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SUMMARY

‘Fixed Point Theory’ is a beautiful mixture of analysis (pure and applied), topology and geometry. Fixed point theorems give the conditions under which mappings (single or multivalued) have solutions. Fixed point theory in probabilistic metric spaces can be considered as a part of Probabilistic Analysis, which is a very dynamic area of mathematical research.

The idea of introducing probabilistic notions into geometry was one of the great thoughts of Karl Menger. His motivation came from the idea that positions, distances, areas, volumes, etc., all are subject to variation in measurement in practice. And, as, e.g., quantum mechanics implies, even in theory some measurements are necessarily inexact. In 1942, Menger [27] published a note entitled Statistical Metrics. The idea of Menger was to use distribution functions instead of nonnegative real numbers as values of the metric. The notion of a probabilistic metric space corresponds to the situations when we do not know exactly the distance between two points, we know only probabilities of possible values of this distance. In this note he explained how to replace the numerical
distance between two points \( u \) and \( v \) by a function \( F_{u,v} \) whose value \( F_{u,v}(x) \) at the real number \( x \) is interpreted as the probability that the distance between \( u \) and \( v \) is less than \( x \). Schweizer and Sklar [38] took up the work, initiated by Menger [27] and developed what is now called the theory of probabilistic metric spaces [see, 39].

Chapter I is introductory in nature. In this chapter, some probabilistic topological preliminaries are collected and a brief survey of the development of fixed point theory in PM-spaces is also presented.

Chapter II is devoted to fixed points of contraction mappings. The first effort of study of contraction mappings in PM-spaces was made by Sehgal [40] in his doctoral dissertation in 1966. Studies by several fixed point theorist have culminated in an elegant theory of fixed point theorems in probabilistic metric spaces which have far reaching consequences and are useful in the study of existence of solutions of operator equations in probabilistic metric spaces and probabilistic functional analysis. In this chapter, we obtain some
common fixed point theorems for triplet and quadruplet of mappings satisfying new contraction conditions in Menger spaces. These results improve and extend some well-known result of Singh and Pant [42] and Singh, Mishra and Pant [41]. In the end of this chapter, we prove a common fixed point theorem for pointwise $R$-weakly commuting mappings having reciprocal continuity and satisfying an implicit relation. Theorems 2.1-2.2 have been published in [32] and Theorems 2.3-2.4 have been published in [25].

Chapter III is intended to the study of fixed points of expansion mappings. Banach contraction principle also yields a fixed point theorem for a diametrically opposite class of mappings viz. expansion mappings. The study of fixed point of single expansion mapping in a metric space is initiated by Wang, Li, Gao and Iseki [43]. In 1987, Pant, Dimri and Singh [31] initiated the study of fixed points of expansion mappings in Menger spaces. The first result in this chapter is for four expansion mappings, two of them being surjective, via compatibility of mappings in Menger spaces. Theorem 3.3 is an improvement of Kumar [23, Theorem 3.2]. Also, Theorems 3.3-3.6 have been proved for non-surjective expansion
mappings. Theorem 3.1 has been published in [24]. Theorems 3.3-3.4 have been accepted for publication in [5].

The purpose of chapter IV is to establish coincidence and common fixed point theorems for certain classes of nearly densifying mappings in complete Menger space. The concept of probabilistic densifying mappings was introduced by Bocşan [1]. In [15], Ganguly, Rajput and Tuteja introduced the notion of probabilistic nearly densifying mappings. Our results extend the results of Khan and Liu [21] to PM-spaces and of Ganguly, Rajput and Tuteja [15] as well. Theorems 4.1-4.3 have been published in [34].

Chapter V is devoted to study of related fixed point theorems. In 2002, Pant [30] initiated the study of the relation between the fixed points of two contraction mappings in two different Menger spaces by generalizing the results of Fisher [12, 13] to PM-spaces. Theorem 5.2 extends the results of Pant [30] to two pairs of mappings. Theorem 5.3 is an interesting generalization of the results of Fisher and Murthy [14], Jain [18] and Jain, Sahu and Fisher [19] to PM-spaces. Theorem 5.4 is an extension of Pant [30,
Theorem 2] for three mappings in three different Menger spaces and is a generalization of the result of Nung [29] to PM-spaces. Theorem 5.2 has been published in [33].

In the last chapter, applications of fixed point theory (especially, Menger probabilistic metric spaces) are mentioned. The concept of PM-spaces may have very important applications in quantum particle physics particularly in connections with both string and $\epsilon^\infty$ theory, which were introduced and studied by a well-known scientist, Mohamed Saladin El Naschie [6–9]. It is also of fundamental importance in probabilistic functional analysis, nonlinear analysis and applications [2, 16]. In the theory of PM-spaces, contraction is one of the main tools to prove the existence and uniqueness of a fixed point. In 1996, a group of mathematicians Chang, Lee, Cho, Chen, Kang and Jung [3] presented a research paper in which they obtained a generalized contraction mapping principle in PM-spaces and applied it to prove the existence theorems of solutions to differential equations in these spaces. In 1968, the concept of fuzzy sets was introduced by Zadeh [44]. Various authors, for example, Deng [4], Ereeg [10], Fang [11], Kaleva and Seikkala [20], Kramosil and Michalek [22] have
introduced the concept of fuzzy metric spaces in different ways. Fixed-point theory in fuzzy metric spaces for different contractive-type mappings is closely related to that in probabilistic metric spaces (refer [2, Chapters VIII, IX], [16, Chapters 3–5], [28]). Various authors, for example, Hadžić and Pap [17], Razani and Shirdaryazdi [37], Razani and Kouladgar [36] and Liu and Li [26] have studied the applications of fixed point theorems in PM-spaces to fuzzy metric spaces. As an application of some of our results namely, Theorems 2.1, 2.3, 2.5 and 3.3 to fuzzy metric spaces, we give here Theorems 6.1, 6.2, 6.3 and 6.4 respectively. It is worth mentioning that Theorem 6.3 is an extension of the result of Pant and Jha [35] to implicit relation and Theorem 6.4 is an improvement of Kumar [23, Theorem 4.1] in the sense that we have taken completeness of one of the subspaces, not the whole space.
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