Chapter 5

Laplacian Energy of an Intuitionistic Fuzzy Graph

5.1 INTRODUCTION


Motivated by this, we define the Laplacian energy of the intuitionistic fuzzy graph $G = (V, E, \mu, \gamma)$ and we derive some bounds for the Laplacian energy of the intuitionistic fuzzy graph $G = (V, E, \mu, \gamma)$. Further, we analyze the spreading rate of virus in terms of its Laplacian energy of the intuitionistic fuzzy graph.
5.2 DEGREE MATRIX OF AN INTUITIONISTIC FUZZY GRAPH

Definition 5.1. Let \( v \) be the vertex of an intuitionistic fuzzy graph \( G = (V, E, \mu, \gamma) \). The degree of \( v \) of \( G \) is defined as

\[
d(v) = \left( \sum_{i=1}^{n} d_{\mu}(v_i), \sum_{i=1}^{n} d_{\gamma}(v_i) \right)
\]

where

\[
d_{\mu}(v_i) = \text{indegree}_{\mu}(v_i) + \text{outdegree}_{\mu}(v_i)
\]

and

\[
d_{\gamma}(v_i) = \text{indegree}_{\gamma}(v_i) + \text{outdegree}_{\gamma}(v_i)
\]

are the degree of membership and non-membership value of \( v_i \) respectively.

Definition 5.2. The indegree and outdegree of the membership and non-membership value of \( v_i \) are defined as

\[
\text{indegree}_{\mu}(v_i) = \sum_{v_j} \mu(v_j, v_i) \quad \text{and} \quad \text{outdegree}_{\mu}(v_i) = \sum_{v_j} \mu(v_i, v_j)
\]

\[
\text{indegree}_{\gamma}(v_i) = \sum_{v_j} \gamma(v_j, v_i) \quad \text{and} \quad \text{outdegree}_{\gamma}(v_i) = \sum_{v_j} \gamma(v_i, v_j).
\]

Remark 5.1. (i). Sum of the degree of membership value of all vertices in an intuitionistic fuzzy graph is equal to twice the sum of the membership value of all edges. Similarly sum of the degree of non-membership value of all vertices in an intuitionistic fuzzy graph is equal to twice the sum of the non-membership value of all edges. Here we have,

\[
d_{\mu}(v_1) = \mu_{12} + \mu_{21} + \mu_{13} + \mu_{31} + \mu_{14} + \mu_{41}
\]

\[
d_{\mu}(v_2) = \mu_{21} + \mu_{12} + \mu_{23} + \mu_{32} + \mu_{24} + \mu_{42}
\]

\[
d_{\mu}(v_3) = \mu_{31} + \mu_{13} + \mu_{32} + \mu_{23} + \mu_{34} + \mu_{43}
\]

\[
d_{\mu}(v_4) = \mu_{41} + \mu_{14} + \mu_{42} + \mu_{24} + \mu_{43} + \mu_{34}.
\]
In general, we can write
\[ \sum_{i=1}^{n} d_{\mu}(v_i) = 2 \sum_{1 \leq i < j \leq n} (\mu_{ij} + \mu_{ji}). \]

Similarly, we have
\[ \sum_{i=1}^{n} d_{\gamma}(v_i) = 2 \sum_{1 \leq i < j \leq n} (\gamma_{ij} + \gamma_{ji}). \]

(ii). The maximum degree of any vertex in an intuitionistic fuzzy graph with \( n \) vertices is \( n - 1 \).

**Definition 5.3.** Let \( G = (V, E, \mu, \gamma) \) be an intuitionistic fuzzy graph with \( n \) vertices. The degree matrix of the membership value of \( G = (V, E, \mu, \gamma) \) is denoted by \( D_{\mu}(G) \) and is defined by \( D_{\mu}(G) = [a_{ij}] \) where
\[
a_{ij} = \begin{cases} 
d_{\mu}(v_i), & \text{if } i=j \\
0, & \text{otherwise}. \end{cases}
\]

The degree matrix of the non-membership value of \( G = (V, E, \mu, \gamma) \) is denoted by \( D_{\gamma}(G) \) and is defined by \( D_{\gamma}(G) = [b_{ij}] \) where
\[
b_{ij} = \begin{cases} 
d_{\gamma}(v_i), & \text{if } i=j \\
0, & \text{otherwise}. \end{cases}
\]

**Remark 5.2.** Let \( D_{\mu}(G) \) denote the \( n \times n \) diagonal matrix whose diagonal entries are the sum of \( i^{th} \) row and \( i^{th} \) column of \( A_{\mu}(G) \) and \( D_{\gamma}(G) \) denote the \( n \times n \) diagonal matrix whose diagonal entries are the sum of \( i^{th} \) row and \( i^{th} \) column of \( A_{\gamma}(G) \).

### 5.3 LAPLACIAN MATRIX OF AN INTUITIONISTIC FUZZY GRAPH

**Definition 5.4.** An intuitionistic fuzzy Laplacian matrix is defined as \( L(G) = (L_{\mu}(G), L_{\gamma}(G)) \) where \( L_{\mu}(G) = (D_{\mu}(G) - A_{\mu}(G)) \) and \( L_{\gamma}(G) = (D_{\gamma}(G) - A_{\gamma}(G)) \).
Here $D_{\mu}(G)$ and $D_{\gamma}(G)$ are the degree matrices of membership value and non-membership value respectively and $A_{\mu}(G), A_{\gamma}(G)$ are the adjacency matrices of membership value and the non-membership value respectively.

**Theorem 5.1.** If $\theta_1, \theta_2, ..., \theta_n$ are the real or complex eigen values of $L_{\mu}(G)$ and $\phi_1, \phi_2, ..., \phi_n$ are the real or complex eigen values of $L_{\gamma}(G)$, then

\[
(i) \sum_{i=1}^{n} \theta_i = 2 \sum_{1 \leq i < j \leq n} (\mu_{ij} + \mu_{ji}) \\
(ii) \sum_{i=1}^{n} \theta_i^2 = 2 \sum_{1 \leq i < j \leq n} \mu_{ij}\mu_{ji} + \sum_{i=1}^{n} (d_{\mu}(v_i))^2
\]

\[
(iii) \sum_{i=1}^{n} \phi_i = 2 \sum_{1 \leq i < j \leq n} (\gamma_{ij} + \gamma_{ji}) \\
(iv) \sum_{i=1}^{n} \phi_i^2 = 2 \sum_{1 \leq i < j \leq n} \gamma_{ij}\gamma_{ji} + \sum_{i=1}^{n} (d_{\gamma}(v_i))^2.
\]

**Proof.** (i) We know that sum of the eigen values of $L_{\mu}(G)$ is equal to the trace of $L_{\mu}(G)$. That is,

\[
\sum_{i=1}^{n} \theta_i = \text{trace}(L_{\mu}(G)) = \sum_{i=1}^{n} d_{\mu}(v_i) = 2 \sum_{1 \leq i < j \leq n} (\mu_{ij} + \mu_{ji}).
\]

(ii) Also the sum of the squares of the eigen values of $L_{\mu}(G)$ is equal to the trace of $(L_{\mu}(G))^2$. That is,

\[
\sum_{i=1}^{n} \theta_i^2 = \text{trace}(L_{\mu}(G))^2
\]

\[
= (d_{\mu}(v_1))^2 + \mu_{12}\mu_{21} + ... + \mu_{1n}\mu_{n1} +
\]

\[
\mu_{21}\mu_{12} + (d_{\mu}(v_2))^2 + ... + \mu_{2n}\mu_{n2} +
\]

\[
................................. +
\]

\[
\mu_{n1}\mu_{1n} + \mu_{n2}\mu_{2n} + ... + (d_{\mu}(v_n))^2
\]

\[
\Rightarrow \sum_{i=1}^{n} \theta_i^2 = 2 \sum_{1 \leq i < j \leq n} \mu_{ij}\mu_{ji} + \sum_{i=1}^{n} (d_{\mu}(v_i))^2.
\]

Similarly we can prove (iii) and (iv).

**Definition 5.5.** Let $G = (V, E, \mu, \gamma)$ be an intuitionistic fuzzy graph with $n$ vertices. Let $\theta_1, \theta_2, ..., \theta_n$ be the eigen values of $L_{\mu}(G)$ and $\phi_1, \phi_2, ..., \phi_n$ be the
eigen values of $L_\gamma(G)$. The Laplacian energy of an intuitionistic fuzzy graph $G = (V, E, \mu, \gamma)$ is defined as $LE(G) = (LE_\mu(G), LE_\gamma(G))$ where

$$LE_\mu(G) = \sum_{i=1}^{n} \left| \theta_i - \frac{2}{1 \leq i < j \leq n} (\mu_{ij} + \mu_{ji}) \right| = \sum_{i=1}^{n} |\alpha_i|$$

is Laplacian energy of membership value and

$$LE_\gamma(G) = \sum_{i=1}^{n} \left| \phi_i - \frac{2}{1 \leq i < j \leq n} (\gamma_{ij} + \gamma_{ji}) \right| = \sum_{i=1}^{n} |\beta_i|$$

is Laplacian energy of non-membership value where

$$\alpha_i = \theta_i - \frac{2}{1 \leq i < j \leq n} (\mu_{ij} + \mu_{ji}) \text{ and } \beta_i = \phi_i - \frac{2}{1 \leq i < j \leq n} (\gamma_{ij} + \gamma_{ji}).$$

**Theorem 5.2.** Let $G = (V, E, \mu, \gamma)$ be an intuitionistic fuzzy graph with $n$ vertices. If $\alpha_1, \alpha_2, ..., \alpha_n$ are the real or complex eigen values of $LE_\mu(G)$ and $\beta_1, \beta_2, ..., \beta_n$ are the real or complex eigen values of $LE_\gamma(G)$, then

1. $\sum_{i=1}^{n} \alpha_i = 0$
2. $\sum_{i=1}^{n} \alpha_i^2 = 2M$
3. $\sum_{i=1}^{n} \beta_i = 0$
4. $\sum_{i=1}^{n} \beta_i^2 = 2N$

where

$$M = \sum_{1 \leq i < j \leq n} \mu_{ij} \mu_{ji} + \frac{1}{2} \sum_{i=1}^{n} \left( d_{\mu}(v_i) - \frac{2}{1 \leq i < j \leq n} (\mu_{ij} + \mu_{ji}) \right)^2$$

and

$$N = \sum_{1 \leq i < j \leq n} \gamma_{ij} \gamma_{ji} + \frac{1}{2} \sum_{i=1}^{n} \left( d_{\gamma}(v_i) - \frac{2}{1 \leq i < j \leq n} (\gamma_{ij} + \gamma_{ji}) \right)^2.$$
Proof. (i) By Definition 5.5, we have

\[
\sum_{i=1}^{n} \alpha_i = \sum_{i=1}^{n} \left( \theta_i - \frac{2}{n} \sum_{1 \leq i < j \leq n} (\mu_{ij} + \mu_{ji}) \right)
\]

\[= \sum_{i=1}^{n} \theta_i - 2n \left( \sum_{1 \leq i < j \leq n} \frac{(\mu_{ij} + \mu_{ji})}{n} \right)\]

\[= 2 \sum_{1 \leq i < j \leq n} (\mu_{ij} + \mu_{ji}) - 2 \sum_{1 \leq i < j \leq n} (\mu_{ij} + \mu_{ji})\]

\[\Rightarrow \sum_{i=1}^{n} \alpha_i = 0.\]

(ii) By Definition 5.5, we have

\[
\sum_{i=1}^{n} \alpha_i^2 = \sum_{i=1}^{n} \left( \theta_i - \frac{2}{n} \sum_{1 \leq i < j \leq n} (\mu_{ij} + \mu_{ji}) \right)^2
\]

\[= \sum_{i=1}^{n} \left( \theta_i^2 + \frac{2}{n} \sum_{1 \leq i < j \leq n} (\mu_{ij} + \mu_{ji})^2 \right) - 2 \sum_{i=1}^{n} \theta_i \left( \frac{2}{n} \sum_{1 \leq i < j \leq n} (\mu_{ij} + \mu_{ji}) \right)\]

\[= \sum_{i=1}^{n} \theta_i^2 + \sum_{i=1}^{n} \left( \frac{2}{n} \sum_{1 \leq i < j \leq n} (\mu_{ij} + \mu_{ji})^2 \right) - 2 \sum_{i=1}^{n} \theta_i \left( \frac{2}{n} \sum_{1 \leq i < j \leq n} (\mu_{ij} + \mu_{ji}) \right)\]

\[= 2 \sum_{1 \leq i < j \leq n} \mu_{ij} \mu_{ji} + \sum_{i=1}^{n} (d_{\mu}(v_i))^2 + \sum_{i=1}^{n} \left( \frac{2}{n} \sum_{1 \leq i < j \leq n} (\mu_{ij} + \mu_{ji}) \right)^2
\]

\[= 2 \sum_{1 \leq i < j \leq n} \mu_{ij} \mu_{ji} + \sum_{i=1}^{n} (d_{\mu}(v_i))^2 - 2 \sum_{i=1}^{n} \left( \frac{2}{n} \sum_{1 \leq i < j \leq n} (\mu_{ij} + \mu_{ji}) \right)^2\]
\[
\sum_{i=1}^{n} \alpha_i^2 = 2 \left[ \sum_{1 \leq i < j \leq n} \mu_{ij}\mu_{ji} + \frac{1}{2} \sum_{i=1}^{n} \left( d_{\mu}(v_i) - \frac{2}{n} \sum_{1 \leq i < j \leq n} (\mu_{ij} + \mu_{ji}) \right)^2 \right].
\]

Similarly we can prove (iii) and (iv).

**Theorem 5.3.** Let \( G = (V, E, \mu, \gamma) \) be an intuitionistic fuzzy graph with \( n \) vertices. If \( \alpha_1, \alpha_2, \ldots, \alpha_n \) are the real eigen values of \( LE_\mu(G) \) and \( \beta_1, \beta_2, \ldots, \beta_n \) are the real eigen values of \( LE_\gamma(G) \), then

\[
(i) \quad LE_\mu(G) \leq \sqrt{2n \left( \sum_{1 \leq i < j \leq n} \mu_{ij}\mu_{ji} + \frac{1}{2} \sum_{i=1}^{n} \left( d_{\mu}(v_i) - \frac{2}{n} \sum_{1 \leq i < j \leq n} (\mu_{ij} + \mu_{ji}) \right)^2 \right)}.
\]

\[
(ii) \quad LE_\gamma(G) \leq \sqrt{2n \left( \sum_{1 \leq i < j \leq n} \gamma_{ij}\gamma_{ji} + \frac{1}{2} \sum_{i=1}^{n} \left( d_\gamma(v_i) - \frac{2}{n} \sum_{1 \leq i < j \leq n} (\gamma_{ij} + \gamma_{ji}) \right)^2 \right)}.
\]

Proof. (i) In Cauchy-Schwarz inequality

\[
\left( \sum_{i=1}^{n} a_i b_i \right)^2 \leq \left( \sum_{i=1}^{n} a_i^2 \right) \left( \sum_{i=1}^{n} b_i^2 \right)
\]

put \( a_i = 1, b_i = |\alpha_i| \), we get

\[
\left( \sum_{i=1}^{n} |\alpha_i| \right)^2 \leq \left( \sum_{i=1}^{n} 1 \right) \left( \sum_{i=1}^{n} \alpha_i^2 \right).
\]

By Theorem 5.2, we have

\[
(LE_\mu(G))^2 \leq n \left( 2 \left( \sum_{1 \leq i < j \leq n} \mu_{ij}\mu_{ji} + \frac{1}{2} \sum_{i=1}^{n} \left( d_{\mu}(v_i) - \frac{2}{n} \sum_{1 \leq i < j \leq n} (\mu_{ij} + \mu_{ji}) \right)^2 \right) \right)
\]

\[
\Rightarrow \quad LE_\mu(G) \leq \sqrt{2n \left( \sum_{1 \leq i < j \leq n} \mu_{ij}\mu_{ji} + \frac{1}{2} \sum_{i=1}^{n} \left( d_{\mu}(v_i) - \frac{2}{n} \sum_{1 \leq i < j \leq n} (\mu_{ij} + \mu_{ji}) \right)^2 \right)}.
\]

Similarly we can prove the second inequality.
Theorem 5.4. Let $G = (V, E, \mu, \gamma)$ be an intuitionistic fuzzy graph with $n$ vertices. If $\alpha_1, \alpha_2, \ldots, \alpha_n$ are the complex eigen values of $LE_{\mu}(G)$ and $\beta_1, \beta_2, \ldots, \beta_n$ are the complex eigen values of $LE_{\gamma}(G)$, then

$$(i) \ LE_{\mu}(G) \leq \sqrt{n \left( \sum_{i=1}^{n} |\alpha_i|^2 \right)} \quad (ii) \ LE_{\gamma}(G) \leq \sqrt{n \left( \sum_{i=1}^{n} |\beta_i|^2 \right)} .$$

Proof. (i) Again in Cauchy-Schwarz inequality, put $a_i = 1$, $b_i = |\alpha_i|$, we get

$$\left( \sum_{i=1}^{n} |\alpha_i| \right)^2 \leq \left( \sum_{i=1}^{n} \right) \left( \sum_{i=1}^{n} |\alpha_i|^2 \right)$$

$$\Rightarrow \sum_{i=1}^{n} |\alpha_i| \leq \sqrt{\left( \sum_{i=1}^{n} \right) \left( \sum_{i=1}^{n} |\alpha_i|^2 \right)}$$

$$\Rightarrow \sum_{i=1}^{n} |\alpha_i| \leq \sqrt{n \left( \sum_{i=1}^{n} |\alpha_i|^2 \right)}$$

$$\Rightarrow LE_{\mu}(G) \leq \sqrt{n \left( \sum_{i=1}^{n} |\alpha_i|^2 \right)} .$$

Similarly we can prove (ii). \qed

Theorem 5.5. Let $G = (V, E, \mu, \gamma)$ be an intuitionistic fuzzy graph with $n$ vertices. If $\alpha_1, \alpha_2, \ldots, \alpha_n$ are the real eigen values of $LE_{\mu}(G)$ and $\beta_1, \beta_2, \ldots, \beta_n$ are the real eigen values of $LE_{\gamma}(G)$, then

$$(i) \ LE_{\mu}(G) \geq \sqrt{2M + n(n - 1)C^2_\mu} \quad (ii) \ LE_{\gamma}(G) \geq \sqrt{2N + n(n - 1)F^2_\gamma}$$

where

$$M = \sum_{1 \leq i < j \leq n} \mu_{ij}\mu_{ji} + \frac{1}{2} \sum_{i=1}^{n} \left( d_{\mu}(v_i) - \frac{2}{n} \sum_{1 \leq i < j \leq n} (\mu_{ij} + \mu_{ji}) \right)^2$$

$$N = \sum_{1 \leq i < j \leq n} \gamma_{ij}\gamma_{ji} + \frac{1}{2} \sum_{i=1}^{n} \left( d_{\gamma}(v_i) - \frac{2}{n} \sum_{1 \leq i < j \leq n} (\gamma_{ij} + \gamma_{ji}) \right)^2$$
\[ C = \left| L_\mu(G) - \frac{2}{n} \sum_{1 \leq i < j \leq n} (\mu_{ij} + \mu_{ji}) \right| \quad \text{and} \quad F = \left| L_\gamma(G) - \frac{2}{n} \sum_{1 \leq i < j \leq n} (\gamma_{ij} + \gamma_{ji}) \right| \]

where \( I \) is the unit matrix.

Proof. We know that

\[ (LE_\mu(G))^2 = \left( \sum_{i=1}^{n} |\alpha_i| \right)^2 \]

\[ = \sum_{i=1}^{n} |\alpha_i|^2 + 2 \sum_{1 \leq i < j \leq n} |\alpha_i| |\alpha_j| \]

\[ = 2 \left[ \sum_{1 \leq i < j \leq n} \mu_{ij}\mu_{ji} + \frac{1}{2} \sum_{i=1}^{n} \left( d_\mu(v_i) - \frac{2}{n} \sum_{1 \leq i < j \leq n} (\mu_{ij} + \mu_{ji}) \right)^2 \right] \]

\[ + \frac{2n(n-1)}{2} AM \{ |\alpha_i| |\alpha_j| \} . \]

Since

\[ AM \{ |\alpha_i| |\alpha_j| \} \geq GM \{ |\alpha_i| |\alpha_j| \}, \quad 1 \leq i < j \leq n, \]

we get

\[ LE_\mu(G) \geq \sqrt{2 \left[ \sum_{1 \leq i < j \leq n} \mu_{ij}\mu_{ji} + \frac{1}{2} \sum_{i=1}^{n} \left( d_\mu(v_i) - \frac{2}{n} \sum_{1 \leq i < j \leq n} (\mu_{ij} + \mu_{ji}) \right)^2 \right] } \]

\[ + \sqrt{n(n-1)GM \{ |\alpha_i| |\alpha_j| \}}. \]

But

\[ GM \{ |\alpha_i| |\alpha_j| \} = \left( \prod_{i=1}^{n} |\alpha_i| \right)^{\frac{2}{n}}. \]

That is

\[ GM \{ |\alpha_i| |\alpha_j| \} = \left( \left| L_\mu(G) - \frac{2}{n} \sum_{1 \leq i < j \leq n} (\mu_{ij} + \mu_{ji}) I \right|^2 \right)^{\frac{n}{2}} = C^{\frac{2}{n}}. \]
Hence, we have

\[
LE_\mu(G) \geq \sqrt{2 \left[ \sum_{1 \leq i < j \leq n} \mu_{ij}\mu_{ji} + \frac{1}{2} \sum_{i=1}^{n} \left( d_\mu(v_i) - \frac{2}{1 \leq i < j \leq n} \left( \mu_{ij} + \mu_{ji} \right) n \right) \right]^2} + \sqrt{n(n-1)C^2_n}
\]

\[
\Rightarrow LE_\mu(G) \geq \sqrt{2M + n(n-1)C^2_n}
\]

where

\[
M = \sum_{1 \leq i < j \leq n} \mu_{ij}\mu_{ji} + \frac{1}{2} \sum_{i=1}^{n} \left( d_\mu(v_i) - \frac{2}{1 \leq i < j \leq n} \left( \mu_{ij} + \mu_{ji} \right) \frac{n}{n} \right)
\]

and

\[
C = \left| L_\mu(G) - \frac{2}{1 \leq i < j \leq n} \left( \mu_{ij} + \mu_{ji} \right) \frac{n}{n} I \right|.
\]

Similarly we can prove (ii).

**Theorem 5.6.** Let \( G = (V, E, \mu, \gamma) \) be an intuitionistic fuzzy graph with \( n \) vertices. If \( \alpha_1, \alpha_2, ..., \alpha_n \) are the complex eigen values of \( LE_\mu(G) \) and \( \beta_1, \beta_2, ..., \beta_n \) are the complex eigen values of \( LE_\gamma(G) \), then

(i) \( LE_\mu(G) \geq \sqrt{\sum_{i=1}^{n} |\alpha_i|^2 + n(n-1)C^2_n} \)
(ii) \( LE_\gamma(G) \geq \sqrt{\sum_{i=1}^{n} |\beta_i|^2 + n(n-1)F^2_n} \).

Proof. We know that

\[
(LE_\mu(G))^2 = \left( \sum_{i=1}^{n} |\alpha_i| \right)^2
\]

\[
= \sum_{i=1}^{n} |\alpha_i|^2 + 2 \sum_{1 \leq i < j \leq n} |\alpha_i| |\alpha_j|
\]

94
\[
LE_{\mu}(G) \geq \sqrt{\sum_{i=1}^{n} |\alpha_i|^2 + n(n-1)GM \{|\alpha_i| |\alpha_j|\}}
\]

\[
GM \{|\alpha_i| |\alpha_j|\} = \left( \left| L_{\mu}(G) - \frac{2}{n} \sum_{1 \leq i < j \leq n} (\mu_{ij} + \mu_{ji}) I \right| \right)^{\frac{2}{n}} = C_{\mu}^2
\]

\[
\Rightarrow LE_{\mu}(G) \geq \sqrt{\sum_{i=1}^{n} |\alpha_i|^2 + n(n-1)C_{\mu}^2}.
\]

Similarly we can prove (ii). \(\square\)

**Corollary 5.1.** Let \(G = (V, E, \mu, \gamma)\) be an intuitionistic fuzzy graph with \(n\) vertices. If \(\alpha_1, \alpha_2, ..., \alpha_n\) are the real eigen values of \(LE_{\mu}(G)\) and \(\beta_1, \beta_2, ..., \beta_n\) are the real eigen values of \(LE_{\gamma}(G)\) and if \(LE_{\mu}(G) \geq LE_{\gamma}(G)\), then

\[
2nM \geq 2N + n(n-1)F^2
\]

where

\[
M = \sum_{1 \leq i < j \leq n} \mu_{ij}\mu_{ji} + \frac{1}{2} \sum_{i=1}^{n} \left( d_{\mu}(v_i) - \frac{2}{n} \sum_{1 \leq i < j \leq n} (\mu_{ij} + \mu_{ji}) \right)^2
\]

\[
N = \sum_{1 \leq i < j \leq n} \gamma_{ij}\gamma_{ji} + \frac{1}{2} \sum_{i=1}^{n} \left( d_{\gamma}(v_i) - \frac{2}{n} \sum_{1 \leq i < j \leq n} (\gamma_{ij} + \gamma_{ji}) \right)^2
\]

and

\[
F = \left| L_{\gamma}(G) - \frac{2}{n} \sum_{1 \leq i < j \leq n} (\gamma_{ij} + \gamma_{ji}) I \right|.
\]

**Proof.** By Theorem 5.3, we have

\[
\sqrt{2nM} \geq LE_{\mu}(G)
\]

\[
\Rightarrow \sqrt{2nM} \geq LE_{\gamma}(G).
\]
By Theorem 5.5, we have
\[ LE_\gamma(G) \geq \sqrt{2N + n(n-1)\frac{2}{n}}. \]

Therefore, we get
\[ \sqrt{2nM} \geq \sqrt{2N + n(n-1)\frac{2}{n}} \]
\[ \Rightarrow 2nM \geq 2N + n(n-1)\frac{2}{n}. \]

Hence the proof. \( \square \)

**Corollary 5.2.** Let \( G = (V,E,\mu,\gamma) \) be an intuitionistic fuzzy graph with \( n \) vertices. If \( \alpha_1, \alpha_2, ..., \alpha_n \) are the real eigen values of \( LE_\mu(G) \) and \( \beta_1, \beta_2, ..., \beta_n \) are the real eigen values of \( LE_\gamma(G) \) and if \( LE_\gamma(G) \geq LE_\mu(G) \), then
\[
2nN \geq 2M + n(n-1)C^2
\]

where
\[
M = \sum_{1 \leq i < j \leq n} \mu_{ij}\mu_{ji} + \frac{1}{2} \sum_{i=1}^{n} \left( d_\mu(v_i) - \frac{2}{1 \leq i < j \leq n} (\mu_{ij} + \mu_{ji}) \right)^2
\]
\[
N = \sum_{1 \leq i < j \leq n} \gamma_{ij}\gamma_{ji} + \frac{1}{2} \sum_{i=1}^{n} \left( d_\gamma(v_i) - \frac{2}{1 \leq i < j \leq n} (\gamma_{ij} + \gamma_{ji}) \right)^2
\]

and
\[
C = \left| L_\mu(G) - \frac{2}{1 \leq i < j \leq n} (\mu_{ij} + \mu_{ji}) \right| I
\]

Proof. By Theorem 5.3, we have
\[ \sqrt{2nN} \geq LE_\gamma(G) \]
\[ \Rightarrow \sqrt{2nN} \geq LE_\mu(G) \]

By Theorem 5.5, we have
\[ LE_\mu(G) \geq \sqrt{2M + n(n-1)C^2}. \]
Therefore, we get
\[
\sqrt{2nN} \geq \sqrt{2M + n(n-1)C^2_2}
\]
\[
\Rightarrow 2nN \geq 2M + n(n-1)C^2_2.
\]

Hence the proof. \(\square\)

**Corollary 5.3.** Let \(G = (V, E, \mu, \gamma)\) be an intuitionistic fuzzy graph with \(n\) vertices. If \(\alpha_1, \alpha_2, ..., \alpha_n\) are the complex eigen values of \(LE_{\mu}(G)\) and \(\beta_1, \beta_2, ..., \beta_n\) are the complex eigen values of \(LE_\gamma(G)\) and if \(LE_{\mu}(G) \geq LE_\gamma(G)\), then
\[
\quad n \left( \sum_{i=1}^{n} |\alpha_i|^2 \right) \geq \sum_{i=1}^{n} |\beta_i|^2 + n(n-1)F^2_2.
\]

**Proof.** By Theorem 5.4, we have
\[
\sqrt{n \left( \sum_{i=1}^{n} |\alpha_i|^2 \right)} \geq LE_{\mu}(G)
\]
\[
\Rightarrow \sqrt{n \left( \sum_{i=1}^{n} |\alpha_i|^2 \right)} \geq LE_\gamma(G).
\]

By Theorem 5.6, we have
\[
LE_\gamma(G) \geq \sqrt{\sum_{i=1}^{n} |\beta_i|^2 + n(n-1)F^2_2}.
\]

Therefore, we get
\[
\sqrt{n \left( \sum_{i=1}^{n} |\alpha_i|^2 \right)} \geq \sqrt{\sum_{i=1}^{n} |\beta_i|^2 + n(n-1)F^2_2}
\]
\[
\Rightarrow n \left( \sum_{i=1}^{n} |\alpha_i|^2 \right) \geq \sum_{i=1}^{n} |\beta_i|^2 + n(n-1)F^2_2.
\]

Hence the proof. \(\square\)

**Corollary 5.4.** Let \(G = (V, E, \mu, \gamma)\) be an intuitionistic fuzzy graph with \(n\) vertices. If \(\alpha_1, \alpha_2, ..., \alpha_n\) are the complex eigen values of \(LE_{\mu}(G)\) and \(\beta_1, \beta_2, ..., \beta_n\) are the complex eigen values of \(LE_\gamma(G)\) and if \(LE_\gamma(G) \geq LE_{\mu}(G)\), then
\[
\quad n \left( \sum_{i=1}^{n} |\beta_i|^2 \right) \geq \sum_{i=1}^{n} |\alpha_i|^2 + n(n-1)C^{n-1}_2.
\]

97
Proof. By Theorem 5.4, we have
\[
\sqrt{n \left( \sum_{i=1}^{n} |\beta_i|^2 \right)} \geq LE_\gamma(G)
\]
\[
\Rightarrow \sqrt{n \left( \sum_{i=1}^{n} |\beta_i|^2 \right)} \geq LE_\mu(G).
\]

By Theorem 5.6, we have
\[
LE_\mu(G) \geq \sqrt{\sum_{i=1}^{n} |\alpha_i|^2 + n(n-1)C_2^n}.
\]

Therefore, we get
\[
\sqrt{n \left( \sum_{i=1}^{n} |\beta_i|^2 \right)} \geq \sqrt{\sum_{i=1}^{n} |\alpha_i|^2 + n(n-1)C_2^n}
\]
\[
\Rightarrow n \left( \sum_{i=1}^{n} |\beta_i|^2 \right) \geq \sum_{i=1}^{n} |\alpha_i|^2 + n(n-1)C_2^n.
\]

Hence the proof. \qed

**Theorem 5.7.** Let \( G = (V, E, \mu, \gamma) \) be an intuitionistic fuzzy graph with \( n \) vertices. If \( \alpha_1, \alpha_2, \ldots, \alpha_n \) are the real eigen values of \( LE_\mu(G) \) and \( \beta_1, \beta_2, \ldots, \beta_n \) are the real eigen values of \( LE_\gamma(G) \), then

\[
(i) \ LE_\mu(G) \geq 2\sqrt{M} \quad (ii) \ LE_\gamma(G) \geq 2\sqrt{N}.
\]

Proof. (i) We know that
\[
\left( \sum_{i=1}^{n} \alpha_i \right)^2 = \sum_{i=1}^{n} \alpha_i^2 + 2 \sum_{1 \leq i < j \leq n} \alpha_i \alpha_j
\]
\[
\Rightarrow 0 = 2M + 2 \sum_{1 \leq i < j \leq n} \alpha_i \alpha_j
\]
\[
\Rightarrow 2M = -2 \sum_{1 \leq i < j \leq n} \alpha_i \alpha_j
\]
\[
\Rightarrow 2M \leq 2 \left| \sum_{1 \leq i < j \leq n} \alpha_i \alpha_j \right|
\]

\[
\Rightarrow 2M \leq 2 \sum_{1 \leq i < j \leq n} |\alpha_i| |\alpha_j|. \quad (5.1)
\]

Now
\[
(LE_\mu(G))^2 = \left( \sum_{i=1}^{n} |\alpha_i| \right)^2
\]

\[
= \sum_{i=1}^{n} |\alpha_i|^2 + 2 \sum_{1 \leq i < j \leq n} |\alpha_i| |\alpha_j|
\]

\[
= 2M + 2 \sum_{1 \leq i < j \leq n} |\alpha_i| |\alpha_j|.
\]

By equation (5.1), we get
\[
(LE_\mu(G))^2 \geq 4M
\]

\[
\Rightarrow LE_\mu(G) \geq 2\sqrt{M}.
\]

Similarly we can prove (ii).

**Theorem 5.8.** Let \( G = (V, E, \mu, \gamma) \) be an intuitionistic fuzzy graph with \( n \) vertices. If \( \alpha_1, \alpha_2, ..., \alpha_n \) are the complex eigen values of \( LE_\mu(G) \) and \( \beta_1, \beta_2, ..., \beta_n \) are the complex eigen values of \( LE_\gamma(G) \), then

\[
(i) \ LE_\mu(G) \geq \sqrt{2M} \quad (ii) \ LE_\gamma(G) \geq \sqrt{2N}.
\]

Proof. (i) We know that
\[
(LE_\mu(G))^2 = \left( \sum_{i=1}^{n} |\alpha_i| \right)^2
\]

\[
= \sum_{i=1}^{n} |\alpha_i|^2 + 2 \sum_{1 \leq i < j \leq n} |\alpha_i| |\alpha_j|
\]

\[
\geq 2 \sum_{1 \leq i < j \leq n} |\alpha_i| |\alpha_j|.
\]
Using equation (5.1), we get
\[(LE_\mu(G))^2 \geq 2M\]
\[\Rightarrow LE_\mu(G) \geq \sqrt{2M}.

Similarly we can prove (ii).

**Theorem 5.9.** Let \( G = (V, E, \mu, \gamma) \) be an intuitionistic fuzzy graph with \( n \) vertices. If \( \alpha_1, \alpha_2, ..., \alpha_n \) are the real eigen values of \( LE_\mu(G) \) and \( \beta_1, \beta_2, ..., \beta_n \) are the real eigen values of \( LE_\gamma(G) \), then
\[
(i) \ LE_\mu(G) \leq \alpha_1 + \sqrt{(n-1)(2M - \alpha_1^2)} \\
(ii) \ LE_\gamma(G) \leq \beta_1 + \sqrt{(n-1)(2N - \beta_1^2)}.
\]

Proof. (i) By the Cauchy-Schwarz inequality, we have
\[
\left( \sum_{i=2}^{n} |\alpha_i| \right)^2 \leq \left( \sum_{i=2}^{n} 1 \right) \left( \sum_{i=2}^{n} \alpha_i^2 \right)
\]
\[\Rightarrow (LE_\mu(G) - \alpha_1)^2 \leq (n-1)(2M - \alpha_1^2)\]
\[\Rightarrow LE_\mu(G) - \alpha_1 \leq \sqrt{(n-1)(2M - \alpha_1^2)}\]
\[\Rightarrow LE_\mu(G) \leq \alpha_1 + \sqrt{(n-1)(2M - \alpha_1^2)}.
\]

Similarly we can prove (ii).

**Theorem 5.10.** Let \( G = (V, E, \mu, \gamma) \) be an intuitionistic fuzzy graph with \( n \) vertices. If \( \alpha_1, \alpha_2, ..., \alpha_n \) are the complex eigen values of \( LE_\mu(G) \) and \( \beta_1, \beta_2, ..., \beta_n \) are the complex eigen values of \( LE_\gamma(G) \), then
\[
(i) \ LE_\mu(G) \leq \alpha_1 + \sqrt{(n-1) \left( \sum_{i=1}^{n} |\alpha_i|^2 - \alpha_1^2 \right)}
\]
\[
(ii) \ LE_\gamma(G) \leq \beta_1 + \sqrt{(n-1) \left( \sum_{i=1}^{n} |\beta_i|^2 - \beta_1^2 \right)}.
\]
Proof. (i) By the Cauchy-Schwarz inequality, we have

\[
\left( \sum_{i=2}^{n} |\alpha_i| \right)^2 \leq \left( \sum_{i=2}^{n} 1 \right) \left( \sum_{i=2}^{n} |\alpha_i|^2 \right)
\]

\[
\Rightarrow (LE_{\mu}(G) - \alpha_1)^2 \leq (n - 1) \left( \sum_{i=1}^{n} |\alpha_i|^2 - \alpha_1^2 \right)
\]

\[
\Rightarrow LE_{\mu}(G) - \alpha_1 \leq \sqrt{(n - 1) \left( \sum_{i=1}^{n} |\alpha_i|^2 - \alpha_1^2 \right)}
\]

\[
\Rightarrow LE_{\mu}(G) \leq \alpha_1 + \sqrt{(n - 1) \left( \sum_{i=1}^{n} |\alpha_i|^2 - \alpha_1^2 \right)}.
\]

Similarly we can prove(ii).

Corollary 5.5. Let \( G = (V, E, \mu, \gamma) \) be an intuitionistic fuzzy graph with \( n \) vertices. If \( \alpha_1, \alpha_2, \ldots, \alpha_n \) are the real eigen values of \( LE_{\mu}(G) \) and \( \beta_1, \beta_2, \ldots, \beta_n \) are the real eigen values of \( LE_{\gamma}(G) \), then we have

(i) if \( LE_{\mu}(G) \geq LE_{\gamma}(G) \) then \( \alpha_1 + \sqrt{(n - 1)(2M - \alpha_1^2)} \geq 2\sqrt{N} \)

(ii) if \( LE_{\gamma}(G) \geq LE_{\mu}(G) \) then \( \beta_1 + \sqrt{(n - 1)(2N - \beta_1^2)} \geq 2\sqrt{M} \).

Proof. (i) By Theorem 5.9, we have

\[
\alpha_1 + \sqrt{(n - 1)(2M - \alpha_1^2)} \geq LE_{\mu}(G)
\]

\[
\Rightarrow \alpha_1 + \sqrt{(n - 1)(2M - \alpha_1^2)} \geq LE_{\gamma}(G).
\]

By Theorem 5.7, we have \( LE_{\gamma}(G) \geq 2\sqrt{N} \).

Therefore, we get

\[
\alpha_1 + \sqrt{(n - 1)(2M - \alpha_1^2)} \geq 2\sqrt{N}.
\]

Similarly we can prove (ii).
**Corollary 5.6.** Let $G = (V, E, \mu, \gamma)$ be an intuitionistic fuzzy graph with $n$ vertices. If $\alpha_1, \alpha_2, ..., \alpha_n$ are the complex eigen values of $LE_\mu(G)$ and $\beta_1, \beta_2, ..., \beta_n$ are the complex eigen values of $LE_\gamma(G)$, then we have

(i) if $LE_\mu(G) \geq LE_\gamma(G)$ then $\alpha_1 + \sqrt{(n-1) \left( \sum_{i=1}^{n} |\alpha_i|^2 - \alpha_1^2 \right)} \geq \sqrt{2N}$

(ii) if $LE_\gamma(G) \geq LE_\mu(G)$ then $\beta_1 + \sqrt{(n-1) \left( \sum_{i=1}^{n} |\beta_i|^2 - \beta_1^2 \right)} \geq \sqrt{2M}$.

**Proof.** (i) By Theorem 5.10, we have

$$\alpha_1 + \sqrt{(n-1) \left( \sum_{i=1}^{n} |\alpha_i|^2 - \alpha_1^2 \right)} \geq LE_\mu(G)$$

$$\Rightarrow \alpha_1 + \sqrt{(n-1) \left( \sum_{i=1}^{n} |\alpha_i|^2 - \alpha_1^2 \right)} \geq LE_\gamma(G).$$

By Theorem 5.8, we have

$$LE_\gamma(G) \geq \sqrt{2N}.$$  

Therefore, we get

$$\alpha_1 + \sqrt{(n-1) \left( \sum_{i=1}^{n} |\alpha_i|^2 - \alpha_1^2 \right)} \geq \sqrt{2N}.$$  

Similarly we can prove (ii).  

5.4 SPREADING RATE OF VIRUS IN LAPLACIAN ENERGY OF AN INTUITIONISTIC FUZZY GRAPH

In this section, we define the infection rate, curing rate and the sharp epidemic threshold of the virus spread in Laplacian energy of an intuitionistic fuzzy graph.
**Definition 5.6.** If $LE_{\mu}(G) > LE_{\gamma}(G)$ then the infection rate of an intuitionistic fuzzy graph is defined as $\beta = \max_{i,j} \mu_{ij}$ and the curing rate of an intuitionistic fuzzy graph is defined as $\delta = \min_{i,j} \gamma_{ij}$. The ratio $\tau = \frac{\beta}{\delta}$, $(\delta \neq 0)$ is the effective spreading rate and $\tau_c = \frac{1}{\lambda_{\text{max}} LE_{\mu}(G)}$ where $\lambda_{\text{max}} LE_{\mu}(G)$ is the largest eigen value of $LE_{\mu}(G)$ of an intuitionistic fuzzy graph.

**Definition 5.7.** If $LE_{\mu}(G) < LE_{\gamma}(G)$ then the infection rate of an intuitionistic fuzzy graph is defined as $\beta = \min_{i,j} \mu_{ij}$ and the curing rate of an intuitionistic fuzzy graph is defined as $\delta = \max_{i,j} \gamma_{ij}$. The ratio $\tau = \frac{\beta}{\delta}$ is the effective spreading rate and $\tau_c = \frac{1}{\lambda_{\text{max}} LE_{\gamma}(G)}$ where $\lambda_{\text{max}} LE_{\gamma}(G)$ is the largest eigen value of $LE_{\gamma}(G)$ of an intuitionistic fuzzy graph.

### 5.5 NUMERICAL EXAMPLE

Let us illustrate the above concepts in the following example.

**Example 5.1.** For the intuitionistic fuzzy graph in Figure 2.1, let us determine the in-degree and out-degree of membership and non-membership value of the each vertex. That is,

- $\text{indegree}_\mu(v_1) = 1.8$, $\text{indegree}_\mu(v_2) = 2.1$, $\text{indegree}_\mu(v_3) = 0.8$, $\text{indegree}_\mu(v_4) = 0.9$,
- $\text{indegree}_\gamma(v_1) = 0.6$, $\text{indegree}_\gamma(v_2) = 0.4$, $\text{indegree}_\gamma(v_3) = 0.6$, $\text{indegree}_\gamma(v_4) = 0.9$,
- $\text{outdegree}_\mu(v_1) = 0.9$, $\text{outdegree}_\mu(v_2) = 0.9$, $\text{outdegree}_\mu(v_3) = 1.8$, $\text{outdegree}_\mu(v_4) = 2.0$,
- $\text{outdegree}_\gamma(v_1) = 0.8$, $\text{outdegree}_\gamma(v_2) = 0.4$, $\text{outdegree}_\gamma(v_3) = 0.6$, $\text{outdegree}_\gamma(v_4) = 0.7$. 

103
By Definition 5.1, the degree of vertex is given by

\[ d(v) = \left( \sum_{i=1}^{n} d_{\mu}(v_i), \sum_{i=1}^{n} d_{\gamma}(v_i) \right) = (11.2, 5). \]

By Remark 5.2, we have the degree matrices of membership value and non-membership values are given by

\[
D_{\mu}(G) = \begin{pmatrix}
2.7 & 0 & 0 & 0 \\
0 & 3 & 0 & 0 \\
0 & 0 & 2.6 & 0 \\
0 & 0 & 0 & 2.9
\end{pmatrix}
\quad \text{and} \quad
D_{\gamma}(G) = \begin{pmatrix}
1.4 & 0 & 0 & 0 \\
0 & 0.8 & 0 & 0 \\
0 & 0 & 1.2 & 0 \\
0 & 0 & 0 & 1.6
\end{pmatrix}.
\]

By Definition 5.4, we have the following Laplacian matrix

\[
L_{\mu}(G) = \begin{pmatrix}
d_{\mu}(v_1) & -\mu_{12} & -\mu_{13} & -\mu_{14} \\
-\mu_{21} & d_{\mu}(v_2) & -\mu_{23} & -\mu_{24} \\
-\mu_{31} & -\mu_{32} & d_{\mu}(v_3) & -\mu_{34} \\
-\mu_{41} & -\mu_{42} & -\mu_{43} & d_{\mu}(v_4)
\end{pmatrix}
\]

\[
L_{\gamma}(G) = \begin{pmatrix}
d_{\gamma}(v_1) & -\gamma_{12} & -\gamma_{13} & -\gamma_{14} \\
-\gamma_{21} & d_{\gamma}(v_2) & -\gamma_{23} & -\gamma_{24} \\
-\gamma_{31} & -\gamma_{32} & d_{\gamma}(v_3) & -\gamma_{34} \\
-\gamma_{41} & -\gamma_{42} & -\gamma_{43} & d_{\gamma}(v_4)
\end{pmatrix}.
\]
\[ L_\mu(G) = \begin{pmatrix} 2.7 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 2.6 & 0 \\ 0 & 0 & 0 & 2.9 \end{pmatrix} - \begin{pmatrix} 0 & 0.6 & 0 & 0.3 \\ 0.4 & 0 & 0.5 & 0 \\ 0.5 & 0.7 & 0 & 0.6 \\ 0.9 & 0.8 & 0 & 0 \end{pmatrix} \]

\[ = \begin{pmatrix} 2.7 & -0.6 & 0 & -0.3 \\ -0.4 & 3 & -0.5 & 0 \\ -0.5 & -0.7 & 2.6 & -0.6 \\ -0.9 & -0.8 & -0.3 & 2.9 \end{pmatrix} \]

\[ L_\gamma(G) = \begin{pmatrix} 1.4 & 0 & 0 & 0 \\ 0 & 0.8 & 0 & 0 \\ 0 & 0 & 1.2 & 0 \\ 0 & 0 & 0 & 1.6 \end{pmatrix} - \begin{pmatrix} 0 & 0.2 & 0 & 0.6 \\ 0.3 & 0 & 0.1 & 0 \\ 0.2 & 0.1 & 0 & 0.3 \\ 0.1 & 0.1 & 0.5 & 0 \end{pmatrix} \]

\[ = \begin{pmatrix} 1.4 & -0.2 & 0 & -0.6 \\ -0.3 & 0.8 & -0.1 & 0 \\ -0.2 & -0.1 & 1.2 & -0.3 \\ -0.1 & -0.1 & -0.5 & 1.6 \end{pmatrix} \]

The eigen values of \( L_\mu(G) \) are 1.5533, 3.0543, 3.111, 3.4813 and \( E(L_\mu(G)) = 11.2 \).

The eigen values of \( L_\gamma(G) \) are 1.7068+0.0861i, 1.7068-0.0861i, 0.5808+0.0000i, 1.0056+0.0000i and \( E(L_\gamma(G)) = 5 \).
By Definition 5.5, we have

\[ LE_\mu(G) = \sum_{i=1}^{n} |\alpha_i| = 1.2467 + 0.2543 + 0.3111 + 0.6813 = 2.4934 \]

and

\[ LE_\gamma(G) = \sum_{i=1}^{n} |\beta_i| = 0.4649 + 0.4649 + 0.6692 + 0.2444 = 1.8434. \]

The Laplacian energy is given by

\[ LE(G) = (LE_\mu(G), LE_\gamma(G)) = (2.4934, 1.8434). \]

Here, we have

\[ LE_\mu(G) > LE_\gamma(G) \]

so the infection rate and curing rate are given by

\[ \beta = \max_{i,j} \mu_{ij} = 0.9 \quad , \quad \delta = \min_{i,j} \gamma_{ij} = 0.1 \]

and the spreading rate of virus and sharp epidemic threshold are given by

\[ \tau = \frac{\beta}{\delta} = 9 \quad , \quad \tau_c = \frac{1}{\lambda_{\max} LE_\mu(G)} = \frac{1}{1.2467} = 0.8021. \]

Here we note that \( \tau > \tau_c \), so the virus continue and a nonzero fraction of the nodes are infected. Hence the spreading rate of virus is maximum.

But by considering the intuitionistic fuzzy adjacency matrix, we have the following

\[ E(A_\mu(G)) = 1.2406 + 0.7153 + 0.3513 + 0.3513 = 2.6585 \]

\[ E(A_\gamma(G)) = 0.6441 + 0.0148 + 0.3543 + 0.3543 = 1.3675 \]

\[ E(A(G)) = (E(A_\mu(G)), E(A_\gamma(G))) = (2.6585, 1.3675). \]

Here, we have

\[ E(A_\mu(G)) > E(A_\gamma(G)) \]
so the infection rate and curing rate are given by

$$\beta = \max_{i,j} \mu_{ij} = 0.9 \quad \delta = \min_{i,j} \gamma_{ij} = 0.1$$

and the spreading rate of virus and sharp epidemic threshold are given by

$$\tau = \frac{\beta}{\delta} = 9 \quad \tau_c = \frac{1}{\lambda_{\text{max}} A_{\mu}(G)} = \frac{1}{1.2406} = 0.8061.$$

Here also $\tau > \tau_c$, so the virus continue and a nonzero fraction of the nodes are infected. Hence the spreading rate of virus is maximum.

The following table represents the comparison of spreading rate of virus and sharp epidemic threshold in $LE(G)$ and $E(A(G))$.

<table>
<thead>
<tr>
<th></th>
<th>Energy</th>
<th>Spreading rate of virus</th>
<th>Sharp epidemic threshold</th>
</tr>
</thead>
<tbody>
<tr>
<td>$E(A(G))$</td>
<td>(2.6585, 1.3675)</td>
<td>9</td>
<td>0.8061</td>
</tr>
<tr>
<td>$LE(G)$</td>
<td>(2.4934, 1.8434)</td>
<td>9</td>
<td>0.8021</td>
</tr>
</tbody>
</table>

From Table 5.1, we can observe that if the sharp epidemic threshold is same then the spreading rate of virus is also same.

Gutman et al. (2008) conjectured that the inequality $E(G) \leq LE(G)$ holds for all graphs but we are conjectured that the inequality $E(G) \leq LE(G)$ does not hold for all graphs because in the given intuitionistic fuzzy directed graph, $E(A_{\mu}(G)) = 2.6585$, $E(A_{\gamma}(G)) = 1.3675$, $LE_{\mu}(G) = 2.4934$ and $LE_{\gamma}(G) = 1.8434$. That is Laplacian energy of membership value is less than the energy of membership value of an intuitionistic fuzzy graph.