Chapter 3

Radio Labeling of Graphs of Diameter Four

In this chapter we exhibit radio labeling of some graphs of diameter four.

3.1 Radio labeling of subdivision of spokes in wheels and $S(K_{n,n})$

Theorem 3.1. Let $SS(W_n)$ be the graph obtained from wheel $W_n$ by subdividing the spokes. Then $\text{rn}(SS(W_n)) = 4n + 2$ for $n \geq 8$.

Proof. Let $v_0$ the centre vertex. The cyclic vertices are $v_1, v_2, \ldots, v_n$ and the vertices of subdivision of spokes are $u_1, u_2, \ldots, u_n$. Clearly $SS(W_n)$ has $2n + 1$ vertices and $\text{diam}(SS(W_n)) = 4$. Define radio labeling $c : V \rightarrow N$ satisfying the condition

$$d(u, v) + |c(u) - c(v)| \geq 1 + \text{diam}(SS(W_n)) = 5,$$

for every pair of distinct vertices $u$ and $v$. 
Case 1: Label \( v_0 \) with 1.

Then \( c(v_0) = 1 \). Label the vertices of cycle and vertices of subdivision which are at distance 3, alternatively. Then \( c(v_i) = 4i \) and \( c(u_{3+i}) = 4i + 2 \), 
\( i = 1, 2, \ldots, n \) where \( u_{n+1} = u_1, u_{n+2} = u_2 \) and \( u_{n+3} = u_3 \).

Hence \( \text{rn}(c) = 4n + 2 \).

Case 2: Label the cyclic vertices with 1.

Let \( c(v_1) = 1 \). Then label the vertices of subdivision and cyclic vertices alternatively, which are at distance 3.

Then \( c(v_i) = 4i - 3 \), \( i = 2, 3, \ldots, n \), and \( c(u_{3+i}) = 4i - 1 \), \( i = 1, 2, \ldots, n \) where \( u_{n+1} = u_1, u_{n+2} = u_2 \) and \( u_{n+3} = u_3 \). Since \( d(u_3, v_0) = 1 \), \( c(v_0) = 4n + 3 \).

Hence \( \text{rn}(c) = 4n + 3 \).

Case 2: Label the vertices of subdivision with one.

Let \( c(u_3) = 1 \). Since \( d(u_3, v_0) = 1 \) and \( d(v_0, u_1) = 2 \), \( c(v_0) = 5 \) and \( c(v_1) = 8 \). Now we label the vertices of subdivision and cycle alternatively with distance three. Then \( c(u_{3+i}) = 4i + 6 \), \( i = 1, 2, \ldots, n \) and \( c(v_i) = 4i + 4 \), \( i = 1, 2, \ldots, n \). Hence \( \text{rn}(c) = 4n + 4 \).

\[ \therefore \text{rn}(SS(W_n)) = \min \{ \text{rn}(c) \} = 4n + 2. \]

Example 3.2. Fig. 3.1 shows that \( \text{rn}(SS(W_8)) = 34. \)

![Figure 3.1](image-url)
Remark 3.3. The radio number for graphs with \( n = 3, 4, 5, 6 \) and 7 are given in Fig. 3.2.

(a) \( \text{rn}(SS(W_3)) = 7 \)

(b) \( \text{rn}(SS(W_4)) = 13 \)

(c) \( \text{rn}(SS(W_5)) = 16 \)

(d) \( \text{rn}(SS(W_6)) = 15 \)

(e) \( \text{rn}(SS(W_7)) = 16 \)

Figure 3.2
**Definition 3.4.** $S(K_{n,n})$ is the subdivision of the complete bipartite graph $K_{n,n}$.

**Theorem 3.5.** $r_n(S(K_{n,n})) = n^2 + 2n + 2$ for all $n \geq 4$.

*Proof. Let $V_1 = \{u_1, u_2, \ldots, u_n\}$ and $V_2 = \{v_1, v_2, \ldots, v_n\}$ be the partite sets. For $i = 1, 2, \ldots, n$, let $w_{i1}, w_{i2}, \ldots, w_{in}$ denote respectively the vertices obtained by subdividing the edges $u_iv_1, u_iv_2, \ldots, u_iv_n$. Clearly the number of vertices in $S(K_{n,n})$ is $n^2 + 2n$ and $\text{diam}(S(K_{n,n})) = 4$ for $n \geq 2$. Any radio labeling $c$ of $S(K_{n,n})$ must satisfy the condition

$$d(u, v) + |c(u) - c(v)| \geq 1 + \text{diam}(S(K_{n,n})) = 5$$  \hspace{1cm} (3.1)$$

for all distinct vertices $u, v \in V(G)$.

We count the number of values needed for labels and the minimum number of forbidden values. Starting with $u_1$, consider $c(u_1) = 1$. $u_i$'s there are no forbidden values, with $w_{11}$, one forbidden value and with $v_i$'s, one forbidden value. So totally two forbidden values. Adding with the $n^2 + 2n$ values, we get $n^2 + 2n + 2$ values.

Hence $r_n(S(K_{n,n})) \geq n^2 + 2n + 2$.

Define a labeling $c : V(S(K_{n,n})) \rightarrow N$ as follows:

\[
x : u_1, u_2, \ldots, u_n, w_{11}, w_{12}, \ldots, w_{1n}, w_{12}, \ldots, w_{n1}, w_{13}, w_{24}, \ldots, \]
\[
w_{n2}, \ldots, w_{1n}, w_{21}, w_{32}, \ldots, w_{n,n-1}, v_1, v_2, \ldots, v_n\]
\[
c(x) : 1, 2, \ldots, n, n + 2, n + 3, \ldots, 2n + 1, 2n + 2, 2n + 3, \ldots,\]
\[
3n + 1, 3n + 2, 3n + 3, \ldots, 4n + 1, \ldots, n^2 + 2, n^2 + 3, \ldots,\]
\[
n^2 + n + 1, n^2 + n + 3, n^2 + n + 4, \ldots n^2 + 2n + 2.\]
Clearly $c$ is a radio labeling of $S(K_{n,n})$ satisfying (3.1) with span $n^2 + 2n + 2$ for $n \geq 4$.

Hence $\text{rn}\left(S(K_{n,n})\right) \leq n^2 + 2n + 2$.

Thus $\text{rn}\left(S(K_{n,n})\right) = n^2 + 2n + 2$. □

**Example 3.6.** Fig. 3.3 shows that $\text{rn}\left(S(K_{4,4})\right) = 26$.

**Remark:** Radio number of $S(K_{2,2})$ and $S(K_{3,3})$ are shown in Fig. 3.4.
3.2 Radio Labeling of the Graph $S(S_n)$

Notation 3.7. The subdivision of star graph $S_n$ is denoted by $S(S_n)$.

Theorem 3.8. $rn(S(S_n)) = 4n + 2, n \geq 3$.

Proof. Let $u_0$ be the centre of $S(S_n)$, $v_1, v_2, \ldots, v_n$ be the pendant vertices and $u_1, u_2, \ldots, u_n$ be the vertices of degree $d$ such that $v_i$ is adjacent to $u_{i-1}$.

Clearly $\text{diam}(S(S_n)) = 4$.

Define radio labeling $c : V \to N$ satisfying the condition

$$d(u, v) + |c(u) - c(v)| \geq 1 + \text{diam}(S(S_n))$$

for every pair of distinct vertices $u$ and $v$.

Case 1. Label the centre vertex $u_0$ with $c(u_0) = 1$.

Let $c(v_i) = 3 + i$ and $c(u_i) = n + 2 + 3i, i = 1, 2, \ldots, n$.

Then $c(u_n) = 4n + 2$. Hence $rn(c) = 4n + 2$.

Case 2. Label any one of the pendant vertices with $c(v_i) = 1$.

Let $c(v_1) = 1$. Then $c(v_i) = i, c(u_0) = n + 3$ and $c(u_i) = n + 4 + 3i, i = 1, 2, \ldots, n$.

Then $c(u_n) = 4n + 4$. Hence $rn(c) = 4n + 4$.

Case 3. Label any one of $u_i$ with $c(u_i) = 1$.

Let $c(u_1) = 1$. Then $c(u_i) = 1 + 3i, i = 2, \ldots, n, c(u_0) = 3n + 5$ and $c(v_i) = 3n + 7 + i, i = 1, 2, \ldots, n$.

Then $c(v_n) = 4n + 7$. Hence $rn(c) = 4n + 7$.

Case 4. Label alternatively $v_i$ and $u_i$.

Label the vertices $v_1, u_1, v_3, u_3, \ldots, v_n, u_n, v_2, u_2, \ldots, v_{n-1}, u_{n-1}$ with $1, 3, 5, 7, \ldots, 2n - 1, 2n + 1, 2n + 3, 2n + 5, \ldots, 4n - 3, 4n - 1$ if $n$ is odd.

Label the vertices $v_1, u_1, v_3, u_3, \ldots, v_{n-1}, u_{n-1}, v_2, u_2, \ldots, v_n, u_n$ with $1, 3, 5, 7, \ldots, 2n - 3, 2n - 1, 2n + 1, 2n + 3, \ldots, 4n - 3, 4n - 1$ if $n$ is even.
Then \( c(u_0) = 4n + 3 \). Hence \( \text{rn}(c) = 4n + 3 \).

\[ \text{rn}(G) = \min\{\text{rn}(c)\} = 4n + 2. \]

**Example 3.9.** Fig. 3.5 gives the radio number of \( S(S_4) \).

![Radio Number of S(S_4)](image)

\( \text{rn} S(S_4) = 18 \)

**Figure 3.5**

**Remark:** Fig. 3.6 shows that \( \text{rn} S(S_2) = 11 \).

![Radio Number of S(S_2)](image)

\( \text{rn} S(S_2) = 11 \)

**Figure 3.6**

### 3.3 Radio Labeling of Helm \( H_n \)

**Definition 3.10.** The helm \( H_n \) is the graph obtained from a wheel by attaching pendant edge at each vertex of the \( n \)-cycle.

**Theorem 3.11.** \( \text{rn}(H_n) = 4n + 2 \) for \( n \geq 5 \).
Proof. Let $u_0$ be the centre vertex and $v_1, v_2, \ldots, v_n$ be the pendant vertices. The vertices of degree 4 are $u_i, i = 1, 2, \ldots, n$, where $u_i$ and $v_i$ are adjacent if $n$ is odd, $v_i$ and $u_{i-1}$ are adjacent if $n$ is even.

Clearly $H_n$ has $2n + 1$ vertices and $\text{diam}(H_n) = 4$ for $n \geq 4$.

Define radio labeling $c : V \to N$ satisfying the condition

$$d(u, v) + |c(u) - c(v)| \geq 1 + \text{diam}(H_n)$$

for every pair of distinct vertices $u$ and $v$.

Case 1. $c(u_0) = 1$.

Label the vertices $v_i$ alternatively so that $c(v_1) = 4, c(v_3) = 5, \ldots, c(v_n) = n + 3$ if $n$ is even or $c(v_{n-1}) = n + 3$ if $n$ is odd.

Then label $u_i$’s alternatively so that $c(u_1) = n + 5, c(u_3) = n + 8, \ldots, c(u_n) = 4n + 2$ if $n$ is even or $c(u_{n-1}) = 4n + 2$ if $n$ is odd.

Hence $\text{rn}(c) = 4n + 2$.

Case 2. $c(v_i) = 1$, for any $i = 1, 2, \ldots, n$.

Label $v_i$’s alternatively with $c(v_1) = 1, c(v_3) = 2, \ldots, c(v_n) = n$ if $n$ is even or $c(v_{n-1}) = n$ if $n$ is odd. Label $u_i$’s alternatively as $c(u_1) = n + 2, c(u_3) = n + 5, \ldots, c(u_n) = 4n - 1$ if $n$ is even or $c(u_{n-1}) = 4n - 1$ if $n$ is odd.

Then $c(u_0) = 4n + 3$.

Hence $\text{rn}(c) = 4n + 3$.

Case 3. $c(u_i) = 1$, for any $i = 1, 2, \ldots, n$.

Let $c(u_1) = 1$. Label $u_i$’s alternatively as $c(u_3) = 4, c(u_5) = 7, \ldots, c(u_n) = 3n - 2$ if $n$ is even or $c(u_{n-1}) = 3n - 2$ if $n$ is odd. If $n$ is even, label $c(v_3) = 3n, c(v_5) = 3n + 1, \ldots$ alternatively and $c(v_2) = 4n - 1$. If $n$ is odd, label $c(v_1) = 3n, c(v_3) = 3n + 1, \ldots, c(v_{n-1}) = 4n - 1$.

Then $c(u_0) = 4n + 2$.

Hence $\text{rn}(c) = 4n + 2$. 
Case 4. Label the vertices $v_i$ and $u_i$ alternatively.

Let $c(v_1) = 1$. Label $c(u_1) = 4$, $c(v_3) = 7$, $c(u_3) = 10$, \ldots, $c(u_{n-1}) = 3n - 2$, $c(v_2) = 3n$, $c(u_2) = 3n + 3$, \ldots, $c(u_n) = 6n - 3$ and $c(u_0) = 6n + 1$, if $n$ is even.

Label $c(u_1) = 4$, $c(v_3) = 7$, $c(v_n) = 3n - 2$, $c(u_1) = 3n + 1$, $c(v_2) = 3n + 4$, \ldots, $c(u_n) = 6n - 2$, and $c(u_0) = 6n + 2$ if $n$ is odd.

Then $\text{rn}(c) = \begin{cases} 
6n + 1, & \text{if } n \text{ is even}, \\
6n + 2, & \text{if } n \text{ is odd}.
\end{cases}$

Hence $\text{rn}(G) = \min \text{rn}(c) = 4n + 2$.

Example 3.12. Fig. 3.7 shows that $\text{rn}(H_5) = 22$. 

Figure 3.7
**Remark:** Fig. 3.8 shows that $rn(H_4) = 21.$

![Figure 3.8](image)

#### 3.4 Radio Labeling of Closed Helms

**Definition 3.13.** Closed Helm is the graph obtained from a Helm by joining the pendant vertices to form a cycle. It is denoted by $\text{cl} H_n$.

**Theorem 3.14.** Radio number of $\text{cl} H_n$ is $4n + 24$ for $n \geq 8$.

**Proof.** Label the outer cycle vertices as $v_1, v_2, \ldots, v_n$ and inner cycle vertices as $u_1, u_2, \ldots, u_n$ so that $u_i, v_i$ are adjacent. Label the centre vertex as $u_0$.

Clearly $\text{cl} H_n$ has $2n + 1$ vertices and $\text{diam (cl} H_n) = 4$ for $n \geq 8$.

Define radio labeling $c : V \rightarrow N$ satisfying the condition

$$d(u, v) + |c(u) - c(v)| \geq 1 + \text{diam (cl} H_n)$$

for every pair of distinct vertices $u$ and $v$.

**Case 1.** $c(u_0) = 1.$
Label the vertices $v_i$ and $u_i$ alternatively at distance 3 so that $c(v_1) = 4$, $c(u_{\lfloor \frac{n}{2} \rfloor}) = 6$, $c(v_2) = 8$, $c(u_{\lfloor \frac{n}{2} \rfloor + 1}) = 10$, \ldots, $c(u_{\lfloor \frac{n}{2} \rfloor - 1}) = 4n + 2$.

Therefore $\text{rn}(c) = 4n + 2$.

**Case 2.** $c(v_i) = 1$.

Let $c(v_1) = 1$. Label the vertices $v_i$ with $c(v_i) = 4i - 3$, $i = 1, 2, \ldots, n$ and $u_i$ with $c(u_i) = 4n - 4 + 4i$, $i = 1, 2, \ldots, n$.

Then $c(u_0) = 8n$.

Therefore $\text{rn}(c) = 8n$.

**Case 3.** $c(u_i) = 1$.

Label $c(u_i) = 4i - 3$, $i = 1, 2, \ldots, n$ and $c(v_i) = 4n - 4 + 4i$, $i = 1, 2, \ldots, n$.

Then $c(u_0) = 8n - 1$.

Therefore $\text{rn}(c) = 8n - 1$.

**Case 4.** $c(u_0) = 1$.

Label the vertices $u_i$ and $v_i$ alternatively at distance 3 so that $c(u_1) = 5$, $c(v_{\lfloor \frac{n}{2} \rfloor}) = 7$, $c(u_2) = 9$, $c(v_{\lfloor \frac{n}{2} \rfloor + 1}) = 11$, \ldots, $c(u_n) = 4n + 1$, $c(v_{\lfloor \frac{n}{2} \rfloor - 1}) = 4n + 3$.

Therefore $\text{rn}(c) = 4n + 3$.

**Case 5.** $c(u_1) = 1$.

Label the vertices at distance 3 starting with $c(v_{\lfloor \frac{n}{2} \rfloor}) = 3$, $c(u_2) = 5$, $c(v_{\lfloor \frac{n}{2} \rfloor + 1}) = 7$, \ldots, $c(u_n) = 4n - 3$, $c(v_{\lfloor \frac{n}{2} \rfloor - 1}) = 4n - 1$.

Then $c(u_0) = 4n + 2$.

Therefore $\text{rn}(c) = 4n + 2$.

From all the above cases, $\text{rn}(\text{cl } H_n) = \min\{\text{rn}(c)\} = 4n + 2$. \hfill \Box
Example 3.15. Fig. 3.9 shows that \( r_n(\text{cl} \, H_8) = 34 \).

![Figure 3.9]

Remark 3.16. Closed Helm \( \text{cl} \, H_7, \text{cl} \, H_6, \text{cl} \, H_5, \text{cl} \, H_4 \) are of diameter three and \( \text{cl} \, H_3 \) is of diameter 2. Radio labeling of the above graphs are given in Fig. 3.10.
(a) \( \text{rn}(\text{cl}H_7) = 22. \)

(b) \( \text{rn}(\text{cl}H_6) = 19. \)

(c) \( \text{rn}(\text{cl}H_5) = 16. \)

(d) \( \text{rn}(\text{cl}H_4) = 13. \)

(e) \( \text{rn}(\text{cl}H_3) = 7. \)

Figure 3.10