CHAPTER 4
HYPERCUBIC DESIGNS AND E-OPTIMALITY

4.0 Introduction

In the early stages of experimentation, only a few experimental designs were at once disposal. But now-a-days, an experimenter with modern statistical equipments at his disposal, can use a variety of experimental designs as various complete and incomplete block designs, designs with variable replications, factorial designs and others. Hence, whenever the conditions of an experiment allow the possibility of simultaneous existence of a number of experimental designs, the question of selection of an appropriate design, a design that is easy to analyze and which has some optimum properties, naturally arises. A systematic study of the specification of optimum experimental designs was undertaken by Kiefer (1958, 1959) in a series of papers, where he introduced various optimality criteria (A, D, E, L, M), discussed interrelations amongst these and established the optimality property of some well-known designs in some particular problems.

4.1 Measures of Optimality

We shall define various measures of optimality in terms of the variance-covariance matrix $\Omega$ for the treatment parameters, as given by Tocher (1952).
\[
\Omega^{-1} = rI - \left(\frac{1}{K}\right) N_d N_d^T + (r^2/bk) J.
\]

Here \(N_d\) is the \(v \times b\) incidence matrix of the design, \(N_d N_d^T\) is called the concurrence matrix and \(J\) is a \(v \times v\) matrix of 1's.

Let the eigen values of \(\Omega^{-1}\) be \((Z_{d1}, Z_{d2}, \ldots, Z_{dv})\). Then the three criteria we shall consider here are:

\[
C_1 = \pi \ Z_{di}, \quad \quad \quad \quad \quad C_2 = \left(\sum_i Z_{di}\right)^{-1}, \quad \quad \quad \quad \quad C_3 = \min_i Z_{di}
\]

An optimal design for any of these criteria is one, which maximizes the criterion value.

Since \(C_1\) is the determinant of \(\Omega^{-1}\), a design which maximizes \(C_1\) is D-optimal.

Designs which maximize \(C_2\) are sometimes called A-optimal (Kiefer, 1959). These designs also maximize the conventional efficiency \(E\), since \(E = (v - 1) \ C_2/(r - C_2)\). The efficiency \(E\) is the average variance of all pairs of treatment differences, divided into the (hypothetical) minimum, which would be achieved by a randomized block design, if it exists.

Designs which maximizes \(C_3\) are sometimes called “E-optimal” (Kiefer, 1959). We shall refrain from using this nomenclature here to avoid confusion with the efficiency \(E\). A \(C_3\)-optimal design minimizes the maximum variance of any linear combination of treatment effects, scaled so that the sum of squared coefficients is unity.
4.2 Preliminary Results

Let $D$ denote the class of all connected block designs having $v$ treatments arranged in $b$ blocks of size $k$. Let $\mathcal{D}_r$ be the subclass of designs in $D$ having all treatments equally replicated. For a design $d \in \mathcal{D}_r$, there are $v$ treatments arranged in $b$ blocks, each having $k$ plots, $k < v$. Each treatment is replicated $r$ times, $vr = bk$.

Let $N_d = (n_{ij})$ denote the $v \times b$ incidence matrix of $d$, where $n_{ij} = 1$, if the treatment $i$ occurs in block $j$, and $n_{ij} = 0$, otherwise. $d$ is called a binary design. Let $N_d N_d^\dagger$ be the concurrence matrix of the design. If $N_d N_d^\dagger$ has all of its diagonal elements are equal and all of its off diagonal elements differing by at most one, then the design is called a Regular Graph Design (RGD) (Jacroux, 1980; Cheng, 1980).

The coefficient matrix of the reduced normal equations for estimating the treatment effects in the usual two-way additive model is

$$C_d = \text{diag}(r_1, r_2, \ldots, r_v) - (1/k) N_d N_d^\dagger,$$

where diag$(r_1, r_2, \ldots, r_v)$ is a $v \times v$ diagonal matrix $(r_1 = r_2 = \ldots = r_v = r)$, and has $v - 1$ positive eigenvalues $Z_{d1} \leq Z_{d2} \leq \ldots \leq Z_{dv-1}$ arranged in the order of magnitude.

To satisfy the E-optimality criterion, those designs in $\mathcal{D}_r$, which minimize the maximum variance of the estimates of all normalized estimable functions of the treatment effects are chosen. Such designs have $C$-matrices whose minimum non-zero eigenvalues have maximum size.

4.2.1 Optimality of Incomplete Block Designs

John and Mitchell (1977) defined the class of regular graph designs, conjectured to be optimal among all incomplete block designs.
They have considered incomplete block designs in the sense that every pair of treatments appears in $\lambda_1$ or $\lambda_2$ blocks where $\lambda_2 = \lambda_1 + 1$. They gave the class of RG designs for all cases in which $v \leq 12$, $r \leq 10$ and $v \leq b$ and in every case, the optimal design is determined for each of the three optimality criteria; A, D and E. The following conjecture is due to John and Mitchell (1977).

**Conjecture:** If an incomplete block design is D-optimal (or A-optimal or E-optimal) it is an RG design (if an RG design exists).

They have also shown that if their conjecture is correct, the incomplete block design with $v \leq b$ are optimal in the entire class of designs.

With regard to the above conjecture, several results are known. Takenchi (1961), Conniffe and Stone (1975), Shah, Raghavarao and Khatri (1976), Williams, Patterson and John (1976, 1977) and Cheng (1978) have obtained results, which imply the optimality of certain types of regular graph designs under various criteria.

### 4.2.2 E-optimality of Block Designs

Cheng (1980) investigated the E-optimality of some block designs and established some conditions under which there is a regular graph design. Immediate applications of these results demonstrates that a great majority of designs obtained by John and Mitchell (1977) given in sec. 4.2.1 are E-optimal.
The following results (Theorem 4.2.2.1, Theorem 4.2.2.2 and Theorem 4.2.2.3) were established by Cheng (1980) and also obtained independently by Jacroux (1980).

**Theorem 4.2.2.1**

For any \( d \in D_r \), if \( d \) is not an RGD, then

\[
Z_{d_1} \leq \max \left\{ \frac{r(k-1) - \lambda - 2}{v}, \frac{r(k-1) + \lambda - 1}{k} \right\}
\]

Thus, if there is an RGD, \( d \in D \) with

\[
Z_{d_1} \geq \max \left\{ \frac{r(k-1) - \lambda - 2}{v}, \frac{r(k-1) + \lambda - 1}{k} \right\}
\]

then there is an RGD which is E-optimal over \( D_r \).

**Theorem 4.2.2.2**

For any \( d \in D \setminus D_r \) (i.e. \( d \) is not equi-replicated)

\[
Z_{d_1} \leq \frac{\{r-1\}(k-1)}{\{(v-1)k\}}
\]

Thus if there is an RGD \( d \in D \) with

\[
Z_{d_1} \geq \max \left\{ \frac{r(k-1) - \lambda - 2}{v}, \frac{r(k-1) + \lambda - 1}{k}, \frac{\{r-1\}(k-1)}{\{(v-1)k\}} \right\}
\]

then there is an RGD which is E-optimal over \( D \).

**Theorem 4.2.2.3**

Suppose \( k \geq 3 \) and there is an RGD \( d \in D \), such that

\[
Z_{d_1} \geq \max \left\{ \frac{r(k-1) - \lambda - 2}{v}, \frac{\{(v-2)k\}}, \frac{r(k-1) + \lambda - 1}{k} \right\}
\]

then there is an RGD which is E-optimal over \( D \).
The following are some regular graph designs, which satisfy the condition

\[Z_d \geq \max \{ \{r(k-1) - \lambda - 2\} v / (v-2)k, \ \{r(k-1) + \lambda - 1\} / k\}.\]

(i) Two-associate-class PBIB designs with triangular schemes having \(\lambda_2 = \lambda_1 + 1\) and \(v = 6, 10, 15, 21, 28\).

(ii) Two-associate-class PBIB designs with triangular schemes having \(\lambda_1 = \lambda_2 + 1\) and \(v = 10\).

(iii) Two-associate-class PBIB designs with \(L_2\) schemes having \(\lambda_2 = \lambda_1 + 1\) and \(v = 9, 16\).

Thus if \(D\) contains any of the designs listed above, then there is an RGD which is E-optimal over \(D_r\).

Using the above theorems Cheng (1980) shown that a group divisible design with \(\lambda_1 = \lambda_2 + 1 > 1\) and group size 2 is E-optimal. The same result also hold for a PBIB design with a cyclic scheme having \(\lambda_2 = \lambda_1 \pm 1\) and \(v = 5\). He also shown that if \(d\) is a BIB design, a group divisible design with \(\lambda_2 = \lambda_1 + 1\), a group divisible design with \(\lambda_1 = \lambda_2 + 1 > 1\) and group size 2 or a PBIB design with a cyclic scheme having \(\lambda_2 = \lambda_1 \pm 1\) and \(v = 5\), then the dual design of \(d\) is also E-optimal.

4.2.3 E-optimality of Regular Graph Designs

Jacroux (1980) investigated the E-optimality of Regular graph designs, within various classes of proper block designs and several sufficient conditions are given for the existence of an E-optimal
RG design, within the classes considered. It is shown that when a RG design exists whose minimum non-zero eigenvalue is at least as large as a given lower bound, then an E-optimal RGD exists.

The following result is due to Jacroux (1980).

**Theorem 4.2.3.1**

If there exists an RGD, \( d \in D_r \) having

\[
Z_{d_1} \geq \max \left\{ \frac{(r^k - r - \lambda - 2)}{v} v \left/ \left( v - 2 \right) \right. k, \frac{(r^k - r + \lambda - 1)}{v} k \right\} \quad (4.2.3.1)
\]

where \( r^k - r = (v-1) \lambda + m^l \) for some non-negative integer \( m^l \) such that \( m^l \leq v - 1 \) for which \( \lambda \) is the greatest integer not exceeding \( \frac{r^k - r}{v - 1} \); then there exists an E-optimal RGD in \( D_r \). When the inequality given above in strict, an E-optimal design in \( D_r \) must be an RGD.

Jacroux (1980) showed that, the minimum non-zero eigenvalues of the following well-known PBIB designs with two associate classes satisfy the condition (4.2.3.1).

(i) Group divisible designs having \( \lambda_2 = \lambda_1 + 1 \)

(ii) Designs with a triangular scheme having \( \lambda_1 = \lambda_2 + 1 \) and \( n \leq 8 \)

(iii) Designs with a triangular scheme having \( \lambda_1 = \lambda_2 + 1 \) and \( n \leq 5 \)

(iv) Designs with a Latin square scheme having \( \lambda_2 = \lambda_1 + 1 \) and \( n \leq 4 \)

Thus, if \( D \) contains any of the designs listed above, then there is an RGD which is E-optimal over \( D_r \).
4.2.4 E-optimality of Cubic Designs

Cubic designs were introduced by Raghavarao and Chandrasekhararao (1964). There are \( v = t^3 \), \( v \geq 2 \) treatments which are divided into three classes and arranged in blocks such that any given treatment appears at most once in a block and appears with every other treatment in the \( i^{th} \) class in \( \lambda_i \) blocks (\( \lambda_i \) is constant for all treatments), \( (i = 1, 2, 3) \).

Duthie (1991) examined the E-optimality of designs which have the cubic association scheme and which are regular graph designs.

By modifying the Theorem 3.3 of Jacroux (1980), Duthie (1991) has given the following condition for the E-optimality of Cubic designs.

If a cubic design is an RGD and satisfies the condition

\[
Z_{d1} \geq \max [a, b] \quad (4.2.4.1)
\]

where \( a = \left[ 1/ k(t^3-1) \right] \left( \frac{(rk-r)t^3 - (t^3 / (t^3-2))}{2t^3 - 2 - m^l} \right) \)

and \( b = \left[ 1/ k(t^3-1) \right]\left( \frac{(rk-r)t^3 - (m^l + t^3 - 1)}{2t^3 - 2 - m^l} \right) \)

then the design must be an E-optimal RGD.

Duthie (1991) examined all the cases where a cubic design is an RGD and he has shown that the minimum non-zero eigenvalues of the following cubic designs satisfy the condition (4.2.4.1).

(i) Cubic designs with \( t = 2, \lambda_1 = \lambda_1, \lambda_2 = \lambda_1, \lambda_3 = \lambda_1 - 1 \)

and \( t = 2, \lambda_1 = \lambda_1, \lambda_2 = \lambda_1 - 1, \lambda_3 = \lambda_1 \)

(ii) Cubic designs with \( t = 2, \lambda_1 = \lambda_1, \lambda_2 = \lambda_1, \lambda_3 = \lambda_1 + 1 \) for \( m^l \neq 0 \)

(iii) Cubic designs with \( t = 2, \lambda_1 = \lambda_1, \lambda_2 = \lambda_3 = \lambda_1 - 1 \) for \( m^l = 3, 4, 5, 6 \).
Thus if D contains any of the designs listed above, then there is an RGD which is E-optimal over Dr.

Shah (1995) examined the E-optimality of Cubic designs whose concurrence matrix $N_dN_d^\dagger$ has all diagonal elements equal and all of its off-diagonal elements differing by 0 or 1 or 2. Then Shah (1995) studied the E-optimality of cubic designs for which the concurrence matrix $N_dN_d^\dagger$ has all diagonal elements equal and off-diagonal elements differing by at most $\lambda^* (r^2 - \lambda^2)$ if the cubic design with parameters $v = t^3$, $b = b^3$, $r = r^3$, $k = k^3$, $\lambda_1 = r^2\lambda^*$, $\lambda_2 = r\lambda^*$ and $\lambda_3 = \lambda^*$ is obtained from the BIB design with parameters $(v^*, b^*, r^*, k^*, \lambda^*)$.

### 4.3 E-optimality of Hypercubic Designs

In this investigation we examine the E-optimality of designs which have hypercubic association scheme and which are regular graph designs (RGD'S). We have shown that dual of such designs also yields E-optimal RGD. Some sufficient conditions are given for the existence of an E-optimal regular graph design within the class considered.

#### 4.3.1 Sufficient Conditions for E-optimality

Rao and Das (1988) developed a general method of obtaining eigenvalues of C-matrix of two associate PBIB designs. In this investigation we modify such results and used for the determination of minimum non-zero eigenvalue ($Z_{d1}$) of the information matrix of design d.
4.3.1.1 Modified Results for determining eigenvalues of C-matrices of two associate Hypercubic Designs

From Rao and Das (1988) we have the eigenvalues

\[ \mu_i = \left( \frac{r(k-1) + \lambda_2}{k} - (\lambda_1 - \lambda_2) \left[ p_{11}^1 - p_{11}^2 + \sqrt{\Delta} \right] / 2k \right) \]

for \( i = 1, 2 \).

Putting \( \lambda_2 = \lambda_1 + 1 \), we get the following

(i) For \( k = 2 \)

\[ Z_{di} = \frac{1}{2} \left( r + \lambda_1 + 1 + \frac{1}{2} \left[ (p_{11}^1 - p_{11}^2) + \sqrt{\Delta} \right] \right). \]

(ii) For \( k = 3 \)

\[ Z_{di} = \frac{1}{3} \left( 2r + \lambda_1 + 1 + \frac{1}{2} \left[ (p_{11}^1 - p_{11}^2) + \sqrt{\Delta} \right] \right), \]

for \( i = 1, 2 \) and \( \Delta = (p_{11}^1 - p_{11}^2)^2 + 4p_{12}^2 \).

4.3.1.2 E-optimal Criterion for Hypercubic Designs

Here we consider the following cases (see equation 4.2.3.1).

(i) \( k = 2, v = 2^m \), \( \lambda = (r - m^1) / (2^m - 1) \)

(ii) \( k = 3, v = 2^m \), \( \lambda = (2r - m^1) / (2^m - 1) \)

(iii) \( k = 2, v = 3^m \), \( \lambda = (r - m^1) / (3^m - 1) \)

(iv) \( k = 3, v = 3^m \), \( \lambda = (2r - m^1) / (3^m - 1) \)

where \( m^1 \) is some non-negative integer \( \leq v - 1 \) for which \( \lambda \) is the greatest integer not exceeding \( \frac{r - r}{v - 1} \) and \( m \) and \( m^1 \) are different.

Here \( v \) stands for the number of treatment combinations.
The following statements respectively, hold:

(i) $Z_{d_1} \geq \max \left\{ \frac{(2^m - 1)(r - 2) - (r - m')2^m}{4(2^{m-1} - 1)(2^m - 1)}, \frac{(2^m - 1)(r - 1) + (r - m')}{(2^m - 1)2} \right\}$

(ii) $Z_{d_1} \geq \max \left\{ \frac{[2(2^m - 1)(r - 1) - (2r - m')]2^m}{6(2^{m-1} - 1)(2^m - 1)}, \frac{(2^m - 1)(2r - 1) + (2r - m')}{3(2^m - 1)} \right\}$

(iii) $Z_{d_1} \geq \max \left\{ \frac{[(3^m - 1)(r - 2) - (r - m')]3^m}{2(3^{m-1} - 2)(3^m - 1)}, \frac{(3^m - 1)(2r - 1) + (2r - m')}{2(3^m - 1)} \right\}$

(iv) $Z_{d_1} \geq \max \left\{ \frac{[2(3^m - 1)(r - 1) - (2r - m')]3^m}{3(3^m - 2)(3^m - 1)}, \frac{(3^m - 1)(2r - 1) + (2r - m')}{3(3^m - 1)} \right\}$

Note that $Z_{d_1} \geq \max \left\{ \frac{[(t^m - 1)(r(k - 1) - 2) - ((k - 1)r - m')t^m]}{k(t^m - 1)(t^m - 2)}, \frac{[(t^m - 1)(k - 1)r - 1) + ((k - 1)r - m')]}{k(t^m - 1)} \right\}$ \hspace{1cm} (4.3.1)

The expression (4.3.1) can be written as follows

$$Z_{d_1} \geq \max \{a, b\} \hspace{1cm} (4.3.2)$$

where $a = \frac{[(t^m - 1)(r(k - 1) - 2) - ((k - 1)r - m')t^m]}{k(t^m - 1)(t^m - 2)}$

and $b = \frac{[(t^m - 1)(k - 1)r - 1) + ((k - 1)r - m')]}{k(t^m - 1)}$.

We state our main result in the form of Theorem 4.3.1 below.
Theorem 4.3.1

If a hypercubic design, \( d \) with parameters \( v = t^m \), \( t = 2 \) or \( 3 \), \( k = 2 \) or \( 3 \), \( \lambda = \lambda_1 + 1 \) exists and satisfy the condition (4.3.2), then \( d \) is an E-optimal RGD in Dr.

Proof

Firstly, we prove that the hypercubic design with parameters \( v = t^m \), \( m = 2 \), \( k = 2 \) or \( 3 \), \( \lambda = \lambda_1 + 1 \) is an RGD. To prove this, consider the concurrence matrix \( N_dN_d^t \) of the hypercubic design.

We know that, for a PBIB design \( \sum_{j=1}^{b} n_{ij}^2 = r \) and \( \sum_{j=1}^{b} n_{ij} n_{ij} = \lambda p \) when \( i^{th} \) and \( i^{th} \) treatments are \( p^{th} \) associates for \( p = 1, 2, \ldots, m \).

So \( N_dN_d^t = \sum_{j=1}^{b} n_{ij} n_{ij} = \lambda p \) if \( i \neq i^{1} \)

\[ = r \quad \text{if} \quad i = i^{1} \]

In this case, \( v = 2^2 \) or \( 3^2 \) i.e. \( p = 1 \) or \( 2 \), \( m = 2 \) and \( \lambda = \lambda_1 + 1 \).

Therefore, the diagonal elements of \( N_dN_d^t \) are all equal to \( r \) and off-diagonal elements are \( \lambda_1 \) or \( \lambda_2 \) according as \( i^{th} \) and \( i^{th} \) treatments are first or second associates. Since \( \lambda_2 = \lambda_1 + 1 \), the off-diagonal elements differing by at most one. So the hypercubic design is an RGD.

Now, by following on the similar lines of Theorem 3.3 of Jacroux (1980) (Theorem 4.2.3.1 in this Chapter). The design \( d \) is an E-optimal RGD in Dr.

This Theorem can be illustrated by the following examples.
4.3.2 Numerical Examples

Example 4.3.1

Consider a hypercubic design $d$ with parameters $v = 2^2$, $b = 4$, $r = 2$, $k = 2$, $\lambda_1 = 0$ and $\lambda_2 = 1$ (see example 3.3.2) whose allocation of treatments to blocks is given by

$$(11, 12), (21, 22), (11, 21), (12, 22)$$

Replacing 11 by 1, 12 by 2, 21 by 3 and 22 by 4, the design becomes $(1, 2), (3, 4), (1, 3)$ and $(2, 4)$.

The concurrence matrix $N_dN_d^t$ obtained from $d$ is

$$N_dN_d^t = \begin{pmatrix} 2 & 1 & 1 & 0 \\ 1 & 2 & 0 & 1 \\ 1 & 0 & 2 & 1 \\ 0 & 1 & 1 & 2 \end{pmatrix}$$

Here all diagonal elements of $N_dN_d^t$ are equal and all of its off diagonal elements are differing by at most one. Then the design $d$ is an RGD.

The C-matrix of this design is

$$C_d = \begin{pmatrix} 1 & -1/2 & -1/2 & 0 \\ -1/2 & 1 & 0 & -1/2 \\ -1/2 & 0 & 1 & -1/2 \\ 0 & -1/2 & -1/2 & 1 \end{pmatrix}$$
The eigenvalues of $C_d$ are 0, 1, 1 and 2.

Also note that the minimum non-zero eigenvalue of the C-matrix associated with this design is 1. That is, $Z_{dl} = 1$.

$\lambda$ is the greatest integer not exceeding $\frac{rk-r}{v-1}$ such that

$$\lambda = \frac{rk-r-m^l}{v-1}$$

for some non-negative integral value of $m^l (\leq v - 1)$. Here the possible values of $m^l$ are 1, 2, ..., $v-1$, i.e., 1,2,3 and the corresponding values of $\lambda$ are obtained as $\frac{1}{3}$, 0, $-\frac{1}{3}$. Among these values of $\lambda$, the greatest integer value not exceeding $\frac{rk-r}{v-1}$ is 0 which corresponds to $m^l = 2$.

Then $\max \{a, b\} = \max \{0, 0.5\} = 0.5$, where a and b defined earlier in section 4.3. Therefore by Theorem 4.3.1, since $Z_{dl} > \max (a,b)$, we conclude that the design d is an E-optimal RGD.

Example 4.3.2

Consider the hypercubic design d with parameters $v = 3^2$, $b = 6$, $r = 2$, $k = 3$, $\lambda_1 = 0$ and $\lambda_2 = 1$ (See example 3.3.5) whose allocation of treatments to blocks is given by (1, 5, 9); (2, 6, 7); (3, 4, 8); (1, 6, 8); (2, 4, 9) and (3, 5, 7) the concurrence matrix of the design d is
For the same reason discussed in Example 4.3.1, d is an RGD.

The C-matrix of the design is

\[
C_d = \begin{pmatrix}
\frac{4}{3} & 0 & 0 & 0 & -\frac{1}{3} & -\frac{1}{3} & 0 & -\frac{1}{3} & -\frac{1}{3} \\
0 & \frac{4}{3} & 0 & -\frac{1}{3} & 0 & -\frac{1}{3} & -\frac{1}{3} & 0 & -\frac{1}{3} \\
0 & 0 & \frac{4}{3} & -\frac{1}{3} & -\frac{1}{3} & 0 & -\frac{1}{3} & -\frac{1}{3} & 0 \\
0 & -\frac{1}{3} & -\frac{1}{3} & 4/3 & 0 & 0 & 0 & -\frac{1}{3} & -\frac{1}{3} \\
-\frac{1}{3} & 0 & -\frac{1}{3} & 0 & 4/3 & 0 & -\frac{1}{3} & 0 & -\frac{1}{3} \\
-\frac{1}{3} & -\frac{1}{3} & 0 & 0 & 0 & 4/3 & -\frac{1}{3} & -\frac{1}{3} & 0 \\
0 & -\frac{1}{3} & -\frac{1}{3} & 0 & -\frac{1}{3} & -\frac{1}{3} & 4/3 & 0 & 0 \\
-\frac{1}{3} & 0 & -\frac{1}{3} & -\frac{1}{3} & 0 & -\frac{1}{3} & 0 & 4/3 & 0 \\
-\frac{1}{3} & -\frac{1}{3} & 0 & -\frac{1}{3} & -\frac{1}{3} & 0 & 0 & 0 & 4/3
\end{pmatrix}
\]}
The eigenvalues of $C_d$ are $0, 1, 1, 1, 2, 2, 2$ and $2$.

Also note that the minimum non-zero eigenvalue of the $C$-matrix associated with this design is $1$, that is $Z_{d1} = 1$. Here the possible values of $m^1$ are $1, 2, 3 \ldots 8$ and the corresponding values of $\lambda$ are $\frac{3}{8}, \frac{2}{8}, \frac{1}{8}, 0, \frac{-1}{8}, \frac{-2}{8}, \frac{-3}{8}, \frac{-4}{8}$. Among these values of $\lambda$, the greatest integer value not exceeding \[ \frac{rk-r}{v-1} \] is $0$ which corresponds to $m^1 = 4$.

Then $\max \{a, b\} = \max \{\frac{6}{7}, 1\} = 1$. Since $Z_{d1} > \max \{a,b\}$, Theorem 4.3.1 implies that $d$ is an E-optimal RGD.

The following Theorem can be established as a generalization of Theorem 4.3.1.

**Theorem 4.3.2**

(a) If a hypercubic design $d$ having parameters $v = t^m$ ($t = 2$ or $3$), $k$, $r$, $\lambda_i$ ($i = 1, 2, \ldots, m$) with all pairs of $\lambda_i$'s differing by at most one exists, then $d$ is an RGD.

(b) If $d$ satisfies the condition (4.3.2) also, then $d$ is an E-optimal RGD in $D_r$.

**Proof**

(a) By definition, if the concurrence matrix $N_d N_d^{-1}$ of the design $d$ has all of its diagonal elements are equal and all of its off-diagonal elements differing by at most one, then $d$ is an RGD.
We have \( N_d N_d^\dagger = \sum_{j=1}^{b} n_{ij} n_{ij} = r \) if \( i = i^\dagger \)
\[
= \lambda_p \quad \text{if} \quad i \neq i^\dagger
\]
when \( i \) and \( i^\dagger \) are \( p^{th} \) associate treatments \( p = 1, 2, \ldots m \).

The diagonal elements of \( N_d N_d^\dagger \) are all equal to \( r \), and off-diagonal elements are \( \lambda i \ (i = 1, 2, \ldots m) \) If all pairs of \( \lambda i \) is differing by at most one, the off diagonal elements of \( N_d N_d^\dagger \) also differing by at most one, then the hypercubic design \( d \) is an RGD by definition.

(b) If the design \( d \) satisfies the condition (4.3.2), it is an E-optimal RGD by Theorem 4.3.1. This completes the proof.

Here we verified that the hypercubic designs constructed by Theorem 3.2.1 of Chapter 3 with parameters

(i) \( t = 2, \ m = 2, \ k = 2 \) \quad (ii) \( t = 2, \ m = 3, \ k = 2 \)
(iii) \( t = 2, \ m = 4, \ k = 2 \) \quad (iv) \( t = 3, \ m = 2, \ k = 3 \)
(v) \( t = 3, \ m = 3, \ k = 3 \) \quad (vi) \( t = 3, \ m = 4, \ k = 3 \) and
(vii) \( t = 3, \ m = 5, \ k = 3 \)

and all the hypercubic designs constructed by Theorem 3.2.2 of Chapter 3 satisfy the condition (a) of Theorem 4.3.2. So these designs are RG Designs. The E-optimality of some of these designs are discussed below.
Example 4.3.3

Consider the hypercubic design, d with parameters $v = 2^3$, $b = 12$, $r = 3$, $k = 2$, $\lambda_1 = 1$, $\lambda_2 = \lambda_3 = 0$ whose allocation of treatments to blocks is given by

$$(1,2); (3,4); (5,6); (7,8),$$
$$(1,3); (2,4); (5,7); (6,8),$$
$$(1,5); (2,6); (3,7); (4,8).$$

The concurrence matrix obtained from d is

$$N_d N_d^\dagger = \begin{pmatrix}
3 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\
1 & 3 & 0 & 1 & 0 & 1 & 0 & 0 \\
1 & 0 & 3 & 1 & 0 & 0 & 1 & 0 \\
0 & 1 & 1 & 3 & 0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 & 3 & 1 & 1 & 0 \\
0 & 1 & 0 & 0 & 1 & 3 & 0 & 1 \\
0 & 0 & 1 & 0 & 1 & 0 & 3 & 1 \\
0 & 0 & 0 & 1 & 0 & 1 & 1 & 3
\end{pmatrix}$$

Here all the diagonal elements of $N_d N_d^\dagger$ are equal and all of its off-diagonal elements are differing by at most one. So it is an RG design. The eigenvalues of the C-matrix associated with this design are $0, 1, 1, 1, 2, 2, 2$ and $3$. Here the minimum non-zero eigenvalue of the C-matrix is $1$, that is $Z_{d1} = 1$ and

$$\text{Max } \{a,b\} = \max \left(\frac{2}{3}, 1\right) = 1.$$ 

So by Theorem 4.3.2, d is an E-optimal RGD.
Example 4.3.4

Consider the hypercubic design $d$ with parameters $v = 2^4$, $b = 32$, $r = 4$, $k = 2$, $\lambda_i = 1$, $\lambda_2 = \lambda_3 = \lambda_4 = 0$, Constructed in example 3.3.1.a., whose allocation of treatments to blocks is given by

\[(1,2), (3,4), (5,6), (7,8), (9,10), (11,12), (13,14), (15,16)\]
\[(1,3), (2,4), (5,7), (6,8), (9,11), (10,12), (13,15), (14,16)\]
\[(1,5), (2,6), (3,7), (4,8), (9,13), (10,14), (11,15), (12,16)\]
\[(1,9), (2,10), (3,11), (4,12), (5,13), (6,14), (7,15), (8,16)\]

The concurrence matrix obtained from the design is

\[
\begin{pmatrix}
4 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 4 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 4 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 1 & 4 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 4 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 1 & 4 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 1 & 0 & 4 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 1 & 1 & 4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 4 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 4 & 0 & 1 & 0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 4 & 1 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 4 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 4 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 4 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 4 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 4 \\
\end{pmatrix}
\]

\[N_dN_d^\perp =
\]
Here all the diagonal elements of $N_dN_d^1$ are equal and all of its off-diagonal elements are differing by at most one. So the design is an RG design. The eigenvalues of the C-matrix associated with this design are 0,1,1,1,1,2,3,3,3,3 and 4. Here the minimum non-zero eigenvalue is 1 that is $Z_{d1}=1$.

Here the possible values of $m^1$ are 1,2, ….,15 and corresponding values of $\lambda$ are $\frac{3}{15},\frac{2}{15},\frac{1}{15},0,-\frac{1}{15},-\frac{2}{15},…….,-\frac{11}{15}$. Among these values of $\lambda$, the greatest integer value not exceeding $\frac{rk-r}{v-1}$ is 0 which corresponds to $m^1=4$.

In this case $\max\{a,b\}=\max\left\{\frac{8}{7},\frac{3}{2}\right\}=\frac{3}{2}$.

But $Z_{d1}<\max\{a,b\}$, so $d$ is not an E-optimal RGD.

4.3.3 Duals

The dual of an incomplete block design is obtained by interchanging its blocks and treatments. So for each $d\in Dr$ there exists a dual design $d^*$ whose incidence matrix is $N_d^* = N_d^1$. Note that each $d^*$ belongs to the class $Dr^*$ having parameters $v^*=b$, $b^*=v$, $k^*=r$ and $r^*=k$. The results of Shah et. al. (1976) imply that a design $d$ is E-optimal in $Dr$ if and only if $d^*$ is E-optimal in $Dr^*$.

In this investigation the duals of the hypercubic designs obtained from Examples 4.3.1, 4.3.2 and 4.3.3 also yield E-optimal RG designs.

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