Chapter 6

Spreading Rate of Virus Between Incoming and Outgoing Links of a Website Through an Intuitionistic Fuzzy Graph

6.1 INTRODUCTION

There are several approaches have been proposed to model and simulate virus spreading in complex networks with different topologies. But epidemiological model is the suitable model to analyze virus spreading in random graphs which is presented by Kephart and White (1991). Wang et al. (2003) proposed a model for virus propagation in arbitrary topologies, epidemic threshold of virus infection and they proved that, under reasonable approximations, the epidemic threshold for a network is closely related to the largest eigenvalue of its adjacency matrix.

In this chapter we analyze the spreading rate of virus between incoming and outgoing links of a website through an intuitionistic fuzzy graph. We have taken the website http://www.pantechsolutions.net/. This website is model as an intuitionistic fuzzy graph $G = (V, E, \mu, \gamma)$ by considering the navigation of the customers. In this intuitionistic fuzzy graph, the links are consider as vertices and the path between the links are consider as edges. The weightage of each edge are consider as number of visitors getting the link from one link to another link.
(membership value), number of visitors not getting the link i.e. under traffic from one link to another link (non-membership value) and drop off case (intuitionistic fuzzy index). For four different time periods, the four links 1. Microcontroller-boards, 2. /Log-in html, 3. / and 4. /Project kits of the given website are taken for our calculation. In this intuitionistic fuzzy graph we constructed two intuitionistic fuzzy matrices using incoming and outgoing links of $G$. The energy of these matrices along with its lower and upper bound are discussed. The spreading rate of virus of the given graph in terms of their energies is also discussed.

6.2 OUT-MIN AND IN-MIN INTUITIONISTIC FUZZY MATRIX OF AN INTUITIONISTIC FUZZY GRAPH

In this section, we consider an intuitionistic fuzzy graph and we construct two intuitionistic fuzzy matrices using incoming and outgoing links and we determine the energies in terms these matrices.

**Definition 6.1.** Let $G = (V, E, \mu, \gamma)$ be an intuitionistic fuzzy graph, for every vertex $i$, define $\alpha_j = \max_i \mu_{ij}$ and $\sigma_j = \min_i \gamma_{ij}$.

**Theorem 6.1.** Let $G = (V, E, \mu, \gamma)$ be an intuitionistic fuzzy graph. If $\alpha_j = \max_i \mu_{ij}$ and $\sigma_j = \min_i \gamma_{ij}$ then $(\alpha_j, \sigma_j)$ is an intuitionistic fuzzy set on $V$.

Proof. If $\alpha_j = \max_i \mu_{ij}$ then $\alpha_j = \mu_{sj}$ for some $s \in V$. Similarly if $\sigma_j = \min_i \gamma_{ij}$ then $\sigma_j = \gamma_{tj}$ for some $t \in V$. Here, we have

$$\gamma_{tj} \leq \gamma_{sj}$$

that is

$$-\gamma_{tj} \geq -\gamma_{sj}.$$  

Now

$$\alpha_j = \mu_{sj} \leq 1 - \gamma_{sj} \leq 1 - \gamma_{tj}$$
which gives \( \mu_{sj} + \gamma_{tj} \leq 1 \)

that is \( \alpha_j + \sigma_j \leq 1 \).

Therefore, we get \( 0 \leq \alpha_j + \sigma_j \leq 1 \).

Hence \( (\alpha_j, \sigma_j) \) is an intuitionistic fuzzy set on \( V \).

**Definition 6.2.** Let \( G = (V, E, \mu, \gamma) \) be an intuitionistic fuzzy graph. An In-Min intuitionistic fuzzy matrix of an intuitionistic fuzzy graph is defined as \( M_I(G) = ([r_{ij}, s_{ij}] ) \). We denote \( R = [r_{ij}] \) is an In-Min intuitionistic fuzzy matrix of membership value, where \( r_{ij} = \min(\alpha_i, \alpha_j) \) and \( S = [s_{ij}] \) is an In-Min intuitionistic fuzzy matrix of non-membership value, where \( s_{ij} = \min(\sigma_i, \sigma_j) \).

**Lemma 6.1.** \( M_I(G) \) is a symmetric matrix.

Proof. \( M_I(G) = ([r_{ij}, s_{ij}] ) \) where \( r_{ij} = \min(\alpha_i, \alpha_j) = \min(\alpha_j, \alpha_i) = r_{ji} \) and \( s_{ij} = \min(\sigma_i, \sigma_j) = \min(\sigma_j, \sigma_i) = s_{ji} \). Hence \( M_I(G) \) is a symmetric matrix.

**Theorem 6.2.** Let \( G = (V, E, \mu, \gamma) \) be an intuitionistic fuzzy graph and \( \alpha_i, \sigma_j \) are as defined above. Let \( M_I(G) = ([r_{ij}, s_{ij}] ) \) and if \( r_{ij} = \min(\alpha_i, \alpha_j) \) and \( s_{ij} = \min(\sigma_i, \sigma_j) \) then \( (r_{ij}, s_{ij}) \) is an intuitionistic fuzzy set.

Proof. Let \( r_{ij} = \min(\alpha_i, \alpha_j) \) and \( s_{ij} = \min(\sigma_i, \sigma_j) \). If \( r_{ij} = \alpha_i \) and \( s_{ij} = \sigma_i \) then \( 0 \leq r_{ij} + s_{ij} \leq 1 \). Now if \( r_{ij} = \alpha_i \) and \( s_{ij} = \sigma_j \) then \( \alpha_i \leq \alpha_j \leq 1 - \sigma_j \Rightarrow \alpha_i + \sigma_j \leq 1 \). Therefore \( 0 \leq \alpha_i + \alpha_j \leq 1 \). Hence \( (\alpha_i, \sigma_j) \) is an intuitionistic fuzzy set. Similarly we can prove the other cases also (if like \( r_{ij} = \alpha_j, s_{ij} = \sigma_i \) and \( r_{ij} = \alpha_j, s_{ij} = \sigma_j \)).

**Definition 6.3.** The energy of an In-Min intuitionistic fuzzy matrix of an intuitionistic fuzzy graph \( G \) is defined as \( E(M_I(G)) = (E(R), E(S)) \), where \( E(R) \) is the sum of the absolute eigen values of the matrix \( R \) and \( E(S) \) is the sum of the absolute eigen values of the matrix \( S \).
Definition 6.4. Let $G = (V, E, \mu, \gamma)$ be an intuitionistic fuzzy graph, for every vertex $i$, define $\theta_i = \max_j \mu_{ij}$ and $\phi_i = \min_j \gamma_{ij}$.

By Theorem 6.1, it can be proved that $(\theta_i, \phi_i)$ is an intuitionistic fuzzy set on $V$.

Definition 6.5. Let $G = (V, E, \mu, \gamma)$ be an intuitionistic fuzzy graph. The Out-Min intuitionistic fuzzy matrix of an intuitionistic fuzzy graph is defined as $M_O(G) = [(u_{ij}, v_{ij})]$. We denote $U = [u_{ij}]$ is an Out-Min intuitionistic fuzzy matrix of membership value, where $u_{ij} = min(\theta_i, \theta_j)$ and $V = [v_{ij}]$ is an Out-Min intuitionistic fuzzy matrix of non-membership value, where $v_{ij} = min(\phi_i, \phi_j)$.

By Theorem 6.2, it can be proved that $(u_{ij}, v_{ij})$ is an intuitionistic fuzzy set and by Lemma 6.1, it can proved that $M_O(G)$ is a symmetric matrix.

Definition 6.6. The energy of the Out-Min intuitionistic fuzzy matrix of an intuitionistic fuzzy graph is defined as, $E(M_O(G)) = (E(U), E(V))$ where $E(U)$ is the sum of the absolute eigen values of the matrix $U$ and $E(V)$ is the sum of the absolute eigen values of the matrix $V$.

Theorem 6.3. Let $G = (V, E, \mu, \gamma)$ be an intuitionistic fuzzy graph and if $r_1, r_2, ..., r_n$ are the real or complex eigen values of the matrix $R$ then
\[ \sum_{i=1}^{n} (r_i)^2 = \sum_{i=1}^{n} (r_{ii})^2 + 2 \sum_{1 \leq i < j \leq n} (r_{ij})^2. \]

Proof. Sum of the square of the eigen values of $R$ is equal to the trace of $R^2$, that is
\[ \sum_{i=1}^{n} (r_i)^2 = trace(R^2). \]
That is
\[ \sum_{i=1}^{n} (r_i)^2 = r_{11}r_{11} + r_{12}r_{21} + ... + r_{1n}r_{n1} + \]
\[ r_{11}r_{12} + r_{22}r_{22} + \ldots + r_{2n}r_{n2} + \ldots \]
\[ r_{nn} + r_{n2}r_{2n} + \ldots + r_{nn}r_{nn} \]
\[ \Rightarrow \sum_{i=1}^{n} (r_{ii})^2 = \sum_{i=1}^{n} (r_{ii})^2 + 2 \sum_{1 \leq i < j \leq n} (r_{ij})^2. \]

Hence the proof. \[\square\]

**Remark 6.1.** Similarly we can prove if \( s_1, s_2, \ldots, s_n \) are the real or complex eigenvalues of the matrix \( S \) then

\[ \sum_{i=1}^{n} (s_{ii})^2 = \sum_{i=1}^{n} (s_{ii})^2 + 2 \sum_{1 \leq i < j \leq n} (s_{ij})^2. \]

If \( u_1, u_2, \ldots, u_n \) are the real or complex eigenvalues of the matrix \( U \) then

\[ \sum_{i=1}^{n} (u_{ii})^2 = \sum_{i=1}^{n} (u_{ii})^2 + 2 \sum_{1 \leq i < j \leq n} (u_{ij})^2. \]

If \( v_1, v_2, \ldots, v_n \) are the real or complex eigenvalues of the matrix \( V \) then

\[ \sum_{i=1}^{n} (v_{ii})^2 = \sum_{i=1}^{n} (v_{ii})^2 + 2 \sum_{1 \leq i < j \leq n} (v_{ij})^2. \]

**Theorem 6.4.** Let \( G = (V, E, \mu, \gamma) \) be an intuitionistic fuzzy graph with \( n \) vertices and if \( r_1, r_2, \ldots, r_n \) are the real eigen values of the matrix \( R \) then

\[
\sqrt{\sum_{i=1}^{n} (r_{ii})^2 + 2 \sum_{1 \leq i < j \leq n} (r_{ij})^2 + n(n - 1) |A|^2} \leq E(R)
\]
\[
\leq \sqrt{n \left( \sum_{i=1}^{n} (r_{ii})^2 + 2 \sum_{1 \leq i < j \leq n} (r_{ij})^2 \right)}
\]

where \( |A| \) is the determinant of \( R \).

Proof. Upper bound: In Cauchy-Schwarz inequality

\[
\left( \sum_{i=1}^{n} a_i b_i \right)^2 \leq \left( \sum_{i=1}^{n} a_i^2 \right) \left( \sum_{i=1}^{n} b_i^2 \right)
\]

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put $a_i = 1$ and $b_i = |r_i|$, we get

$$\left( \sum_{i=1}^{n} |r_i| \right)^2 \leq \left( \sum_{i=1}^{n} 1 \right) \left( \sum_{i=1}^{n} (r_i)^2 \right).$$

By Theorem 6.3, we get

$$(E(R))^2 \leq n \left( \sum_{i=1}^{n} (r_{ii})^2 + 2 \sum_{1 \leq i < j \leq n} (r_{ij})^2 \right)$$

$$\Rightarrow E(R) \leq \sqrt{n \left( \sum_{i=1}^{n} (r_{ii})^2 + 2 \sum_{1 \leq i < j \leq n} (r_{ij})^2 \right)}.$$  \hspace{1cm} (6.1)

Lower bound: We know that

$$(E(R))^2 = \left( \sum_{i=1}^{n} |r_i| \right)^2$$

$$= \sum_{i=1}^{n} |r_i|^2 + 2 \sum_{1 \leq i < j \leq n} |r_i| |r_j|$$

$$= \left[ \sum_{i=1}^{n} (r_{ii})^2 + 2 \sum_{1 \leq i < j \leq n} (r_{ij})^2 \right] + 2 \frac{n(n-1)}{2} AM \{|r_i| |r_j|\}.$$

Since

$$AM \{|r_i| |r_j|\} \geq GM \{|r_i| |r_j|\}, 1 \leq i < j \leq n,$$

we get

$$E(R) \geq \sqrt{ \sum_{i=1}^{n} (r_{ij})^2 + 2 \sum_{1 \leq i < j \leq n} (r_{ij})^2 + n(n-1) GM \{|r_i| |r_j|\}}.$$

But

$$GM \{|r_i| |r_j|\} = \left( \prod_{i=1}^{n} |r_i| \right)^{\frac{2}{n}} = |A|^{\frac{2}{n}}.$$

Hence, we have

$$E(R) \geq \sqrt{ \sum_{i=1}^{n} (r_{ii})^2 + 2 \sum_{1 \leq i < j \leq n} (r_{ij})^2 + n(n-1) |A|^{\frac{2}{n}}.}$$  \hspace{1cm} (6.2)
From equations (6.1) and (6.2), we get
\[
\sqrt{\sum_{i=1}^{n} (r_{ii})^2 + 2 \sum_{1 \leq i < j \leq n} (r_{ij})^2 + n(n-1)|A|^2} \leq E(R)
\]
\[
\leq \sqrt{n \left( \sum_{i=1}^{n} (r_{ii})^2 + 2 \sum_{1 \leq i < j \leq n} (r_{ij})^2 \right)}.
\]
Hence the proof. \(\square\)

**Remark 6.2.** Similarly we can prove if \(s_1, s_2, ..., s_n\) are the real eigen values of the matrix \(S\) then
\[
\sqrt{\sum_{i=1}^{n} (s_{ii})^2 + 2 \sum_{1 \leq i < j \leq n} (s_{ij})^2 + n(n-1)|B|^2} \leq E(S)
\]
\[
\leq \sqrt{n \left( \sum_{i=1}^{n} (s_{ii})^2 + 2 \sum_{1 \leq i < j \leq n} (s_{ij})^2 \right)}
\]
where \(|B|\) is the determinant of \(S\).

If \(u_1, u_2, ..., u_n\) are the real eigen values of the matrix \(U\) then
\[
\sqrt{\sum_{i=1}^{n} (u_{ii})^2 + 2 \sum_{1 \leq i < j \leq n} (u_{ij})^2 + n(n-1)|C|^2} \leq E(U)
\]
\[
\leq \sqrt{n \left( \sum_{i=1}^{n} (u_{ii})^2 + 2 \sum_{1 \leq i < j \leq n} (u_{ij})^2 \right)}
\]
where \(|C|\) is the determinant of \(U\).

If \(v_1, v_2, ..., v_n\) are the real eigen values of the matrix \(V\) then
\[
\sqrt{\sum_{i=1}^{n} (v_{ii})^2 + 2 \sum_{1 \leq i < j \leq n} (v_{ij})^2 + n(n-1)|D|^2} \leq E(V)
\]
\[
\leq \sqrt{n \left( \sum_{i=1}^{n} (v_{ii})^2 + 2 \sum_{1 \leq i < j \leq n} (v_{ij})^2 \right)}
\]
where \(|D|\) is the determinant of \(V\).
**Corollary 6.1.** If \( r_1, r_2, \ldots, r_n \) are the real eigen values of \( R \) and \( s_1, s_2, \ldots, s_n \) are the real eigen values of \( S \) and if \( E(R) \geq E(S) \) then

\[
n \left( \sum_{i=1}^{n} (r_{ii})^2 + 2 \sum_{1 \leq i < j \leq n} (r_{ij})^2 \right) \geq \sum_{i=1}^{n} (s_{ii})^2 + 2 \sum_{1 \leq i < j \leq n} (s_{ij})^2 + n(n - 1) |B|^2 \pi.
\]

**Proof.** From Theorem 6.4, we have,

\[
\sqrt{n \left( \sum_{i=1}^{n} (r_{ii})^2 + 2 \sum_{1 \leq i < j \leq n} (r_{ij})^2 \right)} \geq E(R)
\]

\[
\sqrt{n \left( \sum_{i=1}^{n} (r_{ii})^2 + 2 \sum_{1 \leq i < j \leq n} (r_{ij})^2 \right)} \geq E(S).
\]

Again by Theorem 6.4, we have

\[
E(S) \geq \sqrt{n \sum_{i=1}^{n} (s_{ii})^2 + 2 \sum_{1 \leq i < j \leq n} (s_{ij})^2 + n(n - 1) |B|^2 \pi}.
\]

Therefore, we get

\[
\sqrt{n \left( \sum_{i=1}^{n} (r_{ii})^2 + 2 \sum_{1 \leq i < j \leq n} (r_{ij})^2 \right)} \geq \sqrt{n \sum_{i=1}^{n} (s_{ii})^2 + 2 \sum_{1 \leq i < j \leq n} (s_{ij})^2 + n(n - 1) |B|^2 \pi}
\]

\[
\Rightarrow n \left( \sum_{i=1}^{n} (r_{ii})^2 + 2 \sum_{1 \leq i < j \leq n} (r_{ij})^2 \right) \geq n \sum_{i=1}^{n} (s_{ii})^2 + 2 \sum_{1 \leq i < j \leq n} (s_{ij})^2 + n(n - 1) |B|^2 \pi
\]

Hence the proof.

**Remark 6.3.** Similarly we can prove if \( E(S) \geq E(R) \) then

\[
n \left( \sum_{i=1}^{n} (s_{ii})^2 + 2 \sum_{1 \leq i < j \leq n} (s_{ij})^2 \right) \geq \sum_{i=1}^{n} (r_{ii})^2 + 2 \sum_{1 \leq i < j \leq n} (r_{ij})^2 + n(n - 1) |A|^2 \pi.
\]

If \( u_1, u_2, \ldots, u_n \) are the real eigen values of \( U \) and \( v_1, v_2, \ldots, v_n \) are the real eigen values of \( V \) and if \( E(U) \geq E(V) \) then

\[
n \left( \sum_{i=1}^{n} (u_{ii})^2 + 2 \sum_{1 \leq i < j \leq n} (u_{ij})^2 \right) \geq \sum_{i=1}^{n} (v_{ii})^2 + 2 \sum_{1 \leq i < j \leq n} (v_{ij})^2 + n(n - 1) |D|^2 \pi.
\]
If $E(V) \geq E(U)$ then
\[
n \left( \sum_{i=1}^{n} (v_{ii})^2 + 2 \sum_{1 \leq i < j \leq n} (v_{ij})^2 \right) \geq \sum_{i=1}^{n} (u_{ii})^2 + 2 \sum_{1 \leq i < j \leq n} (u_{ij})^2 + n(n-1)|C|^\frac{1}{2}.
\]

**Theorem 6.5.** Let $G = (V, E, \mu, \gamma)$ be an intuitionistic fuzzy graph with $n$ vertices
and if $r_1, r_2, \ldots, r_n$ are the complex eigen values of the matrix $R$ then
\[
\sqrt{\sum_{i=1}^{n} |r_i|^2 + n(n-1)|A|^\frac{1}{2}} \leq E(R) \leq \sqrt{n \sum_{i=1}^{n} |r_i|^2}
\]
where $|A|$ is the determinant of $R$.

**Proof.** Upper bound: Apply Cauchy-Schwarz inequality to the $n$ numbers 1, 1, ..., 1
and $|r_1|, |r_2|, \ldots, |r_n|$, we get
\[
\sum_{i=1}^{n} |r_i| \leq \sqrt{n} \sqrt{\sum_{i=1}^{n} |r_i|^2}.
\]
That is
\[
E(R) \leq \sqrt{n \sum_{i=1}^{n} |r_i|^2}.
\]
(6.3)

Lower bound: We know that
\[
(E(R))^2 = \left( \sum_{i=1}^{n} |r_i| \right)^2
\]
\[
= \sum_{i=1}^{n} |r_i|^2 + 2 \sum_{1 \leq i < j \leq n} |r_i||r_j|
\]
\[
= \sum_{i=1}^{n} |r_i|^2 + 2 \frac{n(n-1)}{2} AM \{ |r_i| |r_j| \}.
\]
Since
\[
AM \{ |r_i| |r_j| \} \geq GM \{ |r_i| |r_j| \}, 1 \leq i < j \leq n,
\]
we get
\[
E(R) \geq \sqrt{\sum_{i=1}^{n} |r_i|^2 + n(n-1)GM \{ |r_i| |r_j| \}}.
\]
But

\[ \text{GM} \{ |r_i| |r_j| \} = \left( \prod_{i=1}^{n} |r_i| \right)^{\frac{2}{n}} = |A|^{\frac{2}{n}}. \]

Hence, we have

\[ E(R) \geq \sqrt{\sum_{i=1}^{n} |r_i|^2 + n(n-1)|A|^\frac{2}{n}}. \] (6.4)

From equations (6.3) and (6.4) we have,

\[ \sqrt{\sum_{i=1}^{n} |r_i|^2 + n(n-1)|A|^\frac{2}{n}} \leq E(R) \leq \sqrt{\sum_{i=1}^{n} |r_i|^2}. \]

Hence the proof. \[ \square \]

**Remark 6.4.** Similarly, we can prove, if \( s_1, s_2, \ldots, s_n \) are the complex eigen values of the matrix \( S \) then

\[ \sqrt{\sum_{i=1}^{n} |s_i|^2 + n(n-1)|B|^\frac{2}{n}} \leq E(S) \leq \sqrt{\sum_{i=1}^{n} |s_i|^2}. \]

If \( u_1, u_2, \ldots, u_n \) are the complex eigen values of the matrix \( U \) then

\[ \sqrt{\sum_{i=1}^{n} |u_i|^2 + n(n-1)|C|^\frac{2}{n}} \leq E(U) \leq \sqrt{\sum_{i=1}^{n} |u_i|^2}. \]

If \( v_1, v_2, \ldots, v_n \) are the complex eigen values of the matrix \( V \) then

\[ \sqrt{\sum_{i=1}^{n} |v_i|^2 + n(n-1)|D|^\frac{2}{n}} \leq E(V) \leq \sqrt{\sum_{i=1}^{n} |v_i|^2}. \]

**Corollary 6.2.** If \( r_1, r_2, \ldots, r_n \) are the complex eigen values of \( R \) and \( s_1, s_2, \ldots, s_n \) are the complex eigen values of \( S \) and if \( E(R) \geq E(S) \) then

\[ n \sum_{i=1}^{n} |r_i|^2 \geq \sum_{i=1}^{n} |s_i|^2 + n(n-1)|B|^\frac{2}{n}. \]
Proof. From Theorem 6.5, we have

$$\sqrt{n \sum_{i=1}^{n} |r_i|^2} \geq E(R)$$

$$\sqrt{n \sum_{i=1}^{n} |r_i|^2} \geq E(S).$$

Again by Theorem 6.5, we have

$$E(S) \geq \sqrt{\sum_{i=1}^{n} |s_i|^2 + n(n-1)|B|^2.}$$

Therefore, we get

$$\sqrt{n \sum_{i=1}^{n} |r_i|^2} \geq \sqrt{\sum_{i=1}^{n} |s_i|^2 + n(n-1)|B|^2}$$

$$\Rightarrow \ n \sum_{i=1}^{n} |r_i|^2 \geq \sum_{i=1}^{n} |s_i|^2 + n(n-1)|B|^2.$$

Hence the proof. \[ \square \]

**Remark 6.5.** Similarly we can prove if $E(S) \geq E(R)$ then

$$n \sum_{i=1}^{n} |s_i|^2 \geq \sum_{i=1}^{n} |r_i|^2 + n(n-1)|A|^2.$$

If $u_1, u_2, ..., u_n$ are the complex eigen values of $U$ and $v_1, v_2, ..., v_n$ are the complex eigen values of $V$ and if $E(U) \geq E(V)$ then

$$n \sum_{i=1}^{n} |u_i|^2 \geq \sum_{i=1}^{n} |v_i|^2 + n(n-1)|D|^2.$$

If $E(V) \geq E(U)$ then

$$n \sum_{i=1}^{n} |v_i|^2 \geq \sum_{i=1}^{n} |u_i|^2 + n(n-1)|C|^2.$$
Theorem 6.6. Let $G = (V, E, \mu, \gamma)$ be an intuitionistic fuzzy graph with $n$ vertices and if
\[
E(R) \leq 2 \sum_{i=1}^{n} \sum_{j=1}^{n} r_{ij}, \quad E(S) \leq 2 \sum_{i=1}^{n} \sum_{j=1}^{n} s_{ij}
\]
then $E(R) + E(S) \leq 2n^2$.

Proof. By the assumption
\[
E(R) \leq 2 \sum_{i=1}^{n} \sum_{j=1}^{n} r_{ij}
\]
\[
\leq 2 \sum_{i=1}^{n} \sum_{j=1}^{n} (1 - s_{ij})
\]
\[
= 2 \sum_{i=1}^{n} \sum_{j=1}^{n} 1 - 2 \sum_{i=1}^{n} \sum_{j=1}^{n} s_{ij}
\]
\[
\leq 2n^2 - E(S)
\]
\[
E(R) + E(S) \leq 2n^2.
\]
Hence the proof. \qed

6.3 SPREADING RATE OF VIRUS BETWEEN INCOMING AND OUTGOING LINKS

In this section we analyze the spreading rate of virus between incoming and outgoing links of a website through the intuitionistic fuzzy graph.

Definition 6.7. If $E(R) > E(S)$ then the infection rate of an intuitionistic fuzzy graph is defined as $\beta = \max_{i,j} r_{ij}$ and the curing rate of an intuitionistic fuzzy graph is defined as $\delta = \min_{i,j} s_{ij}$. The ratio $\tau = \frac{\beta}{\delta}$, $(\delta \neq 0)$ is the effective spreading rate and $\tau_c = \frac{1}{\lambda_{max} R}$ where $\lambda_{max} R$ is the largest eigen value of $R$ of an intuitionistic fuzzy graph.
Definition 6.8. If $E(R) < E(S)$ then the infection rate of an intuitionistic fuzzy graph is defined as $\beta = \min_{i,j} r_{ij}$ and the curing rate of an intuitionistic fuzzy graph is defined as $\delta = \max_{i,j} s_{ij}$. The ratio $\tau = \frac{\beta}{\delta}$ is the effective spreading rate and $\tau_c = \frac{1}{\lambda_{\text{max}}^S}$ where $\lambda_{\text{max}}^S$ is the largest eigen value of $S$ of an intuitionistic fuzzy graph.

Definition 6.9. If $E(U) > E(V)$ then the infection rate of an intuitionistic fuzzy graph is defined as $\beta = \max_{i,j} u_{ij}$ and the curing rate of an intuitionistic fuzzy graph is defined as $\delta = \min_{i,j} v_{ij}$. The ratio $\tau = \frac{\beta}{\delta}$, $(\delta \neq 0)$ is the effective spreading rate and $\tau_c = \frac{1}{\lambda_{\text{max}}^U}$ where $\lambda_{\text{max}}^U$ is the largest eigen value of $U$ of an intuitionistic fuzzy graph.

Definition 6.10. If $E(U) < E(V)$ then the infection rate of an intuitionistic fuzzy graph is defined as $\beta = \min_{i,j} u_{ij}$ and the curing rate of an intuitionistic fuzzy graph is defined as $\delta = \max_{i,j} v_{ij}$. The ratio $\tau = \frac{\beta}{\delta}$ is the effective spreading rate and $\tau_c = \frac{1}{\lambda_{\text{max}}^V}$ where $\lambda_{\text{max}}^V$ is the largest eigen value of $V$ of an intuitionistic fuzzy graph.

Let us illustrate the above concepts in the following example.

Example 6.1. Let us analyze the spreading rate of virus in terms of incoming and outgoing links for the intuitionistic fuzzy graph in Figure 2.1, the In-Min intuitionistic fuzzy matrix is given by

$$M_I (G) = \begin{pmatrix}
(0.9, 0.1) & (0.8, 0.1) & (0.5, 0.1) & (0.6, 0.1) \\
(0.8, 0.1) & (0.8, 0.1) & (0.5, 0.1) & (0.6, 0.1) \\
(0.5, 0.1) & (0.5, 0.1) & (0.5, 0.1) & (0.5, 0.1) \\
(0.6, 0.1) & (0.6, 0.1) & (0.5, 0.1) & (0.6, 0.3)
\end{pmatrix}$$
where

\[
R = \begin{pmatrix}
0.9 & 0.8 & 0.5 & 0.6 \\
0.8 & 0.8 & 0.5 & 0.6 \\
0.5 & 0.5 & 0.5 & 0.5 \\
0.6 & 0.6 & 0.5 & 0.6
\end{pmatrix}
\quad \text{and} \quad
S = \begin{pmatrix}
0.1 & 0.1 & 0.1 & 0.1 \\
0.1 & 0.1 & 0.1 & 0.1 \\
0.1 & 0.1 & 0.1 & 0.3
\end{pmatrix}.
\]

The energy of the In-Min intuitionistic fuzzy matrix of an intuitionistic fuzzy graph is

\[E(M_I(G)) = (2.8000, 0.6000).\]

Here we note that

\[E(R) > E(S)\]

so, the infection rate and curing rate are given by

\[\beta = \max_{i,j} r_{ij} = 0.9 \quad \delta = \min_{i,j} s_{ij} = 0.1\]

and the spreading rate of virus and sharp epidemic threshold are given by

\[\tau = \frac{\beta}{\delta} = 9 \quad \tau_c = \frac{1}{\lambda_{\max} R} = \frac{1}{2.4958} = 0.4007.\]

For the intuitionistic fuzzy graph in Figure 2.1, the Out-Min intuitionistic fuzzy matrix is given by

\[
M_O(G) = \begin{pmatrix}
(0.6, 0.2) & (0.5, 0.1) & (0.6, 0.1) & (0.6, 0.1) \\
(0.5, 0.1) & (0.5, 0.1) & (0.5, 0.1) & (0.5, 0.1) \\
(0.6, 0.1) & (0.5, 0.1) & (0.7, 0.1) & (0.7, 0.1) \\
(0.6, 0.1) & (0.5, 0.1) & (0.7, 0.1) & (0.9, 0.1)
\end{pmatrix}
\]
where

\[
U = \begin{pmatrix}
0.6 & 0.5 & 0.6 & 0.6 \\
0.5 & 0.5 & 0.5 & 0.5 \\
0.6 & 0.5 & 0.7 & 0.7 \\
0.6 & 0.5 & 0.7 & 0.9
\end{pmatrix}
\quad \text{and} \quad
V = \begin{pmatrix}
0.2 & 0.1 & 0.1 & 0.1 \\
0.1 & 0.1 & 0.1 & 0.1 \\
0.1 & 0.1 & 0.1 & 0.1 \\
0.1 & 0.1 & 0.1 & 0.1
\end{pmatrix}.
\]

The energy of the Out-Min intuitionistic fuzzy matrix of an intuitionistic fuzzy graph is

\[ E(M_O(G)) = (2.7000, 0.5000). \]

Here we note that \( E(U) > E(V) \)

so, the infection rate and curing rate are given by

\[
\beta = \max_{i,j} u_{ij} = 0.9, \quad \delta = \min_{i,j} v_{ij} = 0.1
\]

and the spreading rate of virus and sharp epidemic threshold are given by

\[
\tau = \frac{\beta}{\delta} = 9, \quad \tau_c = \frac{1}{\lambda_{\max} U} = \frac{1}{2.4054} = 0.4157.
\]

### 6.4 NUMERICAL EXAMPLES

In this section, we illustrate the concepts defined in this chapter through real-time example.

**Example 6.2.** In the website http://www.pantechsolutions.net/ we considered the same four links 1./microcontroller-boards, 2./log-in html, 3. / and 4./project kits for the period July 16, 2013 to August 15, 2013 (Period I). For the intuitionistic
fuzzy graph in Figure 2.3, the energy of the In-Min intuitionistic fuzzy matrix of an intuitionistic fuzzy graph is

\[ E(M_I(G)) = (0.6000, 1.4000) . \]

Here note that \( E(R) < E(S) \) so, the infection rate and curing rate are given by

\[ \beta = \min_{i,j} r_{ij} = 0.1 , \quad \delta = \max_{i,j} s_{ij} = 0.6 \]

and the spreading rate of virus and sharp epidemic threshold are given by

\[ \tau = \frac{\beta}{\delta} = 0.1667 , \quad \tau_c = \frac{1}{\lambda_{\text{max}} S} = \frac{1}{1.1630} = 0.8598. \]

For the intuitionistic fuzzy graph in Figure 2.3, the energy of the Out-Min intuitionistic fuzzy matrix of an intuitionistic fuzzy graph is

\[ E(M_O(G)) = (0.5000, 1.5000) . \]

Here note that \( E(U) < E(V) \) so, the infection rate and curing rate are given by

\[ \beta = \min_{i,j} u_{ij} = 0 , \quad \delta = \max_{i,j} v_{ij} = 0.6 \]

and the spreading rate of virus and sharp epidemic threshold are given by

\[ \tau = \frac{\beta}{\delta} = 0 , \quad \tau_c = \frac{1}{\lambda_{\text{max}} V} = \frac{1}{1.2677} = 0.7888. \]

**Example 6.3.** In the same website we consider the same four links for the period August 16, 2013 to September 15, 2013 (Period II). For the intuitionistic fuzzy graph in Figure 2.4, the energy of the In-Min intuitionistic fuzzy matrix of an intuitionistic fuzzy graph is

\[ E(M_I(G)) = (0.5000, 1.3000) . \]

Here note that \( E(R) < E(S) \) so, the infection rate and curing rate are given by

\[ \beta = \min_{i,j} r_{ij} = 0 , \quad \delta = \max_{i,j} s_{ij} = 0.6 \]
and the spreading rate of virus and sharp epidemic threshold are given by

\[ \tau = \frac{\beta}{\delta} = 0 \quad , \quad \tau_c = \frac{1}{\lambda_{\text{max}} S} = \frac{1}{1.0391} = 0.9624. \]

For the intuitionistic fuzzy graph in Figure 2.4, the energy of the Out-Min intuitionistic fuzzy matrix of an intuitionistic fuzzy graph is

\[ E(M_O(G)) = (0.5000, 1.4000). \]

Here note that \( E(U) < E(V) \) so, the infection rate and curing rate are given by

\[ \beta = \min_{i,j} u_{ij} = 0 \quad , \quad \delta = \max_{i,j} v_{ij} = 0.6 \]

and the spreading rate of virus and sharp epidemic threshold are given by

\[ \tau = \frac{\beta}{\delta} = 0 \quad , \quad \tau_c = \frac{1}{\lambda_{\text{max}} V} = \frac{1}{1.1630} = 0.8598. \]

**Example 6.4.** In the same website, we consider the same four links for the period September 16, 2013 to October 15, 2013 (Period III). For the intuitionistic fuzzy graph in Figure 2.5, the energy of the In-Min intuitionistic fuzzy matrix of an intuitionistic fuzzy graph is

\[ E(M_I(G)) = (0.5000, 1.3000). \]

Here note that \( E(R) < E(S) \) so, the infection rate and curing rate are given by

\[ \beta = \min_{i,j} r_{ij} = 0.1 \quad , \quad \delta = \max_{i,j} s_{ij} = 0.5 \]

and the spreading rate of virus and sharp epidemic threshold are given by

\[ \tau = \frac{\beta}{\delta} = 0.2 \quad , \quad \tau_c = \frac{1}{\lambda_{\text{max}} S} = \frac{1}{1.1168} = 0.8954. \]

For the intuitionistic fuzzy graph in Figure 2.5, the energy of the Out-Min intuitionistic fuzzy matrix of an intuitionistic fuzzy graph is

\[ E(M_O(G)) = (0.4000, 1.5000). \]
Here note that

\[ E(U) < E(V) \]

so, the infection rate and curing rate are given by

\[
\beta = \min_{i,j} u_{ij} = 0, \quad \delta = \max_{i,j} v_{ij} = 0.5
\]

and the spreading rate of virus and sharp epidemic threshold are given by

\[
\tau = \frac{\beta}{\delta} = 0, \quad \tau_c = \frac{1}{\lambda_{max}V} = \frac{1}{1.3639} = 0.7332.
\]

**Example 6.5.** In the same website, we consider the same four links for the period of October 16, 2013 to November 15, 2013 (Period IV). For the intuitionistic fuzzy graph in Figure 2.6, the energy of the In-Min intuitionistic fuzzy matrix of an intuitionistic fuzzy graph is

\[ E(M_I(G)) = (0.4000, 1.3000). \]

Here note that \( E(R) < E(S) \) so, the infection rate and curing rate are given by

\[
\beta = \min_{i,j} r_{ij} = 0, \quad \delta = \max_{i,j} s_{ij} = 0.1
\]

and the spreading rate of virus and sharp epidemic threshold are given by

\[
\tau = \frac{\beta}{\delta} = 0, \quad \tau_c = \frac{1}{\lambda_{max}S} = \frac{1}{1.0391} = 0.9624.
\]

For the intuitionistic fuzzy graph in Figure 2.6, the energy of the Out-Min intuitionistic fuzzy matrix of an intuitionistic fuzzy graph is

\[ E(M_O(G)) = (0.5000, 1.5000). \]

Here note that

\[ E(U) < E(V) \]

so, the infection rate and curing rate are given by

\[
\beta = \min_{i,j} u_{ij} = 0.1, \quad \delta = \max_{i,j} v_{ij} = 0.5
\]
and the spreading rate of virus and sharp epidemic threshold are given by

\[
\tau = \frac{\beta}{\delta} = 0.2, \quad \tau_c = \frac{1}{\lambda_{\text{max}}V} = \frac{1}{1.3639} = 0.7332.
\]

From Examples 6.2 to 6.5, we observe that the spreading rate of virus is differ between incoming and outgoing links in terms of their energies. Also we conclude that the spreading rate of virus is maximum through incoming links and minimum through outgoing links.

The following table represents the comparison of energy of \(M_I(G)\) and the energy of \(M_O(G)\).

**Table 6.1: Comparison of Energy \(M_I(G)\) and \(M_O(G)\)**

<table>
<thead>
<tr>
<th>period</th>
<th>Energy of (A(G))</th>
<th>Energy of (M_I(G))</th>
<th>Energy of (M_O(G))</th>
</tr>
</thead>
<tbody>
<tr>
<td>Jul-Aug</td>
<td>(0.4, 2.1078)</td>
<td>(0.6000, 1.4000)</td>
<td>(0.5000, 1.5000)</td>
</tr>
<tr>
<td>Aug-Sep</td>
<td>(0.4, 2.0068)</td>
<td>(0.5000, 1.3000)</td>
<td>(0.5000, 1.4000)</td>
</tr>
<tr>
<td>Sep-Oct</td>
<td>(0.3464, 1.9129)</td>
<td>(0.5000, 1.3000)</td>
<td>(0.4000, 1.5000)</td>
</tr>
<tr>
<td>Oct-Nov</td>
<td>(0.3758, 1.9879)</td>
<td>(0.4000, 1.3000)</td>
<td>(0.5000, 1.5000)</td>
</tr>
</tbody>
</table>

The following table represents the comparison of spreading rate of virus and sharp epidemic threshold of \(M_I(G)\) and \(M_O(G)\).

**Table 6.2: Spreading Rate of Virus and Epidemic Threshold of \(M_I(G)\) and \(M_O(G)\)**

<table>
<thead>
<tr>
<th>period</th>
<th>(\tau) on (M_I(G))</th>
<th>(\tau) on (M_O(G))</th>
</tr>
</thead>
<tbody>
<tr>
<td>Jul-Aug</td>
<td>0.1667</td>
<td>0</td>
</tr>
<tr>
<td>Aug-Sep</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>Sep-Oct</td>
<td>0.2</td>
<td>0</td>
</tr>
<tr>
<td>Oct-Nov</td>
<td>0</td>
<td>0.2</td>
</tr>
</tbody>
</table>
In this chapter we analyzed the spreading rate of virus between incoming and outgoing links of a website http://www.pantechsolutions.net/ for four different time periods.

In this chapter we analyzed the spreading rate of virus between incoming and outgoing links of a website through the intuitionistic fuzzy graph. In this intuitionistic fuzzy graph we constructed two intuitionistic fuzzy matrices using incoming and outgoing links of $G$. The energy of these matrices along with its lower and upper bound are discussed. The spreading rate of virus of the given graph in terms of these two matrices is discussed. These concepts are illustrated with real time examples.