Chapter 1

Preliminaries

In this chapter we collect the basic definitions and results on graphs which are needed for the subsequent chapters.

**Definition 1.1.** A *graph* $G$ is a finite nonempty set of objects called vertices together with a set of unordered pair of distinct vertices of $G$ called edges. The vertex set and the edge set of $G$ are denoted by $V(G)$ and $E(G)$ respectively. $|V(G)|$ is called the order of $G$ and $|E(G)|$ is called the size of $G$. If $e = uv$ is an edge of $G$, we say that $u$ and $v$ are adjacent and $u$ and $v$ are incident with $e$.

**Definition 1.2.** The *degree* of a vertex $v$ in a graph $G$ is defined to be the number of edges incident on $v$ and is denoted by $\text{deg}(v)$. A vertex of degree one is called a **pendant vertex**.

**Definition 1.3.** A graph $H$ is called a *subgraph* of a graph $G$, if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. A **spanning subgraph** of $G$ is a subgraph $H$ with $V(H) = V(G)$.
Definition 1.4. A graph is \textit{simple} if it has no loops and no two of its edges join the same pair of vertices.

Definition 1.5. The \textit{neighbourhood} of a vertex \( u \) is the set \( N(u) \) consisting of all vertices \( v \) which are adjacent to \( u \).

Definition 1.6. A graph \( G \) is \textit{regular} of degree \( r \) if every vertex of \( G \) has degree \( r \). Such a graph is called \( r \)-regular graph.

Definition 1.7. A graph \( G \) is \textit{complete} if every pair of its vertices are adjacent. A complete graph on \( n \) vertices is denoted by \( K_n \).

Definition 1.8. The \textit{complement} \( \overline{G} \), of a graph \( G \) is the graph with vertex set \( V(G) \) such that two vertices are adjacent in \( \overline{G} \), if and only if they are not adjacent in \( G \). It is also denoted by \( G^c \).

Definition 1.9. A \textit{bipartite graph} is a graph whose vertex set \( V(G) \) can be partitioned into two subsets \( V_1 \) and \( V_2 \) such that every edge of \( G \) has one end in \( V_1 \) and the other end in \( V_2 \); \((V_1,V_2)\) is called a bipartition of \( G \). If further, every vertex of \( V_1 \) is joined to all the vertices of \( V_2 \), then \( G \) is called a complete bipartite graph. The complete bipartite graph with bipartition \((V_1,V_2)\) such that \(|V_1|=m\) and \(|V_2|=n\) is denoted by \( K_{m,n} \). The graphs \( K_{1,n} \) are called \textit{star graphs}, \( S_n \).

Definition 1.10. Let \( u \) and \( v \) be (not necessarily distinct) vertices of a graph \( G \). A \textit{u-v walk} of \( G \) is a finite, alternating sequence \( u = u_0, e_1, u_1, e_2, u_2, \ldots, e_n, u_n = v \) of vertices and edges beginning with vertex \( u \) and ending with vertex \( v \) such that \( e_i = u_{i-1}u_i, \ i = 1, 2, \ldots, n. \) The number \( n \) is called the
length of the walk. The walk is said to be open if \( u \) and \( v \) are distinct vertices; it is closed otherwise. If \( G \) is simple, a walk \( u_0, e_1, u_1, e_2, u_2, \ldots, e_n, u_n \) is determined by the sequence \( u_0, u_1, \ldots, u_n \) of its vertices and hence we specify this walk by \((u_0, u_1, u_2, \ldots, u_n)\).

**Definition 1.11.** A walk in which all the edges are distinct is called a **trail**. A walk in which all the vertices are distinct is called a **path**. A closed walk \((u_0, u_1, u_2, \ldots, u_n)\) in which \(u_0, u_1, u_2, \ldots, u_{n-1}\) are distinct is called a **cycle**. A path on \( n \) vertices is denoted by \( P_n \) and a cycle on \( n \) vertices is denoted by \( C_n \).

**Definition 1.12.** Two vertices \( u \) and \( v \) are **connected** in \( G \), if there is a \((u, v)\) path in \( G \). A graph \( G \) is said to be connected if any two distinct vertices of \( G \) are joined by a path. A maximal connected subgraph of \( G \) is called a **component** of \( G \). Thus a disconnected graph has at least two components.

**Definition 1.13.** **One point union** of any number of connected graphs is obtained by identifying one vertex from each graph.

**Definition 1.14.** The join of two graphs \( G_1 \) and \( G_2 \) denoted by \( G_1 + G_2 \) has vertex set \( V = V(G_1) \cup V(G_2) \) and edge set \( E \) consists of edges of \( G_1 \), edges of \( G_2 \) and all edges joining \( V(G_1) \) and \( V(G_2) \).

**Definition 1.15.** The product \( G_1 \times G_2 \) of two graphs \( G_1 \) and \( G_2 \) has vertex set \( V = V_1 \times V_2 \) where \( V_1 = V(G_1) \) and \( V_2 = V(G_2) \) in which two vertices \((u_1, u_2)\) and \((v_1, v_2)\) are adjacent whenever \( u_1 = v_1 \) and \( u_2 \) is adjacent with \( v_2 \) or \( u_2 = v_2 \) and \( u_1 \) is adjacent with \( v_1 \).
Definition 1.16. **Bistar** is a graph obtained by joining the centres of $K_{1,n}$ and $K_{1,m}$ and is denoted by $B_{n,m}$.

Definition 1.17. The **book** $B_n$ is the graph $S_n \times P_2$ where $S_n$ is the star with $n + 1$ vertices.

Definition 1.18. When $n$ copies of $C_m$ share a common edge, it will form $m$-gon book of $n$ pages and is denoted by $B(m, n)$.

Definition 1.19. The graph $C_n + K_1$ is called a **wheel graph** and is denoted by $W_n$.

Definition 1.20. If $e = uv$ is an edge of $G$ and $w$ is not a vertex of $G$, we say that the edge $e$ is **subdivided**, if it is replaced by the edges $uw$ and $wv$.

Definition 1.21. For a vertex $v$ of a connected graph $G$, the **eccentricity** $e(v)$ is the distance between $v$ and a vertex farthest from $v$. The minimum eccentricity among the vertices of $G$ is the **radius**, $\text{rad} \ G$, and the maximum eccentricity is its **diameter**, $\text{diam} \ G$.

Definition 1.22. A graph is said to be **acyclic** if it has no cycles. A **tree** is a connected acyclic graph.

Definition 1.23. A **shell** $S_{n,n-3}$ of width $n$ is a graph obtained by taking $n - 3$ concurrent chords in a cycle $C_n$ on $n$ vertices. The vertex at which all the chords are concurrent is called **apex**. The two vertices adjacent to the apex have degree 2, apex has degree $n - 1$ and all the other vertices have degree 3.

Definition 1.24. A **multiple shell** $\text{MS}\{n_{t_1}^{t_1}, n_{t_2}^{t_2}, \ldots, n_{t_r}^{t_r}\}$ is a graph formed by $t_i$ shells of width $n_i$, $1 \leq i \leq r$, which has a common apex. This graph
The multiple shell has \(\sum_{i=1}^{r} t_i(n_i - 1) + 1\) vertices. A multiple shell is said to be balanced with width \(w\) if it is of the form \(\text{MS}\{w'\}\) or \(\text{MS}\{w',(w+1)^s\}\); that is the cycles involved are more or less of the same size. If a multiple shell has \(k\) shells having a common apex, then it is called a \(k\)-tuple shell; a double shell if \(k = 2\), a triple shell if \(k = 3\) etc. Suppose \(S\) is a balanced shell on \(n\) vertices with \(k\) shells having a common apex. If \(n = kt\), then \(S = \text{MS}\{t,(t+1)^{k-1}\}\). On the other hand if \(n = kt + r, r \neq 0\), then \(S = \text{MS}\{(t+1)^{k-r+1},(t+2)^{r-1}\}\).

**Definition 1.25.** A **snake graph** is formed by taking \(n\)-copies of a cycle \(C_m\) and identifying exactly one edge of each copy to a distinct edge of the path \(P_{n+1}\), which we will call the backbone of the snake. It is denoted by \(T_n^{(m)}\).

**Definition 1.26.** The graph \(P_n \times P_2\) is called a **ladder**.

**Definition 1.27.** The graph \(G\) with vertex set \(\{u_1, u_2, \ldots, u_n, v_1, v_2, \ldots, v_n\}\) and the edge set \(\{u_iu_{i+1}, v_iv_{i+1}, v_iu_{i+1} : 1 \leq i \leq n-1\}\) \(\cup\) \(\{u_iv_i : 1 \leq i \leq n\}\) is called a **semi-ladder** of length \(n\).

**Definition 1.28.** The graph \(P_m \times P_n\) is called a **planar grid**.

**Definition 1.29.** Let \(P_n\) be a path with \(n\) vertices \(u_1, u_2, \ldots, u_n\). The tree obtained by adding \(n\) vertices \(v_1, v_2, \ldots, v_n\) such that for \(i = 1, 2, \ldots, n\), \(v_i\) is adjacent to \(u_i\) is called a **comb**.

**Definition 1.30.** **Klein four group** \(V_4 = Z_2 \oplus Z_2\).

**Definition 1.31.** The graph obtained by attaching a pendant edge at each vertex of a cycle \(C_n\) is called a **crown**. It is denoted by \(C_n \odot K_1\).
**Definition 1.32.** The *sequential join* of graphs $G_1, G_2, \ldots, G_n$ is formed from $G_1 \cup G_2 \cup \cdots \cup G_n$ by adding edges joining each vertex of $G_i$ with each vertex of $G_{i+1}$ for $1 \leq i \leq n - 1$.

**Notation 1.33.** We shall use $\sum\{a_i : 1 \leq i \leq n\}$ to mean $a_1 + a_2 + \cdots + a_n$. 