Chapter 4

The Radio Number of
Subdivision of Wheels

In this chapter, \footnote{Section 4.1 of this chapter has been published in the Journal of Discrete Mathematical Sciences and Cryptography, Vol. 12 No. 6 (2009) 729–736 [13].} we find the radio number of graphs \((W_n : 2)\) and \((W_n : 3)\), which are obtained by subdividing, each edge in the cycle \(C_n\) twice and thrice respectively.

4.1 The Radio Number of \((W_n : 2)\) Graphs

**Definition 4.1.** Let \(W_n = C_n + K_1\) denote the wheel on \(n + 1\) vertices \((n \geq 3)\), with \(V(K_1) = \{u\}\) and \(V(C_n) = \{v_1, v_2, \ldots, v_n\}\). Then the graph obtained from \(W_n\) by subdividing each edge of the cycle \(C_n\) exactly twice is denoted by \((W_n : 2)\). It has \(4n\) edges and \(\text{diam} (W_n : 2) = 4\). The standard drawing of the graph \((W_n : 2)\) is as follows:

The subdivision of vertices are denoted by \(w_1, w_1', w_2, w_2', \ldots, w_n, w_n'\) where \(w_i\)'s and \(w_i'\)'s are adjacent. If \(n\) is odd, then we specify that \(w_1\) is adjacent to \(v_1\), and \(w_1'\) is adjacent to \(v_2\), otherwise \(w_1\) is adjacent to \(v_n\), and \(w_1'\) is
adjacent to $v_1$. The standard drawings of $(W_6 : 2)$ and $(W_7 : 2)$ are shown in Figure 4.1.

![Diagram of $W_6 : 2$](image1)

(a) $(W_6 : 2)$.

![Diagram of $W_7 : 2$](image2)

(b) $(W_7 : 2)$.

Figure 4.1

**Theorem 4.2.** For $n \geq 3$, $rn(W_n : 2) \geq 5n + 2$.

**Proof.** $\text{Diam } (W_n : 2) = 4$ for $n \geq 3$. So any radio labeling $c$ of $(W_n : 2)$ must satisfy the radio condition $d(u, v) + |c(u) - c(v)| \geq 5$ for all distinct $u, v \in V(G)$. We count the number of values needed for labels and add the minimum number of forbidden values - those values precluded by use of a particular value as a label. For instance, if we label the center ‘a’ (i.e., $c(u) = a$), then, as $d(u, r) \leq 2$ for all vertices $u \neq r$, the values $a - 2, a - 1, a + 1,$ and $a + 2$ are forbidden. Similarly, as $d(v_i, r) \leq 3$ for all $v_i$ and for any $r \neq v_i$, $c(v_i) - 1$ and $c(v_i) + 1$ are forbidden. However, as $d(w_i, r) = 4$ and $d(w'_i, r) = 4$ for some vertex $r$, it is possible to use consecutive labels on $w_i$ and $r$ and $w'_i$ and $r$ (i.e. there are no forbidden values associated with the vertices $\{w_1, w_2, \ldots, w_n\}$ and $\{w'_1, w'_2, \ldots, w'_n\}$).

Note that the number of forbidden values are symmetric by any label used for a particular vertex. Thus we find the minimum number of forbidden
values by using the lowest and highest values on the center vertex and on one of \( \{v_1, v_2, v_3, \ldots, v_n\} \). Assume without loss of generality that \( c(u) = 1 \) and \( c(v_n) \) is the span of \( c \). Associated with the center vertex are two forbidden values (2 and 3), with \( v_1 \) is one forbidden value, and with the other \( v_i \) are two forbidden values each. This gives a total of \( 2 + 1 + 2(n - 1) = 2n + 1 \) forbidden values. Adding with the \( 3n + 1 \) values needed to label the \( 3n + 1 \) vertices provides a total of \( 5n + 2 \). Hence \( r_n(W_n : 2) \geq 5n + 2 \).

\[ \square \]

**Theorem 4.3.** For \( n \geq 6 \), \( r_n(W_n : 2) \leq 5n + 2 \).

**Proof.** We provide a radio labeling \( c \) of \( (W_n : 2) \) in two steps. First we define a position function that renames the vertices of \( (W_n : 2) \) using the set \( \{x_0, x_1, \ldots, x_{3n}\} \), then we specify the labels \( c(x_i) \) so that \( i < j \) if and only if \( c(x_i) < c(x_j) \). (This allows us easily to show that \( c \) is indeed a radio labeling.)

The position function \( p : V(W_n : 2) \rightarrow \{x_0, x_1, \ldots, x_{3n}\} \) is defined as follows. For \( n = 2k + 1 \), we define

\[
\begin{align*}
p(u) &= x_0, \\
p(w_{2i-1}) &= x_i & \text{for } i = 1, 2, \ldots, k + 1, \\
p(w_{2i}) &= x_{k+1+i} & \text{for } i = 1, 2, \ldots, k, \\
p(w'_{2i-1}) &= x_{n+i} & \text{for } i = 1, 2, \ldots, k + 1, \\
p(w'_{2i}) &= x_{n+k+1+i} & \text{for } i = 1, 2, \ldots, k, \\
p(v_i) &= x_{2n+i} & \text{for } i = 1, 2, \ldots, n.
\end{align*}
\]

For \( n = 2k \), we define

\[
\begin{align*}
p(u) &= x_0, \\
p(w_{2i-1}) &= x_i & \text{for } i = 1, 2, \ldots, k,
\end{align*}
\]
\[ p(w_{2i}) = x_{k+i} \quad \text{for } i = 1, 2, \ldots, k, \]
\[ p(w'_{2i-1}) = x_{n+i} \quad \text{for } i = 1, 2, \ldots, k, \]
\[ p(w'_{2i}) = x_{n+k+i} \quad \text{for } i = 1, 2, \ldots, k, \]
\[ p(v_i) = x_{2n+i} \quad \text{for } i = 1, 2, \ldots, n. \]

The position function orders the vertices so that \( \{x_0, x_1, x_2, \ldots, x_{3n}\} \) corresponds to \( \{u, w_1, w_3, \ldots, w_n, w_2, w_4, \ldots, w_{n-1}, w'_1, w'_3, \ldots, w'_n, w'_2, w'_4, \ldots, w'_{n-1}, v_1, v_2, \ldots, v_n\} \) when \( n \) is odd and to \( \{u, w_1, w_3, \ldots, w_{n-1}, w_2, w_4, \ldots, w_n, w'_1, w'_3, \ldots, w'_{n-1}, w'_2, w'_4, \ldots, w'_n, v_1, v_2, \ldots, v_n\} \) when \( n \) is even. Figure 4.2 shows the renamed versions of \((W_6:2)\) and \((W_7:2)\).

![Figure 4.2](image)

Define the radio labeling \( c: V(W_n:2) \to N \) by

\[
c(x_i) = \begin{cases} 
1, & \text{if } i = 0, \\
i + 3, & \text{if } 1 \leq i \leq 2n, \\
2n + 2 + 3(i - 2n), & \text{if } 2n + 1 \leq i \leq 3n.
\end{cases}
\]

**Claim:** The labeling \( c \) is a valid radio labeling.

We must show that the radio condition,

\[ d(u, v) + |c(u) - c(v)| \geq \text{diam} (W_n:2) + 1 = 5 \]
holds for all pair of vertices \((u, v)\) where \(u \neq v\).

**Case 1** \((u, r)\) (for any vertex \(r \neq u\)).

Recall \(p(u) = x_0\). As \(c(x_i) \geq 5\) for any \(i \geq 2\), we have \(d(u, x_i) + |c(u) - c(x_i)| \geq 1 + |1 - 5| \geq 5\) for all \(i \geq 2\). This leaves the pair \((u, x_1)\). But \(p^{-1}(x_1) = w_1\), so we calculate \(d(u, w_1) + |c(u) - c(w_1)| = 2 + |1 - 4| = 5\).

**Case 2** \((w_j, w_k)\) (with \(j \neq k\)).

Recall \(p(w_{2i-1}) = x_i\) and note that \(p(w_{2i})\) may be written as \(x_{n-k+i}\) whether \(n\) is even or odd. We have \(d(w_j, w_k) = 3\) for the pair \((w_{2i-1}, w_{2i}), (w_{2i}, w_{2i+1}), (w_n, w_1)\). These are translated respectively to \((x_i, x_{n-k+i}), (x_{n-k+i}, x_i+1)\) and \((x_s, x_1)\) where \(s = 2k\) when \(n\) is even and \(s = k + 1\) when \(n\) is odd.

Examine the label difference for each pair.

\[
|c(x_i) - c(x_{n-k+i})| = n - k,
\]

\[
|c(x_{n-k+i}) - c(x_{i+1})| = n - k - 1
\]

and \(|c(x_s) - c(x_1)|\) is \(k\) when \(s = k + 1\) (\(n\) odd) and is \(2k - 1\) when \(s = 2k\) (\(n\) even). In all cases using the fact that \(n \geq 6\), \(|c(w_j) - c(w_k)| \geq 2\). So the radio condition is satisfied whenever \(d(w_j, w_k) = 3\). Meanwhile, \(j\) and \(k\) not to be consecutive (mod \(n\)), we have \(d(w_j, w_k) \geq 4\). As it is always the case that \(|c(w_j) - c(w_k)| \geq 1\), the radio condition is again satisfied.

**Case 3** \((v_j, v_k)\) (with \(j \neq k\)).

Note that \(d(v_j, v_k) = 2\). Since \(|c(v_j) - c(v_k)| = |c(x_{2n+j}) - c(x_{2n+k})| \geq 3\) for all \(v_j, v_k\), the radio condition is again satisfied.

**Case 4** \((w_j', w_k')\) (with \(j \neq k\)).

Recall \(p(w'_{2i-1}) = x_{n+i}\) and note that \(p(w'_{2i})\) may be written as \(2n-k+i\) where \(n\) is even or odd. We have \(d(w'_j, w'_k) = 3\) for the pairs \((w'_{2i-1}, w'_{2i}), (w'_{2i}, w'_{2i+1})\) and \((w'_n, w'_1)\). These “translate” respectively to \((x_{n+i}, x_{2n-k+i}), (x_{2n-k+i}, x_{n+i+1})\)
and \((x_s, x_{n+1})\) where \(s = n+k+1\) when \(n\) is odd and \(s = n+2k\) when \(n\) is even.

Examine the label difference for each pair: 
\[
|c(x_{n+i}) - c(x_{2n-k+i})| = n - k,  
|c(x_{2n-k+i}), c(x_{n+i+1})| = n - k - 1  
\]
and \(c(x_s) - c(x_{n+1})\) is \(k\) when \(s = n+k+1\) \((n\) odd) and is \(2k - 1\) when \(s = n+2k\) \((n\) even). In all cases \(n \geq 6\), we have that \(|c(w_j) - c(w_k)| \geq 2\). So the radio condition is satisfied whenever 
\[
d(w_j, w_k) = 3. \]
Meanwhile \(j\) and \(k\) not to be consecutive \((\text{mod } n)\), we have 
\[
d(w_j, w_k) \geq 4. \]
As it is always the case that \(|c(w_j) - c(w_k)| \geq 1\), the radio condition is again satisfied.

**Case 5** \((v, w)\), where \(v \in \{v_1, v_2, \ldots, v_n\}\) and \(w \in \{w_1, w_2, \ldots, w_n\}\).

We have \(c(v) \in \{2n+5, 2n+8, \ldots, 5n+2\}\) and \(c(w) \in \{4, 5, 6, \ldots, n+3\}\). For all \(v \neq v_1\), \(|c(v) - c(w)| \geq |(2n+8) - (n+3)| = n+5\), therefore the radio condition is satisfied when \(v \neq v_1\). Meanwhile \(|c(v_1) - c(w)| \geq (2n+5) - (n+3) = n+2\). If \(w\) is at any distance from \(v_1\), the radio condition holds.

**Case 6** \((v, w')\), where \(v \in \{v_1, v_2, \ldots, v_n\}\) and \(w' \in \{w'_1, w'_2, \ldots, w'_n\}\).

We have \(c(v) \in \{2n+5, 2n+8, \ldots, 5n+2\}\) and \(c(w') \in \{n+4, n+5, n+6, \ldots, 2n+3\}\). For all \(v \neq v_1\), \(|c(v) - c(w')| \geq (2n+8) - (2n+3) = 5\). Therefore the radio condition is satisfied when \(v \neq v_1\). Meanwhile \(|c(v_1) - c(w')| \geq (2n+5) - (2n+3) = 2\). If \(w'\) is distance three or greater from \(v_1\), the radio condition holds. If \(d(v_1, w') < 3\), then \(w' = x_{n+1}\) or \(w' = x_{n+\lfloor\frac{n}{2}\rfloor+1}\). Checking the radio condition for each we obtain,

\[
d(v_1, x_{n+1}) + |c(v_1) - c(x_{n+1})| = 1 + |(2n+5) - (n+4)| = n + 2 \geq 5
\]
if \(n\) is even and

\[
d(v_1, x_{n+1}) - |c(v_1) - c(x_{n+1})| = 2 + |(2n+5) - (n+4)| = n + 3 \geq 5
\]
if $n$ is odd.

$$|c(v_1) - c(x_{n+[\frac{n}{2}]+1})| = 2 + \left| (2n + 5) - \left( 3 + n + \frac{n}{2} + 1 \right) \right| = \frac{n}{2} + 3 \geq 5$$

if $n$ is even and

$$|c(v_1) - c(x_{n+[\frac{n}{2}]+1})| = 1 + \left| (2n + 5) - \left( 3 + n + \frac{n}{2} \right) \right|$$

$$= \left\lfloor \frac{n}{2} \right\rfloor + 3 \geq 5$$

if $n$ is odd.

**Case 7** Consider the pair $(w_j, w'_k)$.

Recall $p(w_{2i-1}) = x_i$ and $p(w_{2i})$ may be written as $x_{n-k+i}$. Also $p(w'_{2i-1}) = x_{n+i}$ and $p(w'_{2i})$ may be written as $x_{2n-k+i}$ whether $n$ is odd or even. We have $d(w_j, w'_j) = 1$ for the pair $(w_{2i}, w'_{2i})$ and $(w_{2i-1}, w'_{2i-1})$. These translate respectively to $(x_{n-k+i}, x_{2n-k+i})$ and $(x_i, x_{n+i})$. Examine the label differences for each pair.

$$|c(x_{n-k+i}) - c(x_{2n-k+i})| = n \text{ and } |c(x_i) - c(x_{n+i})| = n.$$  

In all cases using the fact that $n \geq 6$, the radio condition is satisfied whenever $d(w_j, w'_j) = 1$.

We have $d(w_j, w'_k) = 2$ for the pair $(w'_{2i-1}, w_{2i}), (w'_{2i}, w_{2i+1})$ and $(w'_{n}, w_{1})$. These translate respectively to $(x_{n+i}, x_{n-k+i}), (x_{2n-k+i}, x_{i+1})$ and $(x_s, x_1)$ where $s = 2n$ when $n$ is even and $s = n + k + 1$ when $n$ is odd. Examine the label difference for each pair: $|c(x_{n+i}) - c(x_{n-k+i})| = k$, $|c(x_{2n-k+i}) - c(x_{i+1})| = 2n - k - 1$ and $|c(x_s) - c(x_1)|$ is $2n - 1$ when $s = 2n$ (n is even) and is $n + k$ when $s = n + k + 1$ (n is odd).

In all cases using the fact that $n \geq 6, |c(w_j) - c(w_k)| \geq 3$. So the radio condition is satisfied, whenever $d(w_j, w'_k) = 2$. 
Theorem 4.4. The radio number of the graph \((W_n : 2)\) is \(5n+2\) when \(n \geq 4\).

Proof. Theorem 4.2 shows that \(rn(W_n : 2) \geq 5n + 2\) for \(n \geq 3\), and Theorem 4.3 shows that \(rn(W_n : 2) \leq 5n + 2\) for \(n \geq 6\). It remains only to show that \(rn(W_n : 2) \leq 5n + 2\) for \(n = 3, 4, 5\). This is demonstrated by the radio labelings provided in Figure 4.3. Note that the labels with values \(a\) and \(a+2\) are assigned to vertices at distance 4.

4.2 The Radio Number of \((W_n : 3)\) Graphs

Definition 4.5. Let \(W_n = C_n + K_1\) denote the wheel on \(n + 1\) vertices, with \(V(K_1) = \{u\}\) and \(V(C_n) = \{v_1, v_2, \ldots, v_n\}\). Then the graph obtained from \(W_n\) by subdividing each edge of the cycle \(C_n\) exactly thrice is denoted by \((W_n : 3)\). It has \(5n\) edges and \(\text{diam}(W_n : 3) = 6\) for \(n \geq 4\). The standard drawing of the graph \((W_n : 3)\) is as follows:
The subdivision vertices are denoted by \( u_1, w_1, u'_1, u_2, w_2, u'_2, \ldots, u_n, w_n, u'_n \) where \( u_i \) and \( w_i \), \( w_i \) and \( u'_i \) are adjacent. If \( n \) is odd, then we specify that \( u_1 \) is adjacent to \( v_1 \), \( w_1 \) is adjacent to \( u_1 \) and \( u'_1 \) is adjacent to \( v_2 \). If \( n \) is even, \( u_1 \) is adjacent to \( v_n \), \( w_1 \) is adjacent to \( u_1 \) and \( u'_1 \) is adjacent to \( v_1 \). The standard drawings of \((W_7 : 3)\) and \((W_8 : 3)\) are shown in Figure 4.4.

\[
\begin{align*}
\text{Theorem 4.6.} \quad & \text{For } n \geq 4, \\
& r_n(W_n : 3) \geq \begin{cases} 
12n + 2 & \text{if } n \text{ is odd}, \\
12n + 4 & \text{if } n \text{ is even}.
\end{cases}
\end{align*}
\]

\textit{Proof.} Assume \( n \geq 4 \). This gives \( \text{diam} \ (W_n : 3) = 6 \). So any radio labeling \( c \) of \((W_n : 3)\) must satisfy the radio condition

\[
d(u, v) + |c(u) - c(v)| \geq 7
\]

for all distinct \( u, v \in V(G) \). We count the number of values needed for labels and add the minimum number of forbidden values – those values precluded
by use of a particular value as a label. For instance, if we label the center
a (i.e., $c(u) = a$), then, as $d(u, r) \leq 3$ for all vertices $r \neq u$, the values
$a-3, a-2, a-1, a+1, a+2$ and $a+3$ are forbidden. Similarly, as $d(v_i, r) \leq 4$
for all $v_i$ and for any $r \neq v_i$, $c(v_i) - 2, c(v_i) - 1, c(v_i) + 1, c(v_i) + 2$
are forbidden. However, as $d(w_i, r) = 6$ for some vertex $r$, it is possible to use consecutive
labels on $w_i$ and $r$. There are no forbidden values associated with the vertices
$\{w_1, w_2, \ldots, w_n\}$.

The number of forbidden values are symmetric by any label used for a
particular vertex. Thus we find the minimum number of forbidden values
by using the lowest and highest values on the center vertex and on one of
$\{v_1, v_2, \ldots, v_n\}$. Assume without loss of generality that $c(u) = 1$ and $c(v_n)$ is
the span of $c$.

**Case 1.** $n$ is odd.

Associated with the center vertex are the three forbidden values 2, 3 and
4, with $v_n$ are three forbidden values, and with other $v_i$ are four forbidden
values each. With $u_n$ is one forbidden value and with other $u_i$ are two
forbidden values each. With $u'_n$ are two forbidden values and with other
$u'_i$ are two forbidden values each. This gives a total of $3 + 3 + 4(n - 1) +
1 + 2(n - 1) + 2 + 2(n - 1) = 8n + 1$ forbidden values. Adding the $4n + 1$
values needed to label the $4n + 1$ vertices provides a total of $12n + 2$, hence
$r_n(W_n : 3) \geq 12n + 2$.

**Case 2.** $n$ is even.

Associated with the center vertex are the three forbidden values 2, 3, and
4, with $v_n$ are three forbidden values, and with the other $v_i$ are four forbidden
values each. With $u_n$ are three forbidden values and with other $u'_i$s are two
forbidden values each. With $u'_n$s are two forbidden values and with other
$u'_s$ are two forbidden values each. This gives a total of $3 + 3 + 4(n - 1) + 3 + 2(n - 1) + 2 + 2(n - 1) = 8n + 3$ forbidden values. Adding the $4n + 1$ values needed to label the $4n + 1$ vertices provides a total of $12n + 4$, hence $r_n(W_n : 3) \geq 12n + 4$.

\[ \square \]

**Theorem 4.7.** For $n \geq 7$, $r_n(W_n : 3) \leq 12n + 2$, if $n$ is odd.

**Proof.** We provide a radio labeling $c$ of $(W_n : 3)$ in two steps. First we define a position function that renames the vertices of $(W_n : 3)$ using the set $\{x_0, x_1, \ldots, x_{4n}\}$, then we specify the labels $c(x_i)$ so that $i < j$ if and only if $c(x_i) < c(x_j)$.

The position function $p : V(W_n : 3) \rightarrow \{x_0, x_1, \ldots, x_{4n}\}$ is defined as follows. For $n = 2k + 1$, we define

\[
\begin{align*}
p(u) &= x_0, \\
p(w_{2i-1}) &= x_i & \text{for } i = 1, 2, \ldots, k + 1, \\
p(w_{2i}) &= x_{k+1+i} & \text{for } i = 1, 2, \ldots, k, \\
p(u_{2i-1}) &= x_{n+i} & \text{for } i = 1, 2, \ldots, k + 1, \\
p(u_{2i}) &= x_{n+k+1+i} & \text{for } i = 1, 2, \ldots, k, \\
p(u'_{2i-1}) &= x_{2n+i} & \text{for } i = 1, 2, \ldots, k + 1, \\
p(u'_{2i}) &= x_{2n+k+1+i} & \text{for } i = 1, 2, \ldots, k, \\
p(v_i) &= x_{3n+i} & \text{for } i = 1, 2, \ldots, n.
\end{align*}
\]

The position function orders the vertices so that $\{x_0, x_1, \ldots, x_{4n}\}$ corresponds to $\{u, w_1, w_3, \ldots, w_n, w_2, w_4, \ldots, w_{n-1}, u_1, u_3, \ldots, u_n, u_2, u_4, \ldots, u_{n-1}, u'_1, u'_3, \ldots, u'_n u'_2, u'_4, \ldots, u'_{n-1}, v_1, v_2, \ldots, v_n\}$. Figure 4.5 shows the renamed versions of $(W_7 : 3)$. 
Figure 4.5

Now we define the radio labeling \( c : V(G) \to N \) as

\[
c(x_i) = \begin{cases} 
1, & i = 0, \\
i + 4, & 1 \leq i \leq 2n, \\
(n + 3) + 3(i - n), & n + 1 \leq i \leq 2n, \\
(4n + 3) + 3(i - 2n), & 2n + 1 \leq i \leq 3n, \\
(7n + 2) + 5(i - 3n), & 3n + 1 \leq i \leq 4n.
\end{cases}
\]

**Claim:** The labeling \( c \) is a valid radio labeling.

We must show that the radio condition,

\[
d(u, v) + |c(u) - c(v)| \geq \text{diam} (W_n : 3) + 1 = 7,
\]

holds for all pairs of vertices \((u, v)\) where \( u \neq v \).

**Case 1.** \((u, r)\) (for any vertex \( r \neq u \)).

Recall \( p(z) = x_0 \). As \( c(x_i) \geq 7 \) for any \( i \geq 3 \), we have

\[
d(u, x_i) + |c(u) - c(x_i)| \geq 1 + |1 - 7| = 7.
\]
This leaves the pair \((u, x_1)\) and \((u, x_2)\). But \(p^{-1}(x_1) = w_1\) and \(p^{-1}(x_2) = w_3\), so we calculate

\[
d(u, w_1) + |c(u) - c(w_1)| = 3 + |1 - 5| = 7,
\]

and \(d(u, w_3) + |c(u) - c(w_3)| = 3 + |1 - 6| = 8.\)

**Case 2.** \((w_j, w_k)\) with \(j \neq k\).

Recall \(p(w_{2i-1}) = x_i\) and note that \(p(w_{2i})\) may be written as \(x_{n-k+i}\). We have \(d(w_j, w_k) = 4\) for the pair \((w_{2i-1}, w_{2i}), (w_{2i}, w_{2i+1}), \) and \((w_n, w_1)\). These are translated respectively to \((x_i, x_{n-k+i}), (x_{n-k+i}, x_{i+1})\) and \((x_s, x_1)\) where \(s = k + 1\). Examine the label difference for each pair.

\[
|c(x_i) - c(x_{n-k+i})| = n - k, \quad |c(x_{n-k+i}) - c(x_{i+1})| = n - k - 1
\]

and \(|c(x_s) - c(x_1)| = k\) when \(s = k + 1\). Using the fact that \(n \geq 7\) we have that \(|c(w_j) - c(w_k)| \geq 3\), so the radio condition is satisfied whenever \(d(w_j, w_k) = 4.\)

**Case 3.** \((u_j, u_k)\) with \(j \neq k\).

Recall \(p(u_{2i-1}) = x_{n+i}\) and note that \(p(u_{2i})\) may be written as \((x_{2n-k+i})\). We have \(d(u_j, u_k) = 4\) for the pairs \((u_{2i-1}, u_{2i}), (u_{2i}, u_{2i+1}), (u_n, u_i)\). These translate respectively to \((x_{n+i}, x_{2n-k+i}), (x_{2n-k+i}, x_{n+i+1}), (x_s, x_{n+1})\) where \(s = n + k + 1\). Examine the label difference for each pair: \(|c(x_{2n-k+i}) - c(x_{n+i+1})| = n + k - 1, \quad |c(x_{n+i}) - c(x_{2n-k+i})| = 2n - k + 1, \) and \(|c(x_s) - c(x_{n+1})| = n + k - 1\), when \(s = n + k - 1\). In all cases \(n \geq 7\), we have that \(|c(u_j) - c(u_k)| \geq 3\). So the radio condition is satisfied whenever \(d(u_j, u_k) = 4.\)

**Case 4.** \((u'_j, u'_k)\) with \(j \neq k\).

Recall \(p(u'_{2i-1}) = x_{2n+i}\) and note that \(p(u'_{2i})\) may be written as \(3n - k + i\). We have \(d(u'_j, u'_k) = 4\) for the pairs \((u'_{2i-1}, u'_{2i}), (u'_{2i}, u'_{2i+1}), (u'_n, u'_1)\). These
translate respectively to \((x_{2n+i}, x_{3n-k+i}), (x_{3n-k+i}, x_{2n+i+1})\) and \((x_s, x_{2n+1})\) where \(s = 2n + k + 1\). Examine the label difference for each pair:

\[
|c(x_{2n+i}) - c(x_{3n-k+i})| = 2n - k + 1, \quad |c(x_{3n-k+i}) - c(x_{2n+i+1})| = n + k - 1,
\]

and \(|c(x_s) - c(x_{2n+1})| = n + k - 1\) where \(s = 2n + k + 1\).

In all cases \(n \geq 7\), we have that \(|c(u'_j) - c(u'_k)| \geq 3\). So the radio condition is satisfied.

**Case 5.** \((v_j, v_k)\) with \(j \neq k\).

Note that \(d(v_j, v_k) = 2\). Since \(|c(v_j) - c(v_k)| = |c(x_{3n+j}) - c(x_{3n+k})| \geq 5\) for all \(v_j, v_k\), the radio condition is again satisfied.

**Case 6.** \((v, w)\), where \(v \in \{v_1, v_2, \ldots, v_n\}\) and \(w \in \{w_1, w_2, \ldots, w_n\}\).

We have \(c(v) \in \{7n+7, 7n+12, \ldots, 12n+2\}\) and \(c(w) \in \{5, 6, \ldots, n+4\}\). For all \(v\), \(|c(v) - c(w)| \geq |(7n+7) - (n+4)| = 6n + 3\).

Therefore the radio condition is satisfied.

**Case 7.** \((v, u)\) where \(v \in \{v_1, v_2, \ldots, v_n\}\) and \(u \in \{u_1, u_2, \ldots, u_n\}\).

We have \(c(v) \in \{7n+7, 7n+12, \ldots, 12n+2\}\) and \(c(u) \in \{n+6, n+9, \ldots, 4n+3\}\). For all \(v\), \(|c(v) - c(u)| \geq |(7n+7) - (4n+3)| = 3n + 4\).

Therefore the radio condition is satisfied.

**Case 8.** \((v, u')\), where \(v \in \{v_1, v_2, \ldots, v_n\}\) and \(u' \in \{u'_1, u'_2, \ldots, u'_n\}\).

We have \(c(v) \in \{7n+7, 7n+12, \ldots, 12n+2\}\) and \(c(u') \in \{4n+6, 4n+9, \ldots, 7n+3\}\). For all \(v \neq v_1\),

\[
|c(v) - c(u')| \geq |(7n+7) - (7n+3)| = 4.
\]

If \(d(u', v_1) \geq 3\), the radio condition holds. If \(d(v_1, u') < 3\), then \(u' = x_{2n+k+1}\). Checking for the radio condition we obtain,

\[
d(v_1, x_{2n+k+1}) + |c(v_1) - c(x_{2n+k+1})| = 1 + |(7n+7) - (4n + 3k + 6)|
\]
\[= 3n - 3k + 2\]
\[\geq 7.\]

**Case 9.** \((u, w)\), where \(u \in \{u_1, u_2, \ldots , u_n\}\) and \(w \in \{w_1, w_2, \ldots , w_n\}\).

We have \(c(u) \in \{n + 6, n + 9, \ldots , 4n + 3\}\) and \(c(w) \in \{5, 6, \ldots , n + 4\}\).

For all \(u \neq u_1, u_2\), \(|c(u) - c(w)| \geq |(n + 12) - (n + 4)| = 8.\)

Therefore the radio condition is satisfied for \(u \neq u_1, u_2\). Meanwhile
\(|c(u_1) - c(w)| \geq |(n + 6) - (n + 4)| = 2\) and \(|c(u_2) - c(w)| \geq |(n + 9) - (n + 4)| = 5.\)

If \(d(w, u_1) \geq 5\), the radio condition holds.

If \(d(u_1, w) < 5\), then \(w = w_1 = x_1\), or \(w = x_{k+1}\).

Checking for the radio condition, we obtain,
\[d(u_1, x_1) + |c(u_1) - c(x_1)| = 1 + |(n + 6) - 5| = n + 2 \geq 7,\]

and
\[d(u_1, x_{k+1}) + |c(u_1) - c(x_{k+1})| = 3 + |(n + 6) - (k + 5)| = 3 + (n - k + 1) \geq 7.\]

If \(d(w, u_2) \geq 2\), the radio condition holds.

If \(d(u_2, w) < 2\), then \(w = w_2 = x_{k+2}\). Checking for the radio condition, we obtain
\[d(u_2, x_{k+2}) + |c(u_2) - c(x_{k+2})| = 1 + |(n + 9) - (k + 6)| = n - k + 4 \geq 7.\]

**Case 10.** \((u', w)\), where \(u' \in \{u'_1, u'_2, \ldots , u'_n\}\) and \(w \in \{w_1, w_2, \ldots , w_n\}\).

We have \(c(u') \in \{4n + 6, 4n + 9, \ldots , 7n + 3\}\) and \(c(w) \in \{5, 6, \ldots , n + 4\}\).

For all \(u'\),
\[|c(u') - c(w)| \geq |(4n + 6) - (n + 4)| = 3n + 2 \geq 7.\]

Therefore the radio condition is satisfied.
Case 11. \((u', u)\), where \(u' \in \{u'_1, u'_2, \ldots, u'_n\}\) and \(u \in \{u_1, u_2, \ldots, u_n\}\).

We have \(c(u') \in \{4n + 6, 4n + 9, \ldots, 7n + 3\}\) and \(c(u) \in \{n + 6, n + 9, \ldots, 4n + 3\}\). For all \(u' \neq u'_1\),

\[
|c(u') - c(u)| \geq |(4n + 9) - (4n + 3)| = 6.
\]

Since \(d(u, u') \geq 1\), the radio condition is satisfied.

Meanwhile \(|c(u'_1) - c(u)| \geq |(4n + 6) - (4n + 3)| = 3\). If \(d(u, u'_1) \geq 4\) the radio condition holds.

If \(d(u', u) < 4\), then \(u = x_{n+1}\) or \(u = x_{n+k+2}\). Checking for the radio condition, we obtain

\[
d(u'_1, x_{n+1}) + |c(u'_1) - c(x_{n+1})| = 2 + |(4n + 6) - (n + 6)|
\]

\[
= 3n + 2 \geq 7,
\]

and

\[
d(u'_1, x_{n+k+2}) + |c(u'_1) - c(x_{n+k+2})| = 2 + |(4n + 6) - (n + 3k + 9)|
\]

\[
= 2 + (3n - 3k - 3) = 3n - 3k - 1 \geq 7.
\]

\[\square\]

Theorem 4.8. For \(n \geq 7\), \(rn(W_n : 3) \leq 12n + 4\), if \(n\) is even.

Proof. We provide a radio labeling \(c\) of \((W_n : 3)\) in two steps. First we define a position function that rename the vertices of \((W_n : 3)\) using the set \(\{x_0, x_1, \ldots, x_{4n}\}\), then we specify the labels \(c(x_i)\) so that \(i < j\) if and only if \(c(x_i) < c(x_j)\).

The position function \(p : V(W_n : 3) \to \{x_0, x_1, \ldots, x_{4n}\}\) is defined as follows.

For \(n = 2k\), we define

\[p(u) = x_0,\]
The position function orders the vertices so that \( \{x_0, x_1, x_2, \ldots, x_{4n}\} \) corresponds to \( \{u, w_1, w_3, \ldots, w_{n-1}, w_2, w_4, \ldots, w_n, u_1, u_3, \ldots, u_{n-1}, u_2, u_4, \ldots, u_n, u'_1, u'_3, \ldots, u'_{n-1}u'_2, u'_4, \ldots, u'_n, v_1, v_2, \ldots, v_n\} \). Figure 4.6 shows the renamed version of \((W_8 : 3)\).
Now we are define the radio labeling $c : V(G) \to N$ as

$$c(x_i) = \begin{cases} 
1, & i = 0, \\
 i + 4, & 1 \leq i \leq n \\
(n + 5) + 3(i - n), & n + 1 \leq i \leq 2n, \\
(4n + 5) + 3(i - 2n), & 2n + 1 \leq i \leq 3n, \\
(7n + 4) + 5(i - 3n), & 3n + 1 \leq i \leq 4n.
\end{cases}$$

\[\square\]

**Claim:** The labeling $c$ is a valid radio labeling.

We must show that the radio condition

$$d(u, v) + |c(u) - c(v)| \geq \text{diam} (W_n : 3) + 1 = 7$$

holds for all pairs of vertices $(u, v)$ where $u \neq v$.

**Case 1.** $(u, r)$ (for any vertex $r \neq u$).

Recall $p(u) = x_0$. As $c(x_i) \geq 7$ for any $i \geq 3$, we have $d(u, x_i) + |c(u) - c(x_i)| \geq 1 + |1 - 7| = 7$. This leaves the pairs $(u, x_1)$ and $(u, x_2)$. But $p^{-1}(x_1) = w_1$ and $p^{-1}(x_2) = w_3$. So we calculate

$$d(u, w_1) + |c(u) - c(w_1)| = 3 + |1 - 5| = 7,$$

and $d(u, w_3) + |c(u) - c(w_3)| = 3 + |1 - 6| = 8$.

**Case 2.** $(w_j, w_k)$ with $j \neq k$.

Recall $p(w_{2i-1}) = x_i$ and note that $p(w_{2i})$ may be written as $x_{n-k+i}$. We have $d(w_j, w_k) = 4$ for the pairs $(w_{2i-1}, w_{2i}), (w_{2i}, w_{2i+1})$, and $(w_n, w_1)$. These translate respectively to $(x_i, x_{n-k+i}), (x_{n-k+i}, x_{i+1})$, and $(x_s, x_1)$, where $s = 2k$. Examine the label difference for each pair: $|c(x_i) - c(x_{n-k+i})| = n - k, |c(x_{n-k+i}) - c(x_{i+1})| = n - k - 1$, and $|c(x_s) - c(x_1)| = 2k - 1$, when
s = 2k. Since n ≥ 7, we have that |c(w_j) − c(w_k)| ≥ 3, the radio condition is satisfied whenever d(w_j, w_k) = 4.

**Case 3.** (u_j, u_k) with j ≠ k.

Recall p(u_{2i-1}) = x_{n+i} and note that p(u_{2i}) may be written as x_{2n-k+i}. We have d(u_j, u_k) = 4 for the pairs (u_{2i-1}, u_{2i}), (u_{2i}, u_{2i+1}), and (u_n, u_1). These translate respectively to (x_{n+i}, x_{2n-k+i}), (x_{2n-k+i}, x_{n+i+1}), and (x_s, x_{n+i+1}) where s = n + 2k.

Examine the label difference for each pair: |c(x_{n+i}) − c(x_{2n-k+i})| = 2n − k, |c(x_{2n-k+i}) − c(x_{n+i+1})| = n + k − 3, and |c(x_s) − c(x_{n+1})| = 3n − 3, when s = n + 2k.

Since n ≥ 7, we have that |c(u_j) − c(u_k)| ≥ 3. Thus the radio condition is satisfied, whenever d(u_j, u_k) = 4.

**Case 4.** (u'_j, u'_k) with j ≠ k.

Recall p(u'_{2i-1}) = x_{2n+i} and note that p(u'_{2i}) may be written as 3n − k + i. We have d(u'_j, u'_k) = 4 for the pairs (u'_{2i-1}, u'_{2i}), (u'_{2i}, u'_{2i+1}) and (u'_n, u'_1). These translate respectively to (x_{2n+i}, x_{3n-k+i}), (x_{3n-k+i}, x_{2n+i+1}), and (x_s, x_{2n+1}) where s = 2n + k + 1. Examine the label difference for each pair:

|c(x_{2n+i}) − c(x_{3n-k+i})| = 2n − k, |c(x_{3n-k+i}) − c(x_{2n+i+1})| = n + k − 3,

and |c(x_s) − c(x_{2n+1})| = 3n − 3, when s = 2n + k + 1.

Since n ≥ 7, we have that |c(u'_j) − c(u'_k)| ≥ 3. Thus the radio condition is satisfied.

**Case 5.** (v_j, v_k) (with j ≠ k).

Note that d(v_j, v_k) = 2. Since |c(v_j) − c(v_k)| = |c(x_{3n+j}) − c(x_{3n+k})| ≥ 5 for all v_j and v_k, the radio condition is again satisfied.

**Case 6.** (v, w) where v ∈ {v_1, v_2, ..., v_n} and w ∈ {w_1, w_2, ..., w_n}. 
We have \( c(v) \in \{7n+9, 7n+14, \ldots, 12n+4\} \) and \( c(w) \in \{5, 6, \ldots, n+4\} \). For all \( v, |c(v) - c(w)| \geq |(7n + 9) - (n + 4)| = 6n + 5 \).

Therefore, the radio condition is satisfied.

**Case 7.** \((v, u)\) where \( v \in \{v_1, v_2, \ldots, v_n\} \) and \( u \in \{u_1, u_2, \ldots, u_n\} \).

We have \( c(v) \in \{7n + 9, 7n + 14, \ldots, 12n + 4\} \) and \( c(u) \in \{n + 8, n + 11, \ldots, 4n + 5\} \). For all \( v, |c(v) - c(u)| \geq |(7n + 9) - (4n + 5)| = 3n + 4 \).

Therefore the radio condition is satisfied.

**Case 8.** \((v, u')\) where \( v \in \{v_1, v_2, \ldots, v_n\} \) and \( u' \in \{u'_1, u'_2, \ldots, u'_n\} \).

We have \( c(v) \in \{7n + 9, 7n + 14, \ldots, 12n + 4\} \) and \( c(u') \in \{4n + 8, 4n + 11, \ldots, 7n + 5\} \): For all \( v \neq v_1 \),

\[
|c(v) - c(u')| \geq |(7n + 14) - (7n + 5)| = 9.
\]

Therefore the radio condition is satisfied when \( v \neq v_1 \).

Meanwhile \( |c(v_1) - c(u')| \geq |(7n + 9) - (7n + 5)| = 4 \). If \( d(u', v_1) \geq 3 \), the radio condition holds. If \( d(v_1, u') < 3 \), then \( u' = x_{2n+k+1} \).

Checking for the radio condition we obtain, \( d(v_1, x_{2n+k+1}) + |c(v_1) - c(x_{2n+k+1})| = 1 + |(7n + 9) - (4n + 3k + 8)| = 3n - 3k + 2 \geq 7 \).

**Case 9.** \((u, w)\) where \( u \in \{u_1, u_2, \ldots, u_n\} \) and \( w \in \{w_1, w_2, \ldots, w_n\} \).

We have \( c(u) \in \{n + 8, n + 11, \ldots, 4n + 5\} \) and \( c(w) \in \{5, 6, \ldots, n + 4\} \). For all \( u \neq u_1 \), \( |c(u) - c(w)| \geq |(n + 11) - (n + 4)| = 7 \). Therefore the radio condition is satisfied for \( u \neq u_1 \). Meanwhile \( |c(u_1) - c(w)| \geq |(n + 8) - (n + 4)| = 4 \). If \( d(u_1, w) \geq 3 \), the radio condition holds.

If \( d(u_1, w) < 3 \), then \( w = w_1 = x_1 \). Checking for the radio condition we obtain,

\[
d(u_1, x_1) + |c(u_1) - c(x_1)| \geq 1 + |(n + 8) - 5| = n + 4 \geq 7.
\]
Case 10. \((u', w)\) where \(u' \in \{u'_1, u'_2, \ldots, u'_n\}\) and \(w \in \{w_1, w_2, \ldots, w_n\}\).

We have \(c(u') \in \{4n + 8, 4n + 11, \ldots, 7n + 5\}\) and \(c(w) \in \{5, 6, \ldots, n + 4\}\). For all \(u', |c(u') - c(w)| \geq |(4n + 8) - (n - 4)| = 3n + 4 \geq 7\). Therefore the radio condition is satisfied.

Case 11. \((u', u)\) where \(u' \in \{u'_1, u'_2, \ldots, u'_n\}\) and \(u \in \{u_1, u_2, \ldots, u_n\}\).

We have \(c(u') \in \{4n + 8, 4n + 11, \ldots, 7n + 5\}\) and \(c(u) \in \{n + 8, n + 11, \ldots, 4n + 5\}\). For all \(u' \neq u'_1, |c(u') - c(u)| \geq |(4n + 11) - (4n + 5)| = 6\).

Since \(d(u, u') \neq 1\), the radio condition is satisfied for \(u' \neq u'_1\). Meanwhile 

\(|c(u'_1) - c(u)| \geq |(4n + 8) - (4n + 5)| = 3\). If \(d(u'_1, u) \geq 4\), the radio condition holds.

If \(d(u'_1, u) < 4\), then \(u = u_1 = x_{n+1}\) or \(u = u_2 = x_{n+k+1}\). Checking for the radio condition we obtain,

\[d(u'_1, x_{n+1}) + |c(u'_1) - c(x_{n+1})| \geq 2 + |(4n + 8) - (n + 6)| = 3n + 4 \geq 7,\]

and \(d(u'_1, x_{n+k+1}) + |c(u'_1) - c(x_{n+k+1})| \geq 2 + |(4n + 8) - (n + 3k + 8)| = 3n - 3k + 2 \geq 7,\)

Theorem 4.9. The radio number of the graph \((W_n : 3)\) is \(12n + 2\) if \(n\) is odd and \(n \geq 5\).

Proof. Theorem 4.6 shows that \(rn(W_n : 3) \geq 12n + 2\) if \(n\) is odd and \(n \geq 5\), and Theorem 4.7 shows that \(rn(W_n : 3) \leq 12n + 2\) if \(n\) is odd and \(n \geq 7\). It remains only to show that \(rn(W_n : 3) \leq 12n + 2\) for \(n = 5\).

This is demonstrated by the radio labeling provided in Figure 4.7.

Note that the labels with values \(a\) and \(a + 2\) are assigned to vertices at distance 6. \qed
Theorem 4.10. The radio number of the graph \((W_n : 3)\) is \(12n + 4\) if \(n\) is even and \(n \geq 4\).

Proof. Theorem 4.6 shows that \(\text{rn}(W_n : 3) \geq 12n + 4\) if \(n\) is even and \(n \geq 4\), and Theorem 4.8 shows that \(\text{rn}(W_n : 3) \leq 12n + 4\) if \(n\) is even and \(n \geq 7\). It remains only to show that \(\text{rn}(W_n : 3) \leq 12n + 4\) for \(n = 4\) and \(6\). This is demonstrated by the radio labeling provided in Fig. 4.8. Note that the labels with values \(a\) and \(a + 2\) are assigned to vertices at distance 6. \(\square\)