Radio Labeling of Other Classes of Graphs

In this chapter, we find the radio number of $m$-gon books of $n$ pages for $m = 4, 5, 6, 7$ and radio number of the graph $P_n + K_5^c$.

5.1 Radio Labeling of $B(4, n)$

**Definition 5.1.** When $n$ copies of $C_4$ share a common edge, it will form 4-gon book of $n$ pages. It is denoted by $B(4, n)$.

**Theorem 5.2.** $rn(B(4, n)) = 2n + 4$ for all $n \geq 3$.

**Proof.** Label the vertices of common edge as $u_0, v_0$. Label the vertices adjacent to $u_0$ as $u_1, u_2, \ldots, u_n$ and $v_0$ as $v_1, v_2, \ldots, v_n$ and in the same order. Clearly there are $2n + 2$ vertices and $\text{diam}(B(4, n)) = 3$, for $n \geq 2$. Any radio labeling $c$ of $(B(4, n))$ must satisfy the condition,

$$d(u, v) + |c(u) - c(v)| \geq 1 + \text{diam}(B(4, n)) = 4,$$

(5.1)

for all distinct vertices $u, v \in V(G)$. 

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Associated with $u_0$ and $v_0$, there are two different forbidden values. Hence $r_n(B(4, n)) \geq 2n + 4$.

Assume that $n$ is even.

Then the labeling given by $c : V(B(4, n)) \rightarrow N$ as

\[
x : u_0, v_1, u_2, v_3, \ldots, u_{n-2}, v_{n-1}, u_n, v_2, u_1, v_4, \ldots, v_n, u_{n-1}, v_0.
\]

\[c(x) : 1, 3, 4, 5, \ldots, n, n + 1, n + 2, n + 3, n + 4, n + 5, \ldots,
\]

\[2n + 1, 2n + 2, 2n + 4,
\]

is clearly a radio labeling satisfying (5.1) with span $2n + 4$.

Hence $r_n(B(4, n)) \leq 2n + 4$.

Hence $r_n(B(4, n)) = 2n + 4$ when $n$ is even.

The case when $n$ is odd follows similarly.

\[\square\]

**Example 5.3.** Fig. 5.1 shows that $r_n(B(4, 4)) = 12$ and $r_n(B(4, 5)) = 14$. 

Figure 5.1
Remark 5.4. \( rn(B(4, 2)) = 10 \), is shown in Fig. 5.2.

![Figure 5.2](image)

5.2 Radio Labeling of \( B(5, n) \)

Definition 5.5. When \( n \)-copies of \( C_5 \) share a common edge, it will form 5-gon book of \( n \) pages and is denoted by \( B(5, n) \).

Theorem 5.6. \( rn(B(5, n)) = 5n + 6 \) for all \( n \geq 3 \).

Proof. Assume that \( n \) is even.

Label the vertices of common edge as \( u_0 \) and \( v_0 \). Label the vertices adjacent to \( u_0 \) as \( u_1, u_2, \ldots u_n \) and \( v_0 \) as \( v_1, v_2, \ldots, v_n \). Remaining vertices as \( w_1, w_2, \ldots, w_n \) and each \( w_i \) is adjacent to both \( u_i \) and \( v_i \).

Clearly there are \( 3n + 2 \) vertices and \( \text{diam} B(5, n) = 4 \), for \( n \geq 2 \). Any radio labeling \( c \) of \( (B(5, n)) \) must satisfy the condition

\[
d(u, v) + |c(u) - c(v)| \geq 1 + \text{diam} (B(5, n)) = 5, \tag{5.2}
\]

for all distinct vertices \( u, v \in V(G) \).

We count the number of values needed for labels and the minimum number of forbidden values. Consider \( c(u_0) = 1 \). Associated with \( u_0 \) and \( v_0 \), there are four forbidden values. Associated with \( u_i \)'s and \( v_i \)'s, there are \( 2n \)
forbidden values, and with \(w_i\)'s there are no forbidden values. Totally there are \(2n + 4\) forbidden values. Therefore

\[
\text{rn}(B(5, n)) \geq 5n + 6.
\]

Define a labeling \(c : V(B(5, n)) \to N\) as follows:

\[
x: u_0, v_1, u_2, \ldots, v_{n-1}, u_n, v_2, u_1, \ldots, v_n, u_{n-1}, w_1, w_2, \ldots, w_n, v_0
\]

\[
c(x) : 1, 4, 6, \ldots, 2n, 2n + 2, 2n + 4, 2n + 6, \ldots, 4n, 4n + 2,
\]

\[
4n + 4, 4n + 5, \ldots, 5n + 3, 5n + 6,
\]

Clearly \(c\) is a radio labeling of \(B(5, n)\) satisfying (5.2) with span \(5n + 6\).

Hence \(\text{rn}(B(5, n)) \leq 5n + 6\).

Therefore \(\text{rn}(B(5, n)) = 5n + 6\).

The case when \(n\) is odd follows similarly.

**Example 5.7.** \(\text{rn}(B(5, 4)) = 26\) and \(\text{rn}(B(5, 5)) = 31\) as shown in Fig. 5.3.
Remark 5.8. \( \text{rn}(B(5, 2)) = 17 \) as shown in Fig. 5.4.

Figure 5.4

5.3 Radio Labeling of \( B(6, n) \)

Definition 5.9. When \( n \)-copies of \( C_6 \) share a common edge, it will form 6-gon book of \( n \)-pages and is denoted by \( B(6, n) \).

Theorem 5.10. \( \text{rn}(B(6, n)) = 8n + 6 \) for \( n \geq 3 \).

Proof. Assume that \( n \) is even.

Label the vertices of common edge as \( u_0 \) and \( v_0 \). Label the vertices adjacent to \( u_0 \) as \( u_1, u_2, \ldots, u_n \) and adjacent to \( v_0 \) as \( v_1, v_2, \ldots, v_n \). The vertices adjacent to \( u_i' \)'s as \( u_i' \)'s and adjacent to \( v_i' \)'s as \( v_i' \)'s so that \( w_i' \)'s and \( y_i' \)'s are adjacent for \( i = 1, 2, \ldots, n \). There are \( 4n + 2 \) vertices in \( B(6, n) \) and \( \text{diam}(B(6, n)) = 5 \), for \( n \geq 2 \). Any radio labeling \( c \) of \( B(6, n) \) must satisfy the condition

\[
d(u, v) + |c(u) - c(v)| \geq 1 + \text{diam}(B(6, n)) = 6
\]

for all distinct vertices \( u, v \in V(G) \).

Associated with \( u_0 \) and \( v_0 \), there are four forbidden values, with \( u_i' \)'s and \( v_i' \)'s, \( 2(2n - 1) + 2 \) forbidden values. Thus there are \( 4n + 4 \) forbidden values and so \( \text{rn}(B(6, n)) \geq 8n + 6 \).
Then the labeling given by \( c : V(B(6, n)) \to N \) as

\[
\begin{aligned}
x & : u_0, y_1, w_2, y_3, \ldots, w_n, v_1, u_1, \ldots, v_n, u_n, y_2, w_1, \ldots, y_n, w_{n-1}, v_0 \\
c(x) & : 1, 3, 4, 5, \ldots, n + 3, n + 5, n + 8, \ldots, 7n - 1, 7n + 3, 7n + 4, \\
& 7n + 5, \ldots, 8n + 2, 8n + 3, 8n + 6
\end{aligned}
\]

is clearly a radio labeling satisfying (5.3) with span \( 8n + 6 \) so that

\[
\text{rn}(B(6, n)) \leq 8n + 6.
\]

Hence \( \text{rn}(B(6, n)) = 8n + 6. \)

The case when \( n \) is odd follows similarly.

\[ \square \]

**Example 5.11.** The radio number of \( B(6, 4) \) and \( B(6, 5) \) are given in Fig. 5.5.

\[ \begin{array}{l}
\text{(a) } \text{rn}(B(6, 4)) = 38. \\
\text{(b) } \text{rn}(B(6, 5)) = 46.
\end{array} \]

**Figure 5.5**
Remark 5.12. \( \text{rn}(B(6, 2))) = 24 \), shown in Fig. 5.6.

\[ \begin{array}{c}
1 \\
8 \\
19 \\
5 \\
4 \\
22 \\
17 \\
26 \\
14
\end{array} \]

Figure 5.6

5.4 Radio Labeling of \( B(7, n) \)

Definition 5.13. When \( n \)-copies of \( C_7 \) share a common edge, it will form 7-gon book of \( n \)-pages and is denoted by \( B(7, n) \).

Theorem 5.14. \( \text{rn}(B(7, n)) = 13n + 8 \) for \( n \geq 3 \).

Proof. Assume that \( n \) is even.

Label the vertices of common edge as \( u_0 \) and \( v_0 \). Label the vertices adjacent to \( u_0 \) as \( u_1, u_2, \ldots, u_n \) and adjacent to \( v_0 \) as \( v_1, v_2, \ldots, v_n \). Label vertices adjacent to \( u_1, u_2, \ldots, u_n \) as \( u'_1, u'_2, \ldots, u'_n \) and adjacent to \( v_1, v_2, \ldots, v_n \) as \( v'_1, v'_2, \ldots, v'_n \). The vertices adjacent to both set \( u'_1, u'_2, \ldots, u'_n \) and \( v'_1, v'_2, \ldots, v'_n \) are denoted by \( w_1, w_2, \ldots, w_n \). There are \( 5n + 2 \) vertices in \( B(7, n) \) and diam \( (B(7, n)) = 6 \), for \( n \geq 2 \). Any radio labeling \( c \) of \( B(7, n) \) satisfies the condition

\[ d(u, v) + |c(u) - c(v)| \geq 1 + \text{diam} \ (B(7, n)) = 7 \] (5.4)

for all distinct vertices \( u, v \in V(G) \).
Associate with \( u_0 \) and \( v_0 \), there are six forbidden values, with \( u'_i \) and \( v'_i \), \( 2 + (2n - 1) \) and with \( u_i \) and \( v_i \), \( 2 + 3(2n - 1) \) forbidden values. So totally we have \( 8n + 6 \) forbidden values. Hence \( \text{rn}(G) \geq 13n + 8 \).

Then the labeling given by \( c : V(B(7, n)) \rightarrow N \) as

\[
x : u_0, v'_1, u'_2, \ldots, v'_n, u'_n, v'_1, \ldots, v'_n, u'_n-1, v_1, u_1, \ldots, v_n, u_n, w_1, w_2, \ldots, w_n, v_0
\]

\[
c(x) : 1, 5, 7, \ldots, 2n + 1, 2n + 3, 2n + 5, 2n + 7, \ldots, 4n + 1, 4n + 3, 4n + 6,
4n + 10, \ldots, 12n - 2, 12n + 2, 12n + 5, 12n + 6, \ldots, 13n + 4, 13n + 8
\]

is clearly a radio labeling satisfying (5.4) with span \( 13n + 8 \).

Therefore \( \text{rn}(B(7, n)) \leq 13n + 8 \).

Hence \( \text{rn}(B(7, n)) = 13n + 8 \).

The case when \( n \) is odd follows immediately. \( \square \)

**Example 5.15.** The radio number of \( B(7, 4) \) and \( B(7, 5) \) are shown in Fig. 5.7.

\begin{figure}[h]
\begin{center}
\includegraphics[width=\textwidth]{figure5.7.png}
\end{center}
\caption{Figure 5.7}
\end{figure}

(a) \( \text{rn}(B(7, 4)) = 60 \).

(b) \( \text{rn}(B(7, 5)) = 73 \).
Remark 5.16. \(\text{rn}(B(7, 2)) = 36\) as given in Fig. 5.8.

\[
\begin{array}{c}
1 & 16 & 10 & 7 \\
24 & 16 & 10 & 7 \\
31 & 16 & 10 & 7 \\
28 & 16 & 10 & 7 \\
36 & 16 & 10 & 7 \\
20 & 16 & 10 & 7 \\
5 & 16 & 10 & 7 \\
12 & 16 & 10 & 7 \\
\end{array}
\]

Figure 5.8

5.5 Radio Labeling of \(P_n + K^c_t\)

Theorem 5.17. \(\text{rn}(P_n + K^c_t) = n + t + 1\)

Proof. Assume that \(n\) is even. Label the vertices of \(K^c_t\) as \(u_1, u_2, \ldots, u_t\) and the vertices of \(P_n\) as \(v_1, v_2, \ldots, v_n\). \(P_n + K^c_t\) has \(n + t\) vertices and \(\text{diam}(P_n + K^c_t) = 2\). Any radio labeling \(c\) of \(P_n + K^c_t\) must satisfy the condition

\[
d(u, v) + |c(u) - c(v)| \geq 1 + \text{diam}(P_n + K^c_t) = 3 \tag{5.5}
\]

for all distinct vertices \(u, v \in V(G)\). Associated with \(u_t\), there is only one forbidden value, hence \(\text{rn}(P_n + K^c_t) \geq n + t + 1\).

The labeling given by \(c : V(P_n + K^c_t) \to N\) as

\[
x : u_1, u_2, \ldots, u_t, v_{\frac{n}{2}+1}, v_1, v_{\frac{n}{2}+2}, \ldots, v_n, v_{\frac{n}{2}}.
\]

\[
c(x) : 1, 2, \ldots, t, t + 2, t + 3, t + 4, t + 5, \ldots, t + n, t + n + 1.
\]

is clearly a radio labeling satisfying the condition (5.5) with span \(n + t + 1\).

Hence \(\text{rn}(P_n + K^c_t) \leq n + t + 1\).
Thus \(\text{rn}(P_n + K_t^c) = n + t + 1\).

The proof is similar when \(n\) is odd.

**Example 5.18.** The radio number of \(P_6 + K_3^c\) and \(P_5 + K_2^c\) are given in Fig. 5.9.

(a) \(\text{rn}(P_6 + K_3^c) = 10\).

(b) \(\text{rn}(P_5 + K_2^c) = 8\).

**Figure 5.9**

**Remark 5.19.**

(i) Radio number of Fan \(F_n = P_n + K_1\), is \(n + 2\).

(ii) Radio number of double fan \(P_n + K_2^c\) is \(n + 3\).