \frac{2n(n-1)}{2} AM \{|\lambda_i \lambda_j|\}.
\]

Since

\[
AM \{|\lambda_i \lambda_j|\} \geq GM \{|\lambda_i \lambda_j|\}, 1 \leq i < j \leq n,
\]

we get

\[
E(A_\mu(G)) \geq \sqrt{\sum_{i=1}^{n} |\lambda_i|^2 + n(n-1) GM \{|\lambda_i \lambda_j|\}}. \tag{2.13}
\]

But

\[
GM \{|\lambda_i \lambda_j|\} = \left( \prod_{1 \leq i < j \leq n} |\lambda_i \lambda_j| \right)^{2 \frac{n(n-1)}{n-1}}
\]

\[
= \left( \prod_{i=1}^{n} |\lambda_i|^{n-1} \right)^{\frac{2}{n(n-1)}}
\]

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Therefore equation (2.13) becomes

\[
E(A_\mu(G)) \geq \sqrt{\sum_{i=1}^{n} |\lambda_i|^2 + n(n-1)|A|^2}.
\]

(2.14)

From the above equations (2.12) and (2.14) we have,

\[
\sqrt{n \sum_{i=1}^{n} |\lambda_i|^2 + n(n-1)|A|^2} \leq E(A_\mu(G)) \leq \sqrt{n \left( \sum_{i=1}^{n} |\lambda_i|^2 - 2 \sum_{1 \leq i < j \leq n} |\lambda_i \lambda_j| \right)}.
\]

Similarly, we can prove (ii).

Corollary 2.3. If \(\lambda_1, \lambda_2, ..., \lambda_n\) are the complex eigen values of \(A_\mu(G)\) and \(\delta_1, \delta_2, ..., \delta_n\) are the complex eigen values of \(A_\gamma(G)\) and if \(E(A_\mu(G)) \geq E(A_\gamma(G))\), then

\[
n \left[ \sum_{i=1}^{n} |\lambda_i|^2 - 2 \sum_{1 \leq i < j \leq n} |\lambda_i \lambda_j| \right] \geq \sum_{i=1}^{n} |\delta_i|^2 + n(n-1)|B|^2.
\]

Proof. From Theorem 2.3, we have

\[
\sqrt{n \left[ \left( \sum_{i=1}^{n} |\lambda_i|^2 \right)^2 - 2 \sum_{1 \leq i < j \leq n} |\lambda_i \lambda_j| \right]} \geq E(A_\mu(G))
\]

\[
\Rightarrow \sqrt{n \left[ \left( \sum_{i=1}^{n} |\lambda_i|^2 \right)^2 - 2 \sum_{1 \leq i < j \leq n} |\lambda_i \lambda_j| \right]} \geq E(A_\gamma(G))
\]

\[
\Rightarrow \sqrt{n \left[ \left( \sum_{i=1}^{n} |\lambda_i|^2 \right)^2 - 2 \sum_{1 \leq i < j \leq n} |\lambda_i \lambda_j| \right]} \geq \sqrt{\sum_{i=1}^{n} |\delta_i|^2 + n(n-1)|B|^2}
\]

\[
\Rightarrow n \left[ \left( \sum_{i=1}^{n} |\lambda_i|^2 \right)^2 - 2 \sum_{1 \leq i < j \leq n} |\lambda_i \lambda_j| \right] \geq \sum_{i=1}^{n} |\delta_i|^2 + n(n-1)|B|^2.
\]

Hence the proof.
Corollary 2.4. If \( \lambda_1, \lambda_2, \ldots, \lambda_n \) are the complex eigen values of \( A_\mu(G) \) and \( \delta_1, \delta_2, \ldots, \delta_n \) are the complex eigen values of \( A_\gamma(G) \) and if \( E(A_\gamma(G)) \geq E(A_\mu(G)) \), then

\[
 n \left[ \sum_{i=1}^{n} |\delta_i|^2 - 2 \sum_{1 \leq i < j \leq n} |\delta_i \delta_j| \right] \geq \sum_{i=1}^{n} |\lambda_i|^2 + n(n-1)|A|^\frac{2}{n}.
\]

Proof. From Theorem 2.3, we have

\[
 \sqrt{n \left[ \sum_{i=1}^{n} |\delta_i|^2 - 2 \sum_{1 \leq i < j \leq n} |\delta_i \delta_j| \right]} \geq E(A_\gamma(G))
\]

\[
 \Rightarrow \sqrt{n \left[ \sum_{i=1}^{n} |\delta_i|^2 - 2 \sum_{1 \leq i < j \leq n} |\delta_i \delta_j| \right]} \geq E(A_\mu(G))
\]

\[
 \Rightarrow \sqrt{n \left[ \sum_{i=1}^{n} |\delta_i|^2 - 2 \sum_{1 \leq i < j \leq n} |\delta_i \delta_j| \right]} \geq \sqrt{\sum_{i=1}^{n} |\lambda_i|^2 + n(n-1)|A|^\frac{2}{n}}
\]

\[
 \Rightarrow n \left[ \sum_{i=1}^{n} |\delta_i|^2 - 2 \sum_{1 \leq i < j \leq n} |\delta_i \delta_j| \right] \geq \sum_{i=1}^{n} |\lambda_i|^2 + n(n-1)|A|^\frac{2}{n}.
\]

Hence the proof. \( \Box \)

2.3 SPREADING RATE OF VIRUS IN AN INTUITIONISTIC FUZZY GRAPH

In this section, we consider the intuitionistic fuzzy graph \( G = (V, E, \mu, \gamma) \) and we define the infection rate, curing rate and the sharp epidemic threshold of the virus spread for the given intuitionistic fuzzy graph.

Definition 2.6. If \( E(A_\mu(G)) > E(A_\gamma(G)) \) (i.e. if the number of visitors are maximum in an intuitionistic fuzzy graph), then the spreading rate of virus is maximum. Otherwise, we can say that the system or network is unstable.
**Definition 2.7.** If $E(A_\mu(G)) > E(A_\gamma(G))$ then the infection rate of an intuitionistic fuzzy graph is defined as $\beta = \max_{i,j} \mu_{ij}$ and the curing rate of an intuitionistic fuzzy graph is defined as $\delta = \min_{i,j} \gamma_{ij}$. The ratio $\tau = \frac{\beta}{\delta}$ ($\delta \neq 0$) is the effective spreading rate and $\tau_c = \frac{1}{\lambda_{\text{max}} A_\mu(G)}$ where $\lambda_{\text{max}} A_\mu(G)$ is the largest eigen value of the adjacency matrix $A_\mu(G)$ of an intuitionistic fuzzy graph.

**Definition 2.8.** If $E(A_\mu(G)) < E(A_\gamma(G))$ (i.e. if the number of visitors are minimum in an intuitionistic fuzzy graph), then the spreading rate of virus is minimum. Otherwise, we can say that the system or network is stable.

**Definition 2.9.** If $E(A_\mu(G)) < E(A_\gamma(G))$ then the infection rate of an intuitionistic fuzzy graph is defined as $\beta = \min_{i,j} \mu_{ij}$ and the curing rate of an intuitionistic fuzzy graph is defined as $\delta = \max_{i,j} \gamma_{ij}$. The ratio $\tau = \frac{\beta}{\delta}$ is the effective spreading rate and $\tau_c = \frac{1}{\lambda_{\text{max}} A_\gamma(G)}$ where $\lambda_{\text{max}} A_\gamma(G)$ is the largest eigen value of the adjacency matrix $A_\gamma(G)$ of an intuitionistic fuzzy graph.

The above mentioned concepts are illustrated in the following example.

![Intuitionistic Fuzzy Graph](image)

**Figure 2.1: Intuitionistic Fuzzy Graph**
**Example 2.1.** Consider the intuitionistic fuzzy graph as shown in Figure 2.1. For this graph, the intuitionistic fuzzy adjacency matrix is given by

\[
A(G) = \begin{pmatrix}
0 & (0.6, 0.2) & 0 & (0.3, 0.6) \\
(0.4, 0.3) & 0 & (0.5, 0.1) & 0 \\
(0.5, 0.2) & (0.7, 0.1) & 0 & (0.6, 0.3) \\
(0.9, 0.1) & (0.8, 0.1) & (0.3, 0.5) & 0
\end{pmatrix}
\]

where

\[
A_\mu(G) = \begin{pmatrix}
0 & 0.6 & 0 & 0.3 \\
0.4 & 0 & 0.5 & 0 \\
0.5 & 0.7 & 0 & 0.6 \\
0.9 & 0.8 & 0.3 & 0
\end{pmatrix}
\]

and
\[
A_\gamma(G) = \begin{pmatrix}
0 & 0.2 & 0 & 0.6 \\
0.3 & 0 & 0.1 & 0 \\
0.2 & 0.1 & 0 & 0.3 \\
0.1 & 0.1 & 0.5 & 0
\end{pmatrix}
\]

For the intuitionistic fuzzy graph in Figure 2.1, the eigen values of \( A_\mu(G) \) and \( A_\gamma(G) \) are given by

\[
\text{Spec} (A_\mu(G)) = \{1.2406, -0.7153, -0.2627 + 0.2332i, -0.2627 - 0.2332i\}
\]

\[
\text{Spec} (A_\gamma(G)) = \{0.6441, -0.0148, -0.3146 + 0.1629i, -0.3146 - 0.1629i\}
\]

\[
E(A_\mu(G)) = 1.2406 + 0.7153 + 0.3513 + 0.3513 = 2.6585
\]

\[
E(A_\gamma(G)) = 0.6441 + 0.0148 + 0.3543 + 0.3543 = 1.3675.
\]

The lower bound and upper bound of \( E(A_\mu(G)) \) and \( E(A_\gamma(G)) \) are 2.5037, 3.0316 and 1.0401, 1.6324 respectively. From Theorem 2.3, we have

\[
(i) \ 2.5037 \leq 2.6585 \leq 3.0316
\]

\[
(ii) \ 1.0401 \leq 1.3675 \leq 1.6324.
\]
By Corollary 2.3, we have

\[ 3.0316 \geq 1.0401. \]

The energy of the intuitionistic fuzzy graph in Figure 2.1, is given by

\[ E(A(G)) = (2.6585, 1.3675). \]

Here we note that the energy of membership value is greater than the energy of non-membership value. That is,

\[ E(A_\mu(G)) > E(A_\gamma(G)). \]

Hence there are more number of visitors in the link structure, so the spreading rate of virus is maximum.

From Definition 2.7, the infection rate, curing rate and the sharp epidemic threshold are determined as follows.

The infection rate and the curing rate are given by

\[ \beta = \max_{i,j} \mu_{ij} = 0.9, \quad \delta = \min_{i,j} \gamma_{ij} = 0.1. \]

The spreading rate of virus and the sharp epidemic threshold are given by

\[ \tau = \frac{\beta}{\delta} = 9, \quad \tau_c = \frac{1}{\lambda_{\max} A_\mu(G)} = \frac{1}{1.2406} = 0.8060. \]

Here \( \tau > \tau_c \), so the virus continue and a nonzero fraction of the nodes are infected. Hence the spreading rate of virus is maximum.

2.4 NUMERICAL EXAMPLES

In this section, we illustrate the above concepts by taking the website http://www.pantechsolutions.net/ . This website is modeled as an intuitionistic fuzzy graph by considering the navigation of the customers.
The intuitionistic fuzzy graph of this website for four different time periods is taken for each of these periods, the energy of an intuitionistic fuzzy graph, its lower and upper bounds and the spreading rate of viruses are determined in the following examples. We have taken the four links 1. microcontroller-boards, 2. /log-in html, 3. / and 4. /project kits for our calculation.

Figure 2.2: User Flow of the Website http://www.pantechsolutions.net/

Example 2.2. In this website http://www.pantechsolutions.net/, we consider four links 1. microcontroller-boards, 2. /log-in html, 3. / and 4. project kits for the period July 16, 2013 to August 15, 2013 (Period I).
For the intuitionistic fuzzy graph in Figure 2.3, the eigen values of $A_{\mu}(G)$ and $A_{\gamma}(G)$ are given by

\[
\text{Spec}(A_{\mu}(G)) = \{-0.2000, 0.0000, 0.2000, 0.0000\}
\]
\[
\text{Spec}(A_{\gamma}(G)) = \{0.9927, -0.1841, -0.4043 + 0.2306i, -0.4043 - 0.2306i\}
\]
\[
E(A_{\mu}(G)) = 0.2000 + 0.0000 + 0.2000 + 0.0000 = 0.4
\]
\[
E(A_{\gamma}(G)) = 0.9927 + 0.1841 + 0.4655 + 0.4655 = 2.1078.
\]

The lower bound and upper bound of $E(A_{\mu}(G))$ and $E(A_{\gamma}(G))$ are 0.2828, 0.5657 and 1.9598, 2.4106 respectively. From Theorem 2.3, we have

(i) $0.2828 \leq 0.4 \leq 0.5657$

(ii) $1.9598 \leq 2.1078 \leq 2.4106$.

By Corollary 2.4, we have

$2.4106 \geq 0.2828$.

For the intuitionistic fuzzy graph in Figure 2.3, the energy is given by

\[
E(A(G)) = (0.4, 2.1078).
\]
Here we note that the energy of membership value is less than the energy of non-membership value. Hence there are less number of visitors in the link structure so the spreading rate of virus is minimum. From Definition 2.9, the infection rate, curing rate and the sharp epidemic threshold are determined as follows.

The infection rate and the curing rate are given by

\[ \beta = \min_{i,j} \mu_{ij} = 0.0 \quad \text{and} \quad \delta = \max_{i,j} \gamma_{ij} = 0.7. \]

The spreading rate of virus and the sharp epidemic threshold are given by

\[ \tau = \frac{\beta}{\delta} = 0.0 \quad \text{and} \quad \tau_c = \frac{1}{\lambda_{\text{max}} A \gamma(G)} = \frac{1}{0.9927} = 1.007. \]

Here \( \tau < \tau_c \), so the epidemic dies out. Hence the spreading rate of virus is minimum.

**Example 2.3.** In the same website http://www.pantechsolutions.net/, we considered four links 1.microcontroller-boards, 2./log-in html, 3./ and 4. project kits for the period August 16, 2013 to September 15, 2013 (Period II).

![Intuitionistic Fuzzy Graph for the Period II](image-url)
For the intuitionistic fuzzy graph in Figure 2.4, the eigen values of $A_{\mu}(G)$ and $A_{\gamma}(G)$ are given by

\[
Spec(A_{\mu}(G)) = \{-0.2000, 0.0000, 0.2000, 0.0000\}
\]
\[
Spec(A_{\gamma}(G)) = \{0.9279, -0.1412, -0.3933 + 0.2552i, -0.3933 - 0.2552i\}
\]
\[
E(A_{\mu}(G)) = 0.2000 + 0.0000 + 0.2000 + 0.0000 = 0.4
\]
\[
E(A_{\gamma}(G)) = 0.9279 + 0.1412 + 0.4689 + 0.4689 = 2.0068.
\]

The lower bound and upper bound of $E(A_{\mu}(G))$ and $E(A_{\gamma}(G))$ are 0.2828, 0.5657 and 1.8323, 2.2980 respectively. From Theorem 2.3, we have

\( (i) \) 0.2828 \leq 0.4 \leq 0.5657

\( (ii) \) 1.8323 \leq 2.0068 \leq 2.2980.

By Corollary 2.4, we have

\[
2.2980 \geq 0.2828.
\]

For the intuitionistic fuzzy graph in Figure 2.4, the energy is

\[
E(A(G)) = (0.4, 2.0068).
\]

Here the energy of membership values is less than the energy of non-membership values. Hence there are less number of visitors in the link structure so the spreading rate of virus is minimum.

The infection rate and the curing rate are

\[
\beta = \min_{i,j} \mu_{ij} = 0.0 , \ \delta = \max_{i,j} \gamma_{ij} = 0.6.
\]

The spreading rate of virus and the sharp epidemic threshold are

\[
\tau = \frac{\beta}{\delta} = 0.0 \ , \ \tau_c = \frac{1}{\lambda_{max} A_{\gamma}(G)} = \frac{1}{0.9279} = 1.0777.
\]

Here $\tau < \tau_c$, so the epidemic dies out. Hence the spreading rate of virus is minimum.
Example 2.4. In the same website we considered four links 1. microcontroller-boards, 2./log-in html, 3./ and 4. project kits for the period September 16, 2013 to October 15, 2013 (Period III).

![Intuitionistic Fuzzy Graph for the Period III](image)

For the intuitionistic fuzzy graph in Figure 2.5, the eigen values of $A_\mu (G)$ and $A_\gamma (G)$ are given by

$\text{Spec} (A_\mu (G)) = \{-0.1732, 0.1732, 0.0000, 0.0000\}$

$\text{Spec} (A_\gamma (G)) = \{0.9418, -0.2000, -0.3709 + 0.1053i, -0.3709 - 0.1053i\}$

$E (A_\mu (G)) = 0.1732 + 0.1732 + 0.0000 + 0.0000 = 0.3464$

$E (A_\gamma (G)) = 0.9418 + 0.2000 + 0.3856 + 0.3856 = 1.9129.$

The lower bound and upper bound of $E (A_\mu (G))$ and $E (A_\gamma (G))$ are 0.2449, 0.4899 and 1.7979, 2.2127 respectively. From Theorem 2.3, we have

$(i) \ 0.2449 \leq 0.3464 \leq 0.4899$

$(ii) \ 1.7979 \leq 1.9129 \leq 2.2127.$
By Corollary 2.4, we have

\[ 2.2127 \geq 0.2449. \]

In the Figure 2.5, energy of membership value is less than the energy of non-membership value. i.e. there are less number of visitors in the link structure so the spreading rate of virus is minimum. The infection rate and curing rate are

\[ \beta = \min_{i,j} \mu_{ij} = 0.0, \quad \delta = \max_{i,j} \gamma_{ij} = 0.6. \]

The spreading rate of virus and the sharp epidemic threshold are

\[ \tau = \frac{\beta}{\delta} = 0.0, \quad \tau_c = \frac{1}{\lambda_{\text{max}}(A_{\gamma}(G))} = \frac{1}{0.9418} = 1.0618. \]

Here \( \tau < \tau_c \), so the epidemic dies out. Hence the spreading rate of virus is minimum.

**Example 2.5.** In the same website we considered four links 1.microcontroller-boards, 2./log-in.html, 3./ and 4. project kits for the period of October 16, 2013 to November 15, 2013 (Period IV).

![Figure 2.6: Intuitionistic Fuzzy Graph for the Period IV](image-url)
For the intuitionistic fuzzy graph in Figure 2.6, the eigen values of $A_{\mu}(G)$ and $A_{\gamma}(G)$ are given by

$Spec(A_{\mu}(G)) = \{0.0000, 0.1879, -0.1532, -0.0347\}$

$Spec(A_{\gamma}(G)) = \{0.9940, -0.1302, -0.4000, -0.4638\}$

$E(A_{\mu}(G)) = 0.0000 + 0.1879 + 0.1532 + 0.0347 = 0.3758$

$E(A_{\gamma}(G)) = 0.9940 + 0.1302 + 0.4000 + 0.4638 = 1.9879$

The lower bound and upper bound of $E(A_{\mu}(G))$ and $E(A_{\gamma}(G))$ are 0.2449, 0.4900 and 1.7997, 2.3493 respectively. From Theorem 2.2, we have

(i) $0.2449 \leq 0.3758 \leq 0.4900$

(ii) $1.7997 \leq 1.9879 \leq 2.3493$

By Corollary 2.2, we have

$2.3493 \geq 0.2449$

For the intuitionistic fuzzy graph in Figure 2.6, the energy of membership value is less than the energy of non-membership value. The infection rate and the curing rate are

$\beta = \min_{i,j} \mu_{ij} = 0.0 \quad \delta = \max_{i,j} \gamma_{ij} = 0.6$

The spreading rate of virus and the sharp epidemic threshold are

$\tau = \frac{\beta}{\delta} = 0.0 \quad \tau_c = \frac{1}{\lambda_{\max}A_{\gamma}(G)} = \frac{1}{0.9940} = 1.006$

Here $\tau < \tau_c$, so the epidemic dies out. Hence the spreading rate of virus is minimum.

From Examples 2.2 to 2.5, the energy of the defined model for the four time periods is depicted in the following diagram.
From the bar diagram in Figure 2.7, the energy for the period July 16 to August 15 and August 16 to September 15 are very high.

From the above bar diagram in Figure 2.8, the energy for the period July 16 to August 15 is very high.
The following table represents the spreading rate of virus and the epidemic threshold for the four different time periods.

Table 2.1: Spreading Rate of Virus and Epidemic Threshold of $E(A(G))$

<table>
<thead>
<tr>
<th>period</th>
<th>$E(A_\mu(G), A_\gamma(G))$</th>
<th>Spreading rate of virus $\tau$</th>
<th>Epidemic threshold $\tau_c$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Jul-Aug</td>
<td>(0.4, 2.1078)</td>
<td>0</td>
<td>1.007</td>
</tr>
<tr>
<td>Aug-Sep</td>
<td>(0.4, 2.0068)</td>
<td>0</td>
<td>1.0777</td>
</tr>
<tr>
<td>Sep-Oct</td>
<td>(0.3464, 1.9129)</td>
<td>0</td>
<td>1.0618</td>
</tr>
<tr>
<td>Oct-Nov</td>
<td>(0.3758, 1.9879)</td>
<td>0</td>
<td>1.006</td>
</tr>
</tbody>
</table>

In this chapter, we have defined the energy of an intuitionistic fuzzy graph. The lower and upper bound for the energy of an intuitionistic fuzzy graph and its properties are studied. The spreading rate of virus is obtained in terms of the energy of the intuitionistic fuzzy graph. These concepts are illustrated through a real time examples.