"Traditionally, a dynamicist would believe that to write down a system's equations is to understand the system. How better to capture the essential features? But because of the little bits of non-linearity...[he] would find himself helpless to answer the easiest questions about the future of the system. A computer can address the problem by simulating it..."


4.0. Introduction

Physical System Theory has long been used for the modelling and analysis of systems where domain specificity is limited only to that subset of the system description which represents terminal equations characterizing component or phenomenological behavior, while the other subset representing the system interconnection structure, is independent of the domain of application. The form of the "global" system equations tends to remain independent of the system under consideration. However, domain specific characteristics play a role in simulating system behavior under given operating conditions. The physical systems approach to modelling is based on the identification of a system graph embodying the inter-connection structure of components in the system, and the appropriate aggregation of the interconnection constraint equations implied by
the system graph and the constituent component terminal characteristics (behavioral characteristics). Notice that domain specific fundamental through and across variables are employed in the identification of component behavioral characteristics.

Techniques for the formulation of system equations are dependent to some extent upon the form in which the component terminal equations are supplied/expressible and have therefore evoked considerable interest [GIL59] [KOE61]. For many applications, particularly digital ones, time domain behavior analysis is often of critical importance in the course of the design cycle [SPE86]. Numerous efficient system theoretic equation formulation techniques have been proposed, more notable amongst which are the graph theoretic and state variable approaches [KOE61] [KOE66] [KUH65] [ROE67] [ROH70]. In the graph theoretic approach of Physical System Theory, depending upon the form of the terminal characteristics of the system components, one or more of branch, chord, branch-chord and state variable formulations may be adopted as a procedure for the development of system equations.

On the other hand System Dynamics focusses on the structure and behavior of systems composed of interacting feedback loops. Causal loop diagrams identify the principal feedback loops without distinguishing between the nature of the interconnected variables. During model development, they serve as preliminary sketches of causal hypotheses, thereby simplifying the illustration of a model. Underlying structural assumptions in the model are easily communicated. As already mentioned in Chapter 3, the system dynamics flow diagram offers a very straightforward and convenient
approach to modelling physical systems$^{39}$ and lends itself naturally to the simulation approach for the study of such systems [G0074].

Underlying concepts of Linear/Nonlinear System Theory and State Modelling may be employed to gain an improved insight into System Dynamics Modelling. In Physical System Theory, the system structure is assumed to be known in advance and one proceeds to obtain the system model and its solution based on the underlying structure and the constituent terminal characteristics. In System Dynamics, one attempts to identify variables and their inter-dependencies through causal-loop diagramming and then, using flow-diagrams in conjunction with dynamo-equations, simulate the system behavior [G0074]. Correspondence with measured/empirical data may sometimes help refine the model further.

In a broader perspective, the concepts of Physical System Theory (including State Modelling) may be used to identify a certain underlying structure in the basic model formulation process of System Dynamics. This could prove to be useful in the identification of system variables which would form a part of the simulation model, a process which is still largely based on heuristics or group consensus rather than formal structure and theory (such as Physical System Theory). Apart from this, concepts of system stability, observability, controllability, sensitivity and optimality could also be appropriately applied to the system under consideration.

In this chapter we investigate the formulation of a Generalized System Dynamics Model for time-domain simulation of non-degenerate$^{40}$

---

39 Physical systems are all those systems in which the quantities of interest can be measured [KOE61] [KOE66].

40 Non-degenerate systems are those where the system graph may be successfully partitioned into a tree and co-tree, such that all "capacitive" elements together with all across drivers appear in the tree, and all "inductive" elements together with all through
and degenerate circuits and systems by bringing the modelling methodology of Physical System Theory to bear upon System Dynamics. Towards this end the following notation will be consistently followed:

\[
\begin{align*}
\text{e} & : \text{number of edges in the system graph;} \\
\text{n}_a & : \text{number of independent across drivers in the system;} \\
\text{n}_t & : \text{number of independent through drivers in the system;} \\
\text{n}_{bc} & : \text{number of non-degenerate "capacitor" states in the tree;} \\
\text{n}_{cl} & : \text{number of non-degenerate "inductor" states in the co-tree;} \\
\text{n}_{br} & : \text{number of "resistive" elements in the tree;} \\
\text{n}_{cr} & : \text{number of "resistive" elements in the co-tree.}
\end{align*}
\]

4.1. Conventional Graph Theoretic Approach

Conventional graph theoretic approaches start out with an oriented linear graph (or network graph or system graph), which is subsequently partitioned into a tree and a co-tree. Fundamental circuit and cut-set equations are then formulated: one circuit equation for each chord, and one cut-set equation for each branch. As an example consider the branch formulation technique for a network or system whose graph has 'e' edges and 'v' vertices with the following notation:

\[
\begin{align*}
X_{b1}, Y_{b1} & : \text{across and through variables of specified across drivers;} \\
X_{b2}, Y_{b2} & : \text{across and through variables of non-source branches;} \\
X_{c1}, Y_{c1} & : \text{across and through variables of non-source chords;} \\
X_{c2}, Y_{c2} & : \text{across and through variables of specified through drivers.}
\end{align*}
\]

\(^{40}\) ...continued

Drivers appear in the co-tree. Systems where this partitioning is not possible, are called degenerate systems. Degeneracy may arise as a result of a circuit of "capacitors" and/or across drivers, and/or a cutset of "inductors" and/or through drivers.
System equations may then be formulated as follows:

\[
\begin{bmatrix}
  Y_{b1} \\
  U \ 0 \ A_{11} \ A_{12} \ Y_{b2} \\
  0 \ U \ A_{21} \ A_{22} \ Y_{c1} \\
  Y_{c2}
\end{bmatrix}
- \ 0
\] ... (4.1) ....(v-1) fundamental cut-set equations;

\[
\begin{bmatrix}
  X_{b1} \\
  B_{11} \ B_{12} \ U \ 0 \ X_{b2} \\
  B_{21} \ B_{22} \ 0 \ U \ X_{c1} \\
  X_{c2}
\end{bmatrix}
- \ 0
\] ... (4.2) .... (v+1) fundamental circuit equations;

where \( U \) is an identity matrix. \( A_{ij} \)'s are cutset sub-matrices, \( B_{ij} \)'s are circuit sub-matrices, and

\[
\begin{bmatrix}
  B_{11} \ B_{12} \\
  B_{21} \ B_{22}
\end{bmatrix} \ - \ \begin{bmatrix}
  A_{11} \ A_{12} \\
  A_{21} \ A_{22}
\end{bmatrix}^T
\] ... (4.3)

as a result of the orthogonality between cutset-space and circuit-space.

The component terminal equations are required in the form explicit in through variables or what is known as "short circuit" parameter form 41.

\[
\begin{bmatrix}
  Y_{b2} \\
  Y_{c1}
\end{bmatrix}
- \begin{bmatrix}
  G_{b2} \ 0 \\
  0 \ G_{c1}
\end{bmatrix}
\begin{bmatrix}
  X_{b2} \\
  X_{c1}
\end{bmatrix}
+ \begin{bmatrix}
  F_{b2} \\
  F_{c1}
\end{bmatrix}
\]

\( F_{b2}, F_{c1} \) : time dependent functions.

(174)
From the fundamental cut-set equations we have:

\[
\begin{bmatrix} Y_{b2} \\ Y_{c1} \end{bmatrix} = \begin{bmatrix} G_{b2} & 0 \\ 0 & G_{c1} \end{bmatrix} \begin{bmatrix} X_{b2} \\ X_{c1} \end{bmatrix} \quad \ldots (4.4)
\]

Substituting the component terminal equations from (4.4) in (4.5):

\[
\begin{bmatrix} 0 \\ U \end{bmatrix} Y_{b1} + \begin{bmatrix} 0 & A_{11} \\ U & A_{21} \end{bmatrix} \begin{bmatrix} Y_{b2} \\ Y_{c1} \end{bmatrix} + \begin{bmatrix} A_{12} \\ A_{22} \end{bmatrix} Y_{c2} = 0 \quad \ldots (4.5)
\]

From the fundamental circuit equations we have:

\[
\begin{bmatrix} X_{b2} \\ X_{c1} \end{bmatrix} = \begin{bmatrix} 0 & U \\ -B_{11} & -B_{12} \end{bmatrix} \begin{bmatrix} X_{b1} \\ X_{b2} \end{bmatrix} = \begin{bmatrix} 0 & U \\ A_{11}^T & A_{21}^T \end{bmatrix} \begin{bmatrix} X_{b1} \\ X_{b2} \end{bmatrix} \quad \ldots (4.7)
\]

Which upon substitution in (4.6) yield system equations in the final form:

\[
\begin{bmatrix} 0 \\ U \end{bmatrix} Y_{b1} + \begin{bmatrix} 0 & A_{11} \\ U & A_{21} \end{bmatrix} \begin{bmatrix} G_{b2} & 0 \\ 0 & G_{c1} \end{bmatrix} \begin{bmatrix} X_{b1} \\ X_{b2} \end{bmatrix} + \begin{bmatrix} A_{12} \\ A_{22} \end{bmatrix} Y_{c2} = 0 \quad \ldots (4.8)
\]

The complete equation count for branch formulation (when the terminal equations are explicit in the through (current) variables) is \((v-1-n_1)\). When the terminal equations are, however, presented in the explicit across (voltage) variable form, chord formulation techniques may be resorted to. For hybrid forms, branch-chord formulation may be adopted.
4.2. Alternative Formulation

The alternative formulation technique proposed is a modification of the graph theoretic approach outlined above and is suited to the time domain simulation of linear dynamical systems for the non-degenerate case. The concept may however be readily extended to include the degenerate case as well.

In the following equations, the interconnection constraint equations are partitioned according to the variable classification shown:

\[
\begin{bmatrix}
B_{11} & B_{12} & B_{13} & U & 0 & 0 \\
B_{21} & B_{22} & B_{23} & 0 & U & 0 \\
B_{31} & B_{32} & B_{33} & 0 & 0 & U
\end{bmatrix}
\begin{bmatrix}
X_{bc} \\
X_{ac} \\
X_{d}
\end{bmatrix}
- 
\begin{bmatrix}
X_{be} \\
X_{ce} \\
X_{d}
\end{bmatrix}
= 
0 
\tag{4.9}
\]

\((e-v+1)\) fundamental circuit equations;

\[
\begin{bmatrix}
U & 0 & 0 & A_{11} & A_{12} & A_{13} \\
0 & U & 0 & A_{21} & A_{22} & A_{23} \\
0 & 0 & U & A_{31} & A_{32} & A_{33}
\end{bmatrix}
\begin{bmatrix}
Y_{bc} \\
Y_{ac} \\
Y_{d}
\end{bmatrix}
- 
\begin{bmatrix}
Y_{be} \\
Y_{ce} \\
Y_{d}
\end{bmatrix}
= 
0 
\tag{4.10}
\]

\((v-1)\) fundamental cut-set equations;

where

\(X_{be}, Y_{be}: \) across and through variables of independent and/or dependent\(^42\) across drivers;

\(\text{If the dependent driver categories are a part of } e \text{ and } i, \text{ then } X_{be} \text{ and } Y_{ci} \text{ have to be determined as per criteria of determination } [KDBS66].\)

(176)
\(X_{bc}, Y_{bc} \) : across and through variables of branch "capacitors";
\(X_{br}, Y_{br} \) : across and through variables of branch "resistive" elements;
\(X_{cr}, Y_{cr} \) : across and through variables of chord "resistive" elements;
\(X_{cl}, Y_{cl} \) : across and through variables of chord "inductors";
\(X_{ci}, Y_{ci} \) : across and through variables of independent and/or dependent through drivers.

4.2.0. Branch Formulation

According to the foregoing variable classification the following partitioning of the fundamental cutset equations may be considered:

\[
\begin{bmatrix}
U & 0 \\
0 & U \\
0 & 0
\end{bmatrix}
\begin{bmatrix}
Y_{bc} \\
Y_{br} \\
Y_{cr}
\end{bmatrix}
+ 
\begin{bmatrix}
0 & A_{11} \\
0 & A_{21} \\
U & A_{31}
\end{bmatrix}
\begin{bmatrix}
Y_{br} \\
Y_{cr} \\
X_{cr}
\end{bmatrix}
+ 
\begin{bmatrix}
A_{12} & A_{13} \\
A_{22} & A_{23} \\
A_{32} & A_{33}
\end{bmatrix}
\begin{bmatrix}
Y_d \\
Y_d \\
Y_d
\end{bmatrix}
= 0 
\ldots (4.11)
\]

The resistive component terminal equations may be expressed as:

\[
\begin{bmatrix}
Y_{br} \\
Y_{cr}
\end{bmatrix}
= 
\begin{bmatrix}
G_{br} & 0 \\
0 & G_{cr}
\end{bmatrix}
\begin{bmatrix}
X_{br} \\
X_{cr}
\end{bmatrix}
\ldots (4.12)
\]

which when substituted into (4.11) yield:

\[
\begin{bmatrix}
U & 0 \\
0 & U \\
0 & 0
\end{bmatrix}
\begin{bmatrix}
Y_{bc} \\
Y_{br} \\
Y_{cr}
\end{bmatrix}
+ 
\begin{bmatrix}
0 & A_{11} \\
0 & A_{21} \\
U & A_{31}
\end{bmatrix}
\begin{bmatrix}
G_{br} & 0 \\
0 & G_{cr}
\end{bmatrix}
\begin{bmatrix}
X_{br} \\
X_{cr}
\end{bmatrix}
+ 
\begin{bmatrix}
A_{12} & A_{13} \\
A_{22} & A_{23} \\
A_{32} & A_{33}
\end{bmatrix}
\begin{bmatrix}
Y_d \\
Y_d \\
Y_d
\end{bmatrix}
= 0 
\ldots (4.13)
\]

From the fundamental circuit equations we may write:

\(177\)
On substitution into equations (4.13) appropriately, this further yields:

\[
\begin{bmatrix}
U_0 \\
0 \\
0
\end{bmatrix}
\begin{bmatrix}
Y_{bc}
\end{bmatrix}
+ \begin{bmatrix}
0 & A_{11} \\
0 & A_{21} \\
U & A_{31}
\end{bmatrix}
\begin{bmatrix}
G_{bc} \\
0 \\
G_{a1}
\end{bmatrix}
\begin{bmatrix}
0 & 0 & U
\end{bmatrix}
\begin{bmatrix}
X_{bc}
\end{bmatrix}
\]

\[\ldots(4.15)\]

or

\[
\begin{bmatrix}
U_0 \\
0 \\
0
\end{bmatrix}
\begin{bmatrix}
Y_{bc}
\end{bmatrix}
+ \begin{bmatrix}
A_{12} & A_{13} \\
A_{22} & A_{23} \\
A_{32} & A_{33}
\end{bmatrix}
\begin{bmatrix}
Y_d
\end{bmatrix}
= 0
\]

In the conventional branch formulation approach described earlier (equations (4.8)), the final sets of equations are both algebraic and integro-differential in form (often in s-domain or some other operator domain), with a total branch equation count of \(v-1-n_1\). However, in the modified formulation, the equations are purely algebraic in nature, thus requiring no Laplace or other operator

(178)
transforms during the solution process, and the branch count is also reduced to \( n_{br} = (v-1-n_x-n_{bc}) \) (since \( n_{bc} \) "capacitor" state across variables are assumed to be known at the starting instant of each simulation time-point).

It may be noted that the bottom set of \( n_{br} \) equations are completely decoupled from the \( n_x \) top and \( n_{bc} \) middle sets of equations in (4.16). \( X_{be} \) and \( Y_{ci} \) on the one hand, and \( X_{bc} \) and \( Y_{cl} \) on the other may be regarded as known quantities at the beginning of each subsequent time step (since across and through driver variables are specified time functions, and branch "capacitor" across variables and chord "inductor" through variables (i.e., states) are determined through simulation based on information at the preceding step) and the unknowns are the vectors \( Y_{be} \), \( Y_{bc} \) and \( X_{br} \). Once the vector \( X_{br} \) is obtained from the bottom set of \( n_{br} \) equations in (4.16), the chord element across variable vectors \( X_{cr} \), \( X_{cl} \) and \( X_{ci} \) may be obtained by invoking the fundamental circuit equations as follows:

\[
\begin{bmatrix}
X_a \\
X_d \\
X_{ad}
\end{bmatrix} =
\begin{bmatrix}
B_{11} & B_{12} & B_{13} \\
B_{21} & B_{22} & B_{23} \\
B_{31} & B_{32} & B_{33}
\end{bmatrix}
\begin{bmatrix}
X_{be} \\
X_{bc} \\
X_{br}
\end{bmatrix}
\]

\[\ldots(4.17)\]

Also, \( Y_{be} \) and \( Y_{bc} \) are obtained directly from the top and middle sets of equations in (4.16) respectively, while the remaining through variables \( Y_{br} \) and \( Y_{cr} \) are computable from equation (4.12).

4.2.1. Simulation Procedure

The DYNAMO flow diagram for the alternative branch formulation technique is shown in Fig.4.1. This is a generalized flow diagram for the system under consideration. In the model, the level
variables are "capacitor charges" and "inductor flux linkages", and the rate variables naturally become the "capacitor" fundamental through variables and "inductor" fundamental across variables. Inputs to the system model are of three kinds: independent driver vectors; submatrices of the reduced incidence matrix; and branch/chord "admittance" submatrices.

For the time domain simulation process, the vectors $X_{bc}$, $Y_{ci}$, $X_{bc}$, and $Y_{ci}$, are all known quantities at each simulation timepoint as indicated earlier. It can also be observed that it is possible to partition Eqn.(4.16) into three completely decoupled sets of equations. Furthermore, no simultaneous solution of equations is required for the first two sets of equations. However, in the course of solution of the third set of equations (i.e. while solving for $X_{br}$), the coefficient submatrix of the $X_{br}$ vector requires inversion. In the worst case, the dimension of this matrix required to be inverted is $n_{br} \times n_{br}$. However, generally, the matrix may be smaller than this. DYNAMO equations for the flow diagram of Fig.4.1 are shown below as per standard notation given in [G0074]:

\[
\begin{align*}
QBC.K &= QBC.J + DT(YBC.JK) \\
QBC &= QBC(0) \\
PHICL.K &= PHICL.J + DT(XCL.JK) \\
PHICL &= PHICL(0) \\
YBC.KL &= -(A23 YCI.K + A22 YCL.K + A21 GCR A11^T XBE.K + 
\quad A21 GCR A21^T XBC.K + A21 GCR A31^T XBR.K) \\
XCL.KL &= -B21 XBE.K - B22 XBC.K - B23 XBR.K \\
[GBR + A31 GCR A31^T] XBR.K &= -(A31 GCR A11^T XBE.K + 
\quad A31 GCR A21^T XBC.K + A33 YCI.K + A32 YCL.K)
\end{align*}
\]

43 "Capacitor charge" = integral of fundamental "capacitor" through variable and "Inductor flux linkage" = integral of fundamental "inductor" across variable.

44 In the equations presented, DYNAMO symbolism has been used in a generic sense, and the same equations could easily be simulated in a MATLAB simulation environment.

45a Equations of this kind have not been presented in the explicit form to emphasize that the solution could be carried out using a numerical technique such as Gaussian elimination, which is computationally more efficient than matrix inversion.
\[ X_{BC,K} = C^t \, Q_{BC,K} \]  \hspace{1cm} \text{...}(4.23)

\[ Y_{CL,K} = L^t \, \Phi_{ICL,K} \] \hspace{1cm} \text{...}(4.24)

where \( C^t = C^{-1} \) and \( L^t = L^{-1} \).

The fundamental aim of the simulation process is to solve for the dynamic states as represented by "capacitor charges" and "inductor flux linkages". The procedure for solution of these equations is outlined below:

1. Read in the reduced incidence matrix and the component terminal equations in the through variable explicit form (or "short circuit admittance" matrix form) for the system under consideration.
2. Suitably partition \( A \) according to the scheme outlined in section 4.2.0. Set up all initial "capacitor charges" and "inductor flux linkages".
3. Solve for \( X_{BC}, Y_{CL} \) from equations (4.23), (4.24).
4. Obtain \( X_{BR} \) from equation (4.22).
5. Obtain \( Y_{BC}, X_{CL} \) from equations (4.20), (4.21).
6. Update \( Q_{BC} \) and \( \Phi_{ICL} \) using equations (4.18), (4.19).
7. Advance the simulation timestep (of size \( DT \)) if the simulation is incomplete, and repeat steps 4 through 8.

4.2.2. Branch Formulation Example

To demonstrate the branch formulation technique that has been outlined in the earlier section, we take a simple electrical circuit in the form of a neural network, as shown in Fig.4.2, which is an exemplar of feedback Hopfield neural networks.
FIG. 4.2 EXAMPLE: 2 - NEURON HOPFIELD NEURAL NETWORK

FIG. 4.3 FORMAL NEURON WITH A LINEAR CHARACTERISTIC
Without any loss of generality, the neuron transfer characteristic has been assumed to be linear\textsuperscript{45}, as shown in Fig. 4.3, although it could as well have been assumed to be non-linear (sigmoidal) with minor changes in the formulation. Therefore,

\[ v_o - \lambda v_i \]  \hspace{1cm} ...(4.25)

The corresponding terminal graph shown in Fig. 4.4, models the neuronal characteristic on a three terminal basis. The terminal characteristic equations for the linear case are:

\[
\begin{bmatrix}
    l_i \\
    v_o
\end{bmatrix} =
\begin{bmatrix}
    0 & 0 \\
    \lambda & 0
\end{bmatrix}
\begin{bmatrix}
    v_i \\
    i_o
\end{bmatrix}
\]  \hspace{1cm} ...(4.26)

Accordingly, the graph of the network may be drawn as shown in Fig. 4.5 with the tree\textsuperscript{46} selected as indicated in double lines (element numbers have been indicated in circles, and node numbers in triangles). We thus have, \( e = 12 \), and \( v = 5 \). In accordance with this selection of the tree, one can write the fundamental cut-set and circuit equations as shown below.

\textsuperscript{45} A simple linear neuron is assumed to act like a weighted voltage summer, drawing no current on its input terminal, and producing an output voltage which is linearly amplified by some gain scale factor.

\textsuperscript{46} In the selection of the tree, the output of the neuron has been taken as a voltage-controlled-voltage-source (VCVS), and thus absorbed into the tree, while the controlling voltage at the input has been put in the co-tree. The other two elements in the tree are the capacitor elements.
Δ: NODE NUMBERS
〇: ELEMENT NUMBERS

\[
\begin{bmatrix}
i_i \\
v_o
\end{bmatrix} = \begin{bmatrix} 0 & 0 \\ \lambda & 0 \end{bmatrix} \begin{bmatrix} v_i \\ i_o
\end{bmatrix}
\]

FIG. 4.4. TERMINAL NEURONAL MODEL AND ASSOCIATED LINEARIZED CHARACTERISTIC EQUATIONS.

→: BRANCHES OF THE TREE.
←: CHORDS OF THE CO-TREE

FIG. 4.5 GRAPh FOR THE CIRCUIT OF FIG. 4.2 SHOWING THE SELECTED FORMULATION TREE
\[(v-1) = 4 \text{ fundamental cut-set equations:}\]

\[
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & -1 & 0 & 1 & 0 & 1 & 0 & -1 & 0 \\
0 & 0 & 0 & 1 & 0 & -1 & 0 & 1 & 0 & 1 & 0 & -1
\end{bmatrix}
\begin{bmatrix}
i_1 \\
i_2 \\
i_3 \\
i_4 \\
i_5 \\
i_6 \\
i_7 \\
i_8 \\
i_9 \\
i_{10} \\
i_{11} \\
i_{12}
\end{bmatrix}
= 0 \quad \ldots (4.27)

\[(e-v+1) = 8 \text{ fundamental circuit equations:}\]

\[
\begin{bmatrix}
0 & -1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
v_1 \\
v_2 \\
v_3 \\
v_4 \\
v_5 \\
v_6 \\
v_7 \\
v_8 \\
v_9 \\
v_{10} \\
v_{11} \\
v_{12}
\end{bmatrix}
= 0 \quad \ldots (4.28)

(186)
The cut-set equations may be partitioned in the following manner:

\[
\begin{bmatrix}
1 & 0 & 0 & 0 & i_1 \\
0 & 1 & 0 & 0 & i_2 \\
0 & 0 & 1 & 0 & i_3 \\
0 & 0 & 0 & 1 & i_4
\end{bmatrix}
+ \begin{bmatrix}
0 & 1 & 0 & 0 & i_5 \\
1 & 0 & 0 & 0 & i_6 \\
-1 & 0 & 1 & 0 & i_7 \\
0 & -1 & 0 & 1 & i_8
\end{bmatrix}
+ \begin{bmatrix}
0 & 0 & 0 & 0 & i_9 \\
0 & 0 & 0 & 0 & i_{10} \\
1 & 0 & -1 & 0 & i_{11} \\
0 & 1 & 0 & -1 & i_{12}
\end{bmatrix} = 0 \quad (4.29)
\]

From the element characteristics, for the branch and chord resistive elements we may write:

\[
\begin{bmatrix}
i_5 \\
i_6 \\
i_7 \\
i_8
\end{bmatrix}
= \begin{bmatrix}
T_{12} & 0 & 0 & 0 \\
0 & T_{21} & 0 & 0 \\
0 & 0 & g_1 & 0 \\
0 & 0 & 0 & g_2
\end{bmatrix}
\begin{bmatrix}
v_5 \\
v_6 \\
v_7 \\
v_8
\end{bmatrix} \quad \ldots (4.30)
\]

where \( g_1 = 1/r_1 \) and \( g_2 = 1/r_2 \); \( r_1, r_2 \) : neuronal input impedances.

Also, from the fundamental circuit equations, we have:

\[
\begin{bmatrix}
v_5 \\
v_6 \\
v_7 \\
v_8
\end{bmatrix}
= \begin{bmatrix}
0 & 1 & -1 & 0 \\
1 & 0 & 0 & -1 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
v_1 \\
v_2 \\
v_3 \\
v_4
\end{bmatrix} \quad \ldots (4.31)
\]

Using equations (4.30) and (4.31) in equation (4.29), and noting that \( i_9 = i_{10} = 0 \) by definition of the neuronal characteristic, results in the final set of equations:

(187)
Equations (4.32) and (4.33) are completely decoupled and are the final simulation equations. They may be re-expressed in DYNAMO form as:

\[
\begin{bmatrix}
Q_{3.K} \\
Q_{4.K}
\end{bmatrix} = \begin{bmatrix}
Q_{3.J} \\
Q_{4.J}
\end{bmatrix} + DT \begin{bmatrix}
RTQ_{3JK} \\
RTQ_{4JK}
\end{bmatrix} 
\]  
\hspace{1cm} \cdots(4.34)

\[
Q_{3,0} - Q_{3}(0) \\
Q_{4,0} - Q_{4}(0)
\]  
\hspace{1cm} \cdots(4.35)

\[
\begin{bmatrix}
RTQ_{3.KL} \\
RTQ_{4.KL}
\end{bmatrix} - \begin{bmatrix}
^1i_{3.K} \\
^1i_{4.K}
\end{bmatrix} = \begin{bmatrix}
0 & T_{12} - (T_{12} + g_1) & 0 \\
T_{21} & 0 & -(T_{21} + g_2)
\end{bmatrix} \begin{bmatrix}
v_{1,K} \\
v_{2,K} \\
v_{3.K} \\
v_{4.K}
\end{bmatrix} + \begin{bmatrix}
^1i_{11.K} \\
^1i_{12.K}
\end{bmatrix} 
\]  
\hspace{1cm} \cdots(4.36)

(188)
Also,

\[
\begin{bmatrix}
1_{1,K} \\
1_{2,K} \\
1_{3,K} \\
1_{4,K}
\end{bmatrix} =
\begin{bmatrix}
-T_{21} & 0 & 0 & T_{21} \\
0 & -T_{12} & T_{12} & 0
\end{bmatrix}
\begin{bmatrix}
v_{1,K} \\
v_{2,K} \\
v_{3,K} \\
v_{4,K}
\end{bmatrix}
\]  
\( \cdots (4.37) \)

and

\[
\begin{bmatrix}
v_{1,K} \\
v_{2,K} \\
v_{3,K} \\
v_{4,K}
\end{bmatrix} =
\begin{bmatrix}
\lambda & 0 \\
0 & \lambda
\end{bmatrix}
\begin{bmatrix}
v_{3,K} \\
v_{4,K}
\end{bmatrix}
\]  
\( \cdots (4.38) \)

4.2.3. Chord Formulation Technique

According to the six-way variable classification, the following partitioning of the fundamental circuit equations may be considered:

\[
\begin{bmatrix}
B_{11} & B_{12} \\
B_{21} & B_{22} \\
B_{31} & B_{32}
\end{bmatrix}
\begin{bmatrix}
X_{\beta} \\
X_{\alpha}
\end{bmatrix} +
\begin{bmatrix}
B_{13} & U \\
B_{23} & 0 \\
B_{33} & 0
\end{bmatrix}
\begin{bmatrix}
X_{\alpha} \\
X_{\alpha}
\end{bmatrix} +
\begin{bmatrix}
0 & 0 \\
0 & U
\end{bmatrix}
\begin{bmatrix}
X_{d} \\
X_{d}
\end{bmatrix} = 0
\]  
\( \cdots (4.39) \)

The "resistive" component terminal equations in "open circuit" parameter (across variable explicit) form may be expressed as:

\[
\begin{bmatrix}
X_{\alpha} \\
X_{\alpha}
\end{bmatrix} =
\begin{bmatrix}
Z_{\alpha} & 0 \\
0 & Z_{\alpha}
\end{bmatrix}
\begin{bmatrix}
Y_{\alpha} \\
Y_{\alpha}
\end{bmatrix}
\]  
\( \cdots (4.40) \)

which when substituted into (4.39) yield:

(189)
From the fundamental cutset equations we may write:

\[
\begin{bmatrix}
Y_{br} \\
Y_{ae}
\end{bmatrix}
- \begin{bmatrix}
-A_{31} & -A_{32} & -A_{33}
\end{bmatrix}
\begin{bmatrix}
Y_{ae} \\
Y_{ae} \\
Y_{ae}
\end{bmatrix}
- \begin{bmatrix}
B_{13}^T & B_{23}^T & B_{33}^T
\end{bmatrix}
\begin{bmatrix}
Y_{ae} \\
Y_{ae} \\
Y_{ae}
\end{bmatrix}
\ldots \ldots (4.42)
\]

On substitution into equations (4.41) appropriately, this further yields:

\[
\begin{bmatrix}
B_{11} & B_{12} \\
B_{21} & B_{22} \\
B_{31} & B_{32}
\end{bmatrix}
\begin{bmatrix}
X_{bo} \\
X_{bo} \\
X_{bo}
\end{bmatrix}
+ \begin{bmatrix}
B_{13} & 0 \\
B_{23} & 0 \\
B_{33} & 0
\end{bmatrix}
\begin{bmatrix}
Z_{X} & 0 \\
0 & Z_{ae}
\end{bmatrix}
\begin{bmatrix}
Y_{ae} \\
Y_{ae} \\
Y_{ae}
\end{bmatrix}
+ \begin{bmatrix}
U & 0 & 0 \\
U & 0 & 0 \\
U & 0 & 0
\end{bmatrix}
\begin{bmatrix}
X_{d} \\
X_{d} \\
X_{d}
\end{bmatrix}
= 0 \ldots \ldots (4.43)
\]

or,

\[
\begin{bmatrix}
B_{11} & B_{12} \\
B_{21} & B_{22} \\
B_{31} & B_{32}
\end{bmatrix}
\begin{bmatrix}
X_{bo} \\
X_{bo} \\
X_{bo}
\end{bmatrix}
+ \begin{bmatrix}
0 & 0 \\
U & 0 \\
0 & U
\end{bmatrix}
\begin{bmatrix}
X_{d} \\
X_{d} \\
X_{d}
\end{bmatrix}
\begin{bmatrix}
B_{13} & B_{13}^T & B_{13}^T \\
B_{23} & B_{23}^T & B_{23}^T \\
B_{33} & B_{33}^T & B_{33}^T
\end{bmatrix}
\begin{bmatrix}
X_{d} \\
X_{d} \\
X_{d}
\end{bmatrix}
\ldots \ldots (4.44)
\]

Once the vector \( Y_{ae} \) is obtained from the top set of \( n_{c}\) \((= e-v+1-n_{cl}-n_{r})\) equations in (4.44), the branch element through variable vectors \( Y_{br} \), \( Y_{bc} \) and \( Y_{br} \) may be obtained by invoking the fundamental cutset equations as follows:
The system dynamics model for chord formulation is shown in Fig. 4.6. Once again, the equations are decoupled, and a single submatrix inversion of the order \( nc_r \times nc_r \) is required in the worst case. DYNAMO equations may be derived easily, and have been indicated below:

\[
\begin{bmatrix}
Y_w \\
Y_{bc} \\
Y_{bc}
\end{bmatrix} = \begin{bmatrix}
-A_{11} & -A_{12} & -A_{13} \\
-A_{21} & -A_{22} & -A_{23} \\
-A_{31} & -A_{32} & -A_{33}
\end{bmatrix}\begin{bmatrix}
Y_a \\
Y_d \\
Y_d
\end{bmatrix}
\]

\[\cdots (4.45)\]

The simulation procedure is similar to that discussed in the case of the alternative branch formulation above.

4.2.4. Branch-Chord Formulation

The terminal equations are in the mixed or hybrid form, neither explicit in across nor through variables and therefore require the
FIG. 4.6 GENERALIZED SYSTEM DYNAMICS FLOW DIAGRAM: CHORD FORMULATION
use of both "short-circuit admittance" matrix $W$ and "open-circuit impedance" matrix $Z$:
\[
\begin{bmatrix}
W_{11} & W_{12} \\
W_{21} & W_{22}
\end{bmatrix}
\begin{bmatrix}
X_{br} \\
X_{ar}
\end{bmatrix}
- 
\begin{bmatrix}
Z_{11} & Z_{12} \\
Z_{21} & Z_{22}
\end{bmatrix}
\begin{bmatrix}
Y_{br} \\
Y_{ar}
\end{bmatrix}
= \ldots (4.53)
\]

Referring to the fundamental cutset and circuit equations given earlier in (4.9) and (4.10) respectively, we have:
\[
\begin{bmatrix}
X_{br} \\
X_{ar}
\end{bmatrix}
- 
\begin{bmatrix}
0 & 0 & U \\
-B_{11} & -B_{12} & -B_{13}
\end{bmatrix}
\begin{bmatrix}
X_{be} \\
X_{bc} \\
X_{br}
\end{bmatrix}
= \ldots (4.54)
\]
\[
\begin{bmatrix}
Y_{br} \\
Y_{ar}
\end{bmatrix}
- 
\begin{bmatrix}
-A_{31} & -A_{32} & -A_{33} \\
U & 0 & 0
\end{bmatrix}
\begin{bmatrix}
Y_{d} \\
Y_{a}
\end{bmatrix}
= \ldots (4.55)
\]

On substituting these results into the hybrid form of terminal equations, we get:
\[
\begin{bmatrix}
W_{11} & W_{12} \\
W_{21} & W_{22}
\end{bmatrix}
\begin{bmatrix}
0 & 0 & U \\
-B_{11} & -B_{12} & -B_{13}
\end{bmatrix}
\begin{bmatrix}
X_{be} \\
X_{bc} \\
X_{br}
\end{bmatrix}
- 
\begin{bmatrix}
Z_{11} & Z_{12} \\
Z_{21} & Z_{22}
\end{bmatrix}
\begin{bmatrix}
-A_{31} & -A_{32} & -A_{33} \\
U & 0 & 0
\end{bmatrix}
\begin{bmatrix}
Y_{d} \\
Y_{a}
\end{bmatrix}
= \ldots (4.56)
\]

which results in:

\[193\]
In branch-chord formulation the largest dimension of the matrix to be inverted is \((n_{br}+n_{cr})x(n_{br}+n_{cr})\) where \(n_{br}\) and \(n_{cr}\) are as defined already.

4.3. Alternative State Equation Formulation

In state variable formulation we are interested in an analytical formulation of equations which are explicit in the derivatives of the primary variables of the dynamic elements. We may assume that the algebraic or "resistive" component terminal equations are explicit in the primary variables (i.e. branch across and chord through variables):

\[
\begin{bmatrix}
X_{br} \\
Y_{\alpha}
\end{bmatrix} = \begin{bmatrix}
Z_{br} & 0 \\
0 & G_{\alpha}
\end{bmatrix} \begin{bmatrix}
Y_{br} \\
X_{\alpha}
\end{bmatrix}
\] .... (4.58)

\[
X_{br} = Z_{br}Y_{br} - Z_{br}(A_{31}Y_{\alpha} + [A_{32} A_{33}] \begin{bmatrix}
Y_{d}
\end{bmatrix})
\] .... (4.59)

\[
Y_{\alpha} = G_{\alpha}X_{\alpha} - G_{\alpha} ([B_{11} B_{12}] \begin{bmatrix}
X_{bc}
\end{bmatrix} + B_{13}X_{br})
\] .... (4.60)

or

(194)
Defining

\[ \begin{bmatrix} X_{br} \\ Y_{\sigma} \end{bmatrix} = - \begin{bmatrix} Z_{br} & 0 \\ 0 & G_{\sigma} \end{bmatrix} \begin{bmatrix} 0 & A_{32} \\ B_{12} & 0 \end{bmatrix} \begin{bmatrix} X_{bc} \\ Y_{\sigma} \end{bmatrix} + \begin{bmatrix} 0 & A_{33} \\ B_{11} & 0 \end{bmatrix} \begin{bmatrix} X_{be} \\ Y_{d} \end{bmatrix} + \begin{bmatrix} 0 & A_{31} \\ B_{13} & 0 \end{bmatrix} \begin{bmatrix} X_{wr} \\ Y_{\sigma} \end{bmatrix} \quad \ldots \quad (4.61) \]

or

\[ \begin{bmatrix} U & Z_{br}A_{31} \\ G_{B_{13}} & U \end{bmatrix} \begin{bmatrix} X_{br} \\ Y_{\sigma} \end{bmatrix} = - \begin{bmatrix} 0 & Z_{br}A_{32} \\ G_{B_{12}} & 0 \end{bmatrix} \begin{bmatrix} X_{bc} \\ Y_{d} \end{bmatrix} - \begin{bmatrix} 0 & Z_{br}A_{33} \\ G_{B_{11}} & 0 \end{bmatrix} \begin{bmatrix} X_{be} \\ Y_{d} \end{bmatrix} \quad \ldots \quad (4.62) \]

Defining

\[ - \begin{bmatrix} U & Z_{br}A_{31} \\ G_{B_{13}} & U \end{bmatrix}^{-1} \begin{bmatrix} W_{11} & W_{12} \\ W_{21} & W_{22} \end{bmatrix} \quad \ldots \quad (4.63) \]

a matrix of dimension \( e-n_{z}-n_{f}-n \) (s.t. \( n = n_{bc} + n_{cl} \) is the minimal number of states in the system) suitably as shown in (4.63), we have the solution of (4.62) as given by (4.64):

\[ \begin{bmatrix} X_{br} \\ Y_{\sigma} \end{bmatrix} = \begin{bmatrix} W_{12}G_{B_{12}} & W_{11}Z_{br}A_{32} \\ W_{22}G_{B_{12}} & W_{21}Z_{br}A_{32} \end{bmatrix} \begin{bmatrix} X_{bc} \\ Y_{d} \end{bmatrix} + \begin{bmatrix} W_{12}G_{B_{11}} & W_{11}Z_{br}A_{33} \\ W_{22}G_{B_{11}} & W_{21}Z_{br}A_{33} \end{bmatrix} \begin{bmatrix} X_{be} \\ Y_{d} \end{bmatrix} \quad \ldots \quad (4.64) \]

The basic state equations arise from the dynamic terminal characteristics and can be expressed in the following form by substituting for secondary variables (branch through and chord across variables) in terms of primary variables (branch across and chord through variables) by making use of fundamental cutset and circuit equations:

(195)
\[
\frac{d}{dt} [X_{bc}] = - \begin{bmatrix} C^* & 0 \\ 0 & L^* \end{bmatrix} \begin{bmatrix} 0 & A_{22} & X_{bc} \\ B_{22} & 0 & Y_d \end{bmatrix} + \begin{bmatrix} 0 & A_{23} & X_{be} \\ B_{23} & 0 & Y_d \end{bmatrix} + \begin{bmatrix} 0 & A_{21} & X_ar \end{bmatrix} \quad \ldots (4.65)
\]

where \( L^* = L^{-1} \), and \( C^* = C^{-1} \), such that \( L \) and \( C \) are the "capacitance" and "inductance" matrices.

\[
\frac{d}{dt} [X_{bc}] = - \begin{bmatrix} 0 & C^* A_{22} & X_{bc} \\ L^* B_{22} & 0 & Y_d \end{bmatrix} + \begin{bmatrix} 0 & C^* A_{23} & X_{be} \\ L^* B_{23} & 0 & Y_d \end{bmatrix} + \begin{bmatrix} 0 & C^* A_{21} & X_ar \end{bmatrix} \quad \ldots (4.66)
\]

is the final form of the state equations. Based on this alternative state variable model (equation (4.66)), the generalized system dynamics flow diagram can be drawn as indicated in Fig.4.7.

In the course of development of these equations, we see that the only inversion required is in the solution of the \([X_{br}^* \ Y_{cr}^*]^T\) vector (equation (4.64)). The maximum dimension of the matrix required to be inverted in this case is \((e-n_x-n_y-n_{bc}-n_{cl})\). In a vast majority of circuits, the dimension of the matrix to be inverted would be less than this.

DYNAMO equations for the state model are indicated below:

\[
\begin{align*}
XBC.K &= XBC.J + DT(RTXBC.JK) \quad \ldots (4.67) \\
XBC &= XBC(0) \\
YCL.K &= YCL.J + DT(RTYCL.JK) \quad \ldots (4.68) \\
YCL &= YCL(0) \\
RTXBC.KL &= -[C^* A_{23} YCI.K + C^* A_{22} YCL.K + C^* A_{21} YCR.K] \quad \ldots (4.69) \\
YCR.K &= W22 GCR B12 XBC.K + W22 GCR B11 XBE.K + \\
& \quad W21 ZBR A32 YCL.K + W21 ZBR A33 YCI.K \quad \ldots (4.70)
\end{align*}
\]
FIG. 4.7 GENERALIZED SYSTEM DYNAMICS FLOW DIAGRAM: STATE VARIABLE FORMULATION


\[\text{RTYCL.KL} = -(L^1 B22 XBC.K + L^1 B21 XBE.K + L^1 B23 XBR.K) \quad \ldots(4.71)\]

\[\text{XBR.K} = W11 ZBR A32 YCL.K + W12 GCR B11 XBE.K + \]
\[W11 ZBR A33 YCI.K + W12 GCR B12 XBC.K \quad \ldots(4.72)\]

State equations can also be formulated in "charge \((q_{bc})\)" - "flux-linkage \((\phi_{c1})\)" controlled form, and the generalized system dynamics flow diagram for this case is shown in Fig. 4.8.

4.3.0. State Equation Formulation Example

From our earlier example with the tree selection as in the case of branch formulation, we may formulate the \([X_{br}^T Y_{cr}^T]^T\) vector, i.e., \([v_{br}^T i_{cr}^T]^T\) as:

\[
\begin{bmatrix}
  i_5 \\
  i_6 \\
  i_7 \\
  i_8 \\
\end{bmatrix} =
\begin{bmatrix}
  T_{12} & 0 & 0 & 0 \\
  0 & T_{21} & 0 & 0 \\
  0 & 0 & g_1 & 0 \\
  0 & 0 & 0 & g_2 \\
\end{bmatrix}
\begin{bmatrix}
  v_5 \\
  v_6 \\
  v_7 \\
  v_8 \\
\end{bmatrix}
\]

\ldots(4.73)

From the f-circuit equations (4.28) we have:

\[
\begin{bmatrix}
  v_5 \\
  v_6 \\
  v_7 \\
  v_8 \\
\end{bmatrix} =
\begin{bmatrix}
  0 & 1 & -1 & 0 \\
  1 & 0 & 0 & -1 \\
  0 & 0 & 1 & 0 \\
  0 & 0 & 0 & 1 \\
\end{bmatrix}
\begin{bmatrix}
  v_1 \\
  v_2 \\
  v_3 \\
  v_4 \\
\end{bmatrix}
\]

\ldots(4.7)

(198)
FIG. 4.8 GENERALIZED SYSTEM DYNAMICS FLOW DIAGRAM: STATE VARIABLE FORMULATION: "CHARGE" - "FLUX - LINKAGE" CONTROLLED CASE
Substitution of equation (4.74) into (4.73) yields:

\[
\begin{pmatrix}
  i_3 \\
  i_6 \\
  i_7 \\
  i_8
\end{pmatrix}
- \begin{pmatrix}
  0 & T_{12} & -T_{12} & 0 \\
  T_{21} & 0 & 0 & -T_{21} \\
  0 & 0 & g_1 & 0 \\
  0 & 0 & 0 & g_2
\end{pmatrix}
\begin{pmatrix}
  v_1 \\
  v_2 \\
  v_3 \\
  v_4
\end{pmatrix}
= \ldots (4.75)
\]

State equations for simulation may be formulated from the characteristics of the dynamic elements:

\[
\begin{pmatrix}
  C_3 & 0 \\
  0 & C_4
\end{pmatrix}
\frac{d}{dt}
\begin{pmatrix}
  v_3 \\
  v_4
\end{pmatrix}
- \begin{pmatrix}
  i_3 \\
  i_4
\end{pmatrix}
- \begin{pmatrix}
  1 & 0 & -1 & 0 & -1 & 0 & 1 & 0 \\
  0 & 1 & 0 & -1 & 0 & -1 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
  i_5 \\
  i_6 \\
  i_7 \\
  i_8 \\
  i_9 \\
  i_{10} \\
  i_{11} \\
  i_{12}
\end{pmatrix}
= \ldots (4.76)
\]

On substitution of (4.75) into (4.77), and noting that \(i_\delta = i_{10} = 0\), we have:

\[
\begin{pmatrix}
  1 & 0 & -1 & 0 \\
  0 & 1 & 0 & -1
\end{pmatrix}
\begin{pmatrix}
  i_9 \\
  i_{10}
\end{pmatrix}
+ \begin{pmatrix}
  -1 & 0 & 1 & 0 \\
  0 & -1 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
  i_9 \\
  i_{10} \\
  i_{11} \\
  i_{12}
\end{pmatrix}
= \ldots (4.77)
\]
\[
\begin{bmatrix}
C_3 & 0 \\
C_4 & \frac{1}{C_4}
\end{bmatrix}
\begin{bmatrix}
\frac{dv_3}{dt} \\
\frac{dv_4}{dt}
\end{bmatrix}
- \begin{bmatrix}
0 & T_{12} \\
T_{21} & 0
\end{bmatrix}
\begin{bmatrix}
v_1 \\
v_2
\end{bmatrix}
+ \begin{bmatrix}
-T_{12} - g_1 & 0 \\
0 & -T_{21} - g_2
\end{bmatrix}
\begin{bmatrix}
v_3 \\
v_4
\end{bmatrix}
= \begin{bmatrix}
i_{11} \\
i_{12}
\end{bmatrix}
\] 

or

\[
\begin{bmatrix}
\frac{1}{C_3} & 0 \\
0 & \frac{1}{C_4}
\end{bmatrix}
\begin{bmatrix}
\frac{dv_3}{dt} \\
\frac{dv_4}{dt}
\end{bmatrix}
- \begin{bmatrix}
0 & T_{12} \\
T_{21} & 0
\end{bmatrix}
\begin{bmatrix}
v_1 \\
v_2
\end{bmatrix}
+ \begin{bmatrix}
-T_{12} - g_1 & 0 \\
0 & -T_{21} - g_2
\end{bmatrix}
\begin{bmatrix}
v_3 \\
v_4
\end{bmatrix}
= \begin{bmatrix}
i_{11} \\
i_{12}
\end{bmatrix}
\] 

where

\[
\begin{bmatrix}
v_1 \\
v_2
\end{bmatrix}
= \begin{bmatrix}
\lambda & 0 \\
0 & \lambda
\end{bmatrix}
\begin{bmatrix}
v_3 \\
v_4
\end{bmatrix}
\begin{bmatrix}
u_1 \\
u_2
\end{bmatrix}
\] 

such that

\[
\begin{bmatrix}
v_3 \\
v_4
\end{bmatrix}
= \begin{bmatrix}
u_1 \\
u_2
\end{bmatrix}
\] 

Equations (4.79) and (4.80) may be used in the system dynamics simulation in the following form:

\[
\begin{bmatrix}
v_3 \\
v_4
\end{bmatrix}
- \begin{bmatrix}
v_3 \\
v_4
\end{bmatrix}
+ \begin{bmatrix}
RTV_3JK \\
RTV_4JK
\end{bmatrix}
\] 

where

\[
\begin{bmatrix}
RTV_{3KL} \\
RTV_{4KL}
\end{bmatrix}
- \begin{bmatrix}
\frac{1}{C_3} & 0 \\
0 & \frac{1}{C_4}
\end{bmatrix}
\begin{bmatrix}
0 & T_{12} \\
T_{21} & 0
\end{bmatrix}
\begin{bmatrix}
v_1 \\
v_2
\end{bmatrix}
+ \begin{bmatrix}
-T_{12} - g_1 & 0 \\
0 & -T_{21} - g_2
\end{bmatrix}
\begin{bmatrix}
v_3 \\
v_4
\end{bmatrix}
= \begin{bmatrix}
i_{11,K} \\
i_{12,K}
\end{bmatrix}
\] 

\[(201)\]
The state equations in simulation form as depicted in Eqn. (4.79) can be easily generalized for an N-neuron Hopfield network, where each neuron is fed back to every other neuron in the network, except itself. Assuming neurons numbered from 1 through N, we have the following generalized state model:

\[
\begin{bmatrix}
v_1,K
\end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \begin{bmatrix} v_3,K \\ v_4,K \end{bmatrix}
\] ....(4.84)

\[
v_3,0 - v_3(0) \\
v_4,0 - v_4(0)
\] ....(4.83)

and
where \( g_i = 1/r_i \), and \( u_i \) is voltage across the capacitor \( C_i \), s.t. \( v_i = \lambda_i u_i \). These equations could obviously have been written directly in system dynamics simulation form.

#### 4.4. Modelling Degenerate Linear Systems

In the following discussion we consider modelling degenerate physical systems. Throughout the discussion we assume the following notation with respect to the choice of a maximally selected formulation tree\(^\text{47}\):

\(^\text{47}\) A maximally selected formulation tree is one that includes maximal number of "capacitors" besides all across drivers and excludes maximal number of "inductors" besides all through drivers.
\( X_{be}, Y_{be} \): across and through variables of independent and/or dependent across drivers;

\( X_{ci}, Y_{ci} \): across and through variables of independent and/or dependent through drivers;

\( X_{bc}, Y_{bc} \): across and through variables of branch "capacitors" (non-degenerate states);

\( X_{cl}, Y_{cl} \): across and through variables of chord "inductors" (non-degenerate states);

\( X_{cc}, Y_{cc} \): across and through variables of chord "capacitors" (degenerate states);

\( X_{bl}, Y_{bl} \): across and through variables of branch "inductors" (degenerate states);

\( X_{br}, Y_{br} \): across and through variables of branch "resistive" elements;

\( X_{cr}, Y_{cr} \): across and through variables of chord "resistive" elements.

The fundamental circuit and cut-set equations may now be partitioned on the basis of the classification identified as above:

\[
\begin{bmatrix}
X_{be} \\
X_{bc} \\
X_{br} \\
X_{cr}
\end{bmatrix} = \begin{bmatrix}
B_{11} & B_{12} & B_{13} & B_{14} & U & 0 & 0 & 0 \\
B_{21} & B_{22} & B_{23} & B_{24} & 0 & U & 0 & 0 \\
B_{31} & B_{32} & B_{33} & B_{34} & 0 & 0 & U & 0 \\
B_{41} & B_{42} & B_{43} & B_{44} & 0 & 0 & 0 & U
\end{bmatrix} - 0 \quad \ldots (4.86)
\]
These fundamental constraint equations satisfy the orthogonal relationship:

\[
\begin{bmatrix}
B_{11} & B_{12} & B_{13} & B_{14} \\
B_{21} & B_{22} & B_{23} & B_{24} \\
B_{31} & B_{32} & B_{33} & B_{34} \\
B_{41} & B_{42} & B_{43} & B_{44}
\end{bmatrix}
\begin{bmatrix}
A_{11} & A_{12} & A_{13} & A_{14} \\
A_{21} & A_{22} & A_{23} & A_{24} \\
A_{31} & A_{32} & A_{33} & A_{34} \\
A_{41} & A_{42} & A_{43} & A_{44}
\end{bmatrix}^T - 
\begin{bmatrix}
Y_{bc} \\
Y_{bl} \\
Y_{bc} \\
Y_{cl}
\end{bmatrix} = 0 \quad \ldots (4.87)
\]

For the non-degenerate states we have:

\[
\frac{d}{dt} \begin{bmatrix} X_{bc} \\ Y_d \end{bmatrix} - \begin{bmatrix} C_{bc} & 0 \\ 0 & L_d \end{bmatrix} \begin{bmatrix} Y_{bc} \\ X_d \end{bmatrix} = 0 \quad \ldots (4.89)
\]

where \(C_{bc}^{-1} = C_{bc}^T\) and \(L_{cl}^{-1} = L_{cl}^T\). Minimal computation is required in inversion as they are usually diagonal matrices (except in the case of mutually coupled elements).

On the other hand, for the degenerate states we have:
and for the resistive elements we may write:

\[
\begin{bmatrix} X_{br} \\ Y_{\alpha} \end{bmatrix} = \begin{bmatrix} R_{br} & 0 \\ 0 & G_{\alpha} \end{bmatrix} \begin{bmatrix} Y_{br} \\ X_{\alpha} \end{bmatrix} \quad \ldots (4.91)
\]

The fundamental circuit and fundamental cut-set equations may be partitioned and combined into the following form:

\[
\begin{bmatrix} Y_{0} \\ 0 \\ A_{0} \\ 0 \\ A_{1} \\ 0 \\ A_{2} \\ 0 \\ A_{3} \\ 0 \\ A_{4} \\ 0 \\ A_{5} \\ 0 \\ A_{6} \\ 0 \\ A_{7} \\ 0 \\ A_{8} \\ 0 \\ A_{9} \\ 0 \\ A_{10} \\ 0 \\ A_{11} \\ 0 \\ A_{12} \\ 0 \\ A_{13} \\ 0 \\ A_{14} \\ 0 \\ A_{15} \\ 0 \\ A_{16} \\ 0 \\ A_{17} \\ 0 \\ A_{18} \\ 0 \\ A_{19} \\ 0 \\ A_{20} \\ 0 \\ A_{21} \\ 0 \\ A_{22} \\ 0 \\ A_{23} \\ 0 \\ A_{24} \\ 0 \\ A_{25} \\ 0 \\ A_{26} \\ 0 \\ A_{27} \\ 0 \\ A_{28} \\ 0 \\ A_{29} \\ 0 \\ A_{30} \\ 0 \\ A_{31} \\ 0 \\ A_{32} \\ 0 \\ A_{33} \\ 0 \\ A_{34} \\ 0 \\ A_{35} \\ 0 \\ A_{36} \\ 0 \\ A_{37} \\ 0 \\ A_{38} \\ 0 \\ A_{39} \\ 0 \\ A_{40} \\ 0 \\ A_{41} \\ 0 \\ A_{42} \\ 0 \\ A_{43} \\ 0 \\ A_{44} \\ 0 \\ A_{45} \\ 0 \\ A_{46} \\ 0 \\ A_{47} \\ 0 \\ A_{48} \\ 0 \\ A_{49} \\ 0 \\ A_{50} \\ 0 \\ A_{51} \\ 0 \\ A_{52} \\ 0 \\ A_{53} \\ 0 \\ A_{54} \\ 0 \\ A_{55} \\ 0 \\ A_{56} \\ 0 \\ A_{57} \\ 0 \\ A_{58} \\ 0 \\ A_{59} \\ 0 \\ A_{60} \\ 0 \\ A_{61} \\ 0 \\ A_{62} \\ 0 \\ A_{63} \\ 0 \\ A_{64} \\ 0 \\ A_{65} \\ 0 \\ A_{66} \\ 0 \\ A_{67} \\ 0 \\ A_{68} \\ 0 \\ A_{69} \\ 0 \\ A_{70} \\ 0 \\ A_{71} \\ 0 \\ A_{72} \\ 0 \\ A_{73} \\ 0 \\ A_{74} \\ 0 \\ A_{75} \\ 0 \\ A_{76} \\ 0 \\ A_{77} \\ 0 \\ A_{78} \\ 0 \\ A_{79} \\ 0 \\ A_{80} \\ 0 \\ A_{81} \\ 0 \\ A_{82} \\ 0 \\ A_{83} \\ 0 \\ A_{84} \\ 0 \\ A_{85} \\ 0 \\ A_{86} \\ 0 \\ A_{87} \\ 0 \\ A_{88} \\ 0 \\ A_{89} \\ 0 \\ A_{90} \\ 0 \\ A_{91} \\ 0 \\ A_{92} \\ 0 \\ A_{93} \\ 0 \\ A_{94} \\ 0 \\ A_{95} \\ 0 \\ A_{96} \\ 0 \\ A_{97} \\ 0 \\ A_{98} \\ 0 \\ A_{99} \\ 0 \\ A_{100} \end{bmatrix} \begin{bmatrix} X_{0} \\ X_{1} \\ X_{2} \\ X_{3} \\ X_{4} \\ X_{5} \\ X_{6} \\ X_{7} \\ X_{8} \\ X_{9} \\ X_{10} \\ X_{11} \\ X_{12} \\ X_{13} \\ X_{14} \\ X_{15} \\ X_{16} \\ X_{17} \\ X_{18} \\ X_{19} \\ X_{20} \\ X_{21} \\ X_{22} \\ X_{23} \\ X_{24} \\ X_{25} \\ X_{26} \\ X_{27} \\ X_{28} \\ X_{29} \\ X_{30} \\ X_{31} \\ X_{32} \\ X_{33} \\ X_{34} \\ X_{35} \\ X_{36} \\ X_{37} \\ X_{38} \\ X_{39} \\ X_{40} \\ X_{41} \\ X_{42} \\ X_{43} \\ X_{44} \\ X_{45} \\ X_{46} \\ X_{47} \\ X_{48} \\ X_{49} \\ X_{50} \\ X_{51} \\ X_{52} \\ X_{53} \\ X_{54} \\ X_{55} \\ X_{56} \\ X_{57} \\ X_{58} \\ X_{59} \\ X_{60} \\ X_{61} \\ X_{62} \\ X_{63} \\ X_{64} \\ X_{65} \\ X_{66} \\ X_{67} \\ X_{68} \\ X_{69} \\ X_{70} \\ X_{71} \\ X_{72} \\ X_{73} \\ X_{74} \\ X_{75} \\ X_{76} \\ X_{77} \\ X_{78} \\ X_{79} \\ X_{80} \\ X_{81} \\ X_{82} \\ X_{83} \\ X_{84} \\ X_{85} \\ X_{86} \\ X_{87} \\ X_{88} \\ X_{89} \\ X_{90} \\ X_{91} \\ X_{92} \\ X_{93} \\ X_{94} \\ X_{95} \\ X_{96} \\ X_{97} \\ X_{98} \\ X_{99} \\ X_{100} \end{bmatrix} \quad \ldots (4.92)
\]

Consider the non-degenerate states:

\[
\begin{bmatrix} d & [X_{bc}] \\ dt & Y_{d} \end{bmatrix} = - \begin{bmatrix} C_{bc} & 0 \\ 0 & L_{d} \end{bmatrix} \begin{bmatrix} 0 & A_{23} \\ B_{32} & 0 \end{bmatrix} \begin{bmatrix} X_{bc} \\ Y_{d} \end{bmatrix} + \begin{bmatrix} 0 & A_{24} \\ B_{31} & 0 \end{bmatrix} \begin{bmatrix} X_{bc} \\ Y_{d} \end{bmatrix} + \begin{bmatrix} 0 & A_{21} \\ B_{34} & 0 \end{bmatrix} \begin{bmatrix} X_{bc} \\ Y_{d} \end{bmatrix} + \begin{bmatrix} 0 & A_{22} \\ B_{33} & 0 \end{bmatrix} \begin{bmatrix} X_{bc} \\ Y_{d} \end{bmatrix} \quad \ldots (4.93)
\]

For the degenerate states, appropriate substitution from equation (4.92) into equation (4.90) yields:

(206)
In the above equations, the coefficient matrices of \([X_{bl}^\top Y_{cc}^\top]\) and \([X_{br}^\top Y_{cr}^\top]\) are identically zero.\(^{48}\)

From equation (4.91) for the branch and chord resistive elements, we have:

\[
\begin{bmatrix}
X_{br} \\
Y_{cr}
\end{bmatrix} = -
\begin{bmatrix}
0 & 0 \\
0 & 0 
\end{bmatrix}
\begin{bmatrix}
0 & A_{33} \\
0 & A_{34}
\end{bmatrix}
\begin{bmatrix}
X_{bc} \\
X_{po}
\end{bmatrix} +
\begin{bmatrix}
0 & A_{31} \\
0 & A_{32}
\end{bmatrix}
\begin{bmatrix}
X_{bl} \\
X_{tl}
\end{bmatrix} +
\begin{bmatrix}
0 & A_{43} \\
0 & A_{44}
\end{bmatrix}
\begin{bmatrix}
X_{bc} \\
X_{po}
\end{bmatrix}
\]

\[
\begin{bmatrix}
R_{br} \\
G_{cr}
\end{bmatrix}
\begin{bmatrix}
B_{12} \\
B_{11}
\end{bmatrix}
\begin{bmatrix}
Y_{dl} \\
Y_{di}
\end{bmatrix} +
\begin{bmatrix}
B_{13} \\
B_{14}
\end{bmatrix}
\begin{bmatrix}
Y_{dl} \\
Y_{di}
\end{bmatrix}
\]

\[
\ldots (4.95)
\]

Once again, the coefficient matrix of \([X_{bl}^\top Y_{cc}^\top]\) is identically zero.\(^{49}\) Hence, from equations (4.94) and (4.95) we have:

\[
\begin{bmatrix}
X_{bl} \\
Y_{cc}
\end{bmatrix} = -
\begin{bmatrix}
L_{bl} & 0 \\
0 & C_{cc}
\end{bmatrix}
\begin{bmatrix}
0 & A_{43} \\
B_{12} & 0
\end{bmatrix}
\begin{bmatrix}
X_{bc} \\
X_{po}
\end{bmatrix} +
\begin{bmatrix}
0 & A_{44} \\
B_{11} & 0
\end{bmatrix}
\begin{bmatrix}
X_{bc} \\
X_{po}
\end{bmatrix}
\]

\[
\ldots (4.96)
\]

\[
\ldots (4.94)
\]

In the interconnection constraint equations, the cut-set of a degenerate "inductor" cannot contain the through variable of a degenerate "capacitor" for a maximally selected tree, since by definition, a degeneracy is created by a cut-set of "inductors" and/or independent through drivers (i.e., the node could not have been accessed through a "capacitor" or some "resistive" element). Similarly, the circuit of a degenerate "capacitor" cannot contain the across variable of a degenerate "inductor" for a maximally selected tree, since by definition, a degeneracy is created by a circuit of "capacitors" and/or independent across drivers (i.e., the circuit could not have included an "inductor" or some "resistive" element).

\(^{48}\) Again, branch "resistor" f-cut-sets cannot contain chord "capacitor" elements or else the tree selection is not maximal. Similarly, chord resistor f-circuits cannot contain branch inductor elements for the same reason.
\[
\begin{bmatrix}
X_{bc} \\
Y_{\sigma}
\end{bmatrix} = \begin{bmatrix}
I & R_{br}A_{32}^{-1} \\
G_\alpha B_{23} & I
\end{bmatrix} \begin{bmatrix}
0 & R_{br}A_{33} \\
G_\alpha B_{22} & 0
\end{bmatrix} \begin{bmatrix}
X_{bc} \\
Y_{\alpha}
\end{bmatrix} + \begin{bmatrix}
0 & R_{br}A_{34} \\
G_\alpha B_{21} & 0
\end{bmatrix} \begin{bmatrix}
X_{be} \\
Y_{\sigma}
\end{bmatrix}
\]

....(4.97)

Alternatively, we may write
\[
\begin{bmatrix}
X_{br} \\
Y_{\alpha}
\end{bmatrix} = - \begin{bmatrix}
M_{12}G_\alpha B_{22} & M_{11}R_{br}A_{33} \\
M_{22}G_\alpha B_{22} & M_{21}R_{br}A_{33}
\end{bmatrix} \begin{bmatrix}
X_{be} \\
Y_{\alpha}
\end{bmatrix} - \begin{bmatrix}
M_{12}G_\alpha B_{21} & M_{11}R_{br}A_{34} \\
M_{22}G_\alpha B_{21} & M_{21}R_{br}A_{34}
\end{bmatrix} \begin{bmatrix}
X_{be} \\
Y_{\alpha}
\end{bmatrix}
\]

....(4.98)

where
\[
\begin{bmatrix}
I & R_{br}A_{32}^{-1} \\
G_\alpha B_{23} & I
\end{bmatrix} \Delta \begin{bmatrix}
M_{11} & M_{12} \\
M_{21} & M_{22}
\end{bmatrix}
\]

....(4.99)

From Eqn. (4.96),
\[
\begin{bmatrix}
X_{bl} \\
Y_{\infty}
\end{bmatrix} = - \begin{bmatrix}
0 & L_{bl}A_{43} \\
C_\infty B_{12} & 0
\end{bmatrix} \frac{d}{dt} \begin{bmatrix}
X_{bc} \\
Y_{\alpha}
\end{bmatrix} - \begin{bmatrix}
0 & L_{bl}A_{44} \\
C_\infty B_{11} & 0
\end{bmatrix} \frac{d}{dt} \begin{bmatrix}
X_{be} \\
Y_{\alpha}
\end{bmatrix}
\]

....(4.100)

From equation (4.93) the final equations may be formulated as :
\[
\frac{d}{dt} \begin{bmatrix}
X_{bc} \\
Y_{\alpha}
\end{bmatrix} = \begin{bmatrix}
0 & C_{bc}A_{23} \\
L_{bl}B_{32} & 0
\end{bmatrix} \begin{bmatrix}
X_{bc} \\
Y_{\alpha}
\end{bmatrix} - \begin{bmatrix}
0 & C_{bc}A_{24} \\
L_{bl}B_{31} & 0
\end{bmatrix} \begin{bmatrix}
X_{be} \\
Y_{\alpha}
\end{bmatrix}
\]

....(4.101)
$$ - \left[ \begin{array}{cc} 0 & C_{bc}^* A_{23} \\ L_{da} B_{32} & 0 \end{array} \right] [X_{bc}] - \left[ \begin{array}{cc} 0 & C_{bc}^* A_{24} \\ L_{da} B_{31} & 0 \end{array} \right] [X_{be}] $$

$$ + \left[ \begin{array}{cc} 0 & C_{bc}^* A_{21} \\ L_{da} B_{34} & 0 \end{array} \right] \left( \begin{array}{ccc} 0 & L_{ab} A_{43} & \frac{d}{dt} [X_{bc}] \\ C_{cc} B_{12} & 0 & \frac{d}{dt} [Y_{cl}] \\ C_{cc} B_{11} & 0 & \frac{d}{dt} [Y_{dl}] \end{array} \right) $$

$$ + \left[ \begin{array}{cc} 0 & C_{bc}^* A_{22} \\ L_{da} B_{33} & 0 \end{array} \right] \left( \begin{array}{ccc} M_{12} G_{\alpha} B_{22} & M_{11} R_{br} A_{33} & [X_{bc}] \\ M_{22} G_{\alpha} B_{22} & M_{21} R_{br} A_{33} & [Y_{cl}] \end{array} \right) $$

$$ \begin{array}{c} \phantom{\left( \begin{array}{ccc} M_{12} G_{\alpha} B_{21} & M_{11} R_{br} A_{34} & [X_{be}] \\ M_{21} G_{\alpha} B_{21} & M_{22} R_{br} A_{34} & [Y_{dl}] \end{array} \right) +} \end{array} $$

\[ \ldots (4.102) \]

or

$$ \begin{bmatrix} I-C_{bc}^* A_{21} C_{cc} B_{12} & 0 \\ 0 & I-L_{da}^* B_{34} L_{ab} A_{43} \end{bmatrix} \begin{bmatrix} \frac{d}{dt} [X_{bc}] \\ \frac{d}{dt} [Y_{cl}] \end{bmatrix} $$

$$ + \left[ \begin{array}{cc} C_{bc}^* A_{22} M_{22} G_{\alpha} B_{22} & C_{bc}^* A_{22} M_{21} R_{br} A_{33} - C_{bc}^* A_{23} \\ L_{da} B_{33} M_{12} G_{\alpha} B_{22} - L_{da}^* B_{32} & L_{da}^* B_{33} M_{11} R_{br} A_{33} \end{array} \right] \begin{bmatrix} [X_{bc}] \\ [Y_{cl}] \end{bmatrix} $$

$$ \begin{array}{c} \phantom{\left( \begin{array}{ccc} M_{12} G_{\alpha} B_{21} & M_{11} R_{br} A_{34} & [X_{be}] \\ M_{21} G_{\alpha} B_{21} & M_{22} R_{br} A_{34} & [Y_{dl}] \end{array} \right) +} \end{array} $$

\[ \ldots (4.103) \]

or symbolically,
Granting that $R^{-1}$ exists, the following explicit form for the state formulation results:

$$\frac{d(\psi_{pl})}{dt} = P\psi_{pl} + Q_1\psi_{p0} + Q_2\frac{d(\psi_{p0})}{dt} \quad \ldots (4.105)$$

where

$$P^* = R^{-1}P$$
$$Q_1^* = R^{-1}Q_1$$
$$Q_2^* = R^{-1}Q_2 \quad \ldots (4.106)$$

The generalized system dynamics flow diagram for the degenerate systems case is given in Fig. 4.9. Using equation (4.105), the DYNAMO equations can be formulated as:

$$SI_{P1.K} = SI_{P1.J} + DT(RTSI_{P1.JK}) \quad \ldots (4.107)$$
$$SI_{P1.0} = SI_{P1}(0) \quad \ldots (4.108)$$
$$RTSI_{P1.KL} = P^* SI_{P1.K} + Q_1^* SI_{P0.K} + Q_2^* RTSI_{P0.K} \quad \ldots (4.109)$$

4.5. Non-degenerate Case with Non-linear Components

For the non-linear case, the state equation formulation turns out to be the most convenient one from a simulation point of view. In this case, the state variables may be chosen as either the branch "capacitor" across variables and chord "inductor" through variables, or the branch "capacitor charges" and chord "inductor flux-
FIG. 4.9 GENERALIZED SYSTEM DYNAMICS FLOW DIAGRAM: DEGENERATE SYSTEMS
linkages". In either case, Lipschitz sufficiency conditions must be satisfied for the existence of a unique solution\(^{50}\).

Let \(x_{be}\) and \(y_{ci}\) be specified time functions as already defined earlier. The resistive element characteristics may be assumed to be in the following form:

\[
\begin{bmatrix}
X_{br} \\
Y_{\alpha}
\end{bmatrix} - f_r \begin{bmatrix}
Y_{br} \\
X_{\alpha}
\end{bmatrix} = \begin{bmatrix}
f_{br}(\cdot) & 0 \\
0 & f_{\alpha}(\cdot)
\end{bmatrix} \begin{bmatrix}
Y_{br} \\
X_{\alpha}
\end{bmatrix} 
\]

\[\ldots(4.110)\]

where \(f_r\) is any non-linear function.

Upon substitution from fundamental cutset and fundamental circuit equations, Eqn.(4.110) may be re-cast as:

\[
\begin{bmatrix}
X_{br} \\
Y_{\alpha}
\end{bmatrix} - f_r \begin{bmatrix}
0 & -A_{32} & X_{be} \\
-B_{12} & 0 & Y_{ci}
\end{bmatrix} + \begin{bmatrix}
0 & -A_{33} & X_{ba} \\
-B_{11} & 0 & Y_{ci}
\end{bmatrix} + \begin{bmatrix}
0 & -A_{31} & X_{tr} \\
-B_{13} & 0 & Y_{ci}
\end{bmatrix}
\]

\[\ldots(4.111)\]

\[
\begin{bmatrix}
X_{br} \\
Y_{\alpha}
\end{bmatrix} - f_{br}(\cdot) \begin{bmatrix}
0 & -A_{32} & X_{be} \\
-B_{12} & 0 & Y_{ci}
\end{bmatrix} + f_{\alpha}(\cdot) \begin{bmatrix}
0 & -A_{33} & X_{ba} \\
-B_{11} & 0 & Y_{ci}
\end{bmatrix} + f_{\alpha}(\cdot) \begin{bmatrix}
0 & -A_{31} & X_{tr} \\
-B_{13} & 0 & Y_{ci}
\end{bmatrix}
\]

\[\ldots(4.112)\]

\(^{50}\) Lipschitz "sufficiency" conditions for the existence of a unique solution in a non-linear system require that each algebraic component should have a strictly monotonically increasing characteristic with positive bounded slope. However, unlike linear systems, there are no known necessary conditions for the existence and uniqueness of solutions to non-linear algebraic systems of equations, and as such it is not possible to state necessary conditions for the existence of a unique solution in this case.
Also, the state equations would have the form:

\[
\frac{d}{dt} \begin{bmatrix} X_{bc} \\ Y_{cl} \end{bmatrix} = - f_c \begin{bmatrix} Y_{bc} \\ X_{cl} \end{bmatrix} - \begin{bmatrix} f_{bc}(\cdot) & 0 \\ 0 & f_{\alpha c}(\cdot) \end{bmatrix} \begin{bmatrix} Y_{bc} \\ X_{cl} \end{bmatrix} \quad \text{....(4.113)}
\]

where \( f_c \) is any non-linear function. These equations result in the form:

\[
\frac{d}{dt} \begin{bmatrix} X_{bc} \\ Y_{cl} \end{bmatrix} = - f_c \begin{bmatrix} 0 & -A_{22} \\ -B_{22} & 0 \end{bmatrix} \begin{bmatrix} X_{bc} \\ Y_{cl} \end{bmatrix} + \begin{bmatrix} 0 & -A_{23} \\ -B_{21} & 0 \end{bmatrix} \begin{bmatrix} X_{bc} \\ Y_{cl} \end{bmatrix} + \begin{bmatrix} 0 & -A_{21} \\ -B_{23} & 0 \end{bmatrix} \begin{bmatrix} X_{bc} \\ Y_{cl} \end{bmatrix}
\]

\[
- \begin{bmatrix} f_{bc}(\cdot) & 0 \\ 0 & f_{\alpha c}(\cdot) \end{bmatrix} \begin{bmatrix} 0 & -A_{22} \\ -B_{22} & 0 \end{bmatrix} \begin{bmatrix} X_{bc} \\ Y_{cl} \end{bmatrix} + \begin{bmatrix} 0 & -A_{23} \\ -B_{21} & 0 \end{bmatrix} \begin{bmatrix} X_{bc} \\ Y_{cl} \end{bmatrix} + \begin{bmatrix} 0 & -A_{21} \\ -B_{23} & 0 \end{bmatrix} \begin{bmatrix} X_{bc} \\ Y_{cl} \end{bmatrix}
\]

\[
\quad \text{....(4.114)}
\]

For simulation purposes, one may use Katzenelson's algorithm [KAT66]. Fig.4.10 depicts the generalized system dynamics flow diagram for systems with non-linear components.

Non-linear state equations could also be formulated using branch "capacitor charges" and chord "inductor flux linkages" as state variables. In this case:

\[
\begin{bmatrix} X_{bc} \\ Y_{cl} \end{bmatrix} - F_c \begin{bmatrix} q_{bc} \\ \phi_{cl} \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 0 & F_{\alpha c}(\cdot) \end{bmatrix} \begin{bmatrix} q_{bc} \\ \phi_{cl} \end{bmatrix} \quad \text{....(4.115)}
\]

where \( F_c \) is a non-linear function. We therefore have:

\[
\frac{d}{dt} \begin{bmatrix} q_{bc} \\ \phi_{cl} \end{bmatrix} - \begin{bmatrix} Y_{bc} \\ X_{cl} \end{bmatrix} = \begin{bmatrix} 0 & -A_{22} \\ -B_{22} & 0 \end{bmatrix} \begin{bmatrix} X_{bc} \\ Y_{cl} \end{bmatrix} + \begin{bmatrix} 0 & -A_{23} \\ -B_{21} & 0 \end{bmatrix} \begin{bmatrix} X_{bc} \\ Y_{cl} \end{bmatrix} + \begin{bmatrix} 0 & -A_{21} \\ -B_{23} & 0 \end{bmatrix} \begin{bmatrix} X_{bc} \\ Y_{cl} \end{bmatrix}
\]

\[
\text{(213)}
\]
FIG. 4.10 GENERALIZED SYSTEM DYNAMICS FLOW DIAGRAM FOR NON-DEGENERATE SYSTEMS WITH NON-LINEAR COMPONENTS.
The generalized systems dynamics flow diagram for this case is shown in Fig. 4.11.

4.6. Exploiting Model Structure

Looking at the alternative state equations given in (4.65) from a broader point of view, we see that they represent the general structure underlying system dynamics models. If we accept this, then by studying the generalized state model of any particular system, assuming that the model exists, one can gain numerous structural insights into the system dynamics model, and considerably facilitate the modelling and simulation procedure. For linear systems, necessary conditions for the existence of state models are already known in Physical System Theory [KOE66]. To exploit this structural insight to its fullest extent it is important to take note of the following points.

Firstly, in the modelling procedure for system dynamics, there seems to be no formal procedure for the identification of level, rate and other auxiliary variables which should be considered for inclusion into the flow diagrams and hence the DYNAMO equations. If we assume that a state model for the system exists, then by simply identifying the general form of the state equations for the system under consideration one can get a precise idea of the variables that should be taken into account during the process of development of the model. This would considerably ease the model development process. The generalized flow diagram of Fig.4.7 would then hold

\[
- \begin{bmatrix} 0 & -A_{22} & 0 \\ -B_{22} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} F_{bc}(\cdot) \\ F_{dc}(\cdot) \\ F_{ac}(\cdot) \end{bmatrix} + \begin{bmatrix} 0 & -A_{23} & 0 \\ -B_{21} & 0 & 0 \\ 0 & -B_{23} & 0 \end{bmatrix} \begin{bmatrix} X_{bc} \\ Y_{dc} \\ Y_{ac} \end{bmatrix}
\]

\[ (4.116) \]
FIG. 4.11 GENERALIZED SYSTEM DYNAMICS FLOW DIAGRAM: NON DEGENERATE SYSTEMS WITH NON-LINEAR COMPONENTS: "CHARGE"-"FLUX- LINKAGE" CONTROLLED CASE
good and the DYNAMO equations would follow the pattern as indicated in section 4.3.

Secondly, if we consider that the state model is representative of the underlying structure of the system dynamics model, then powerful concepts and results of linear/nonlinear system theory would automatically apply. That is to say, the concept of controllability and observability of states would come into play. Criteria for stability analysis of systems would also apply to the system under consideration and could thus be applied to study the stability of the simulation being performed. Also, sensitivity and optimality criteria could prove valuable in design-by-analysis or synthesis of large and complex systems.

Moving one step further, we may visualize the development of the system model directly through observation and measurement. Consider the state model equations in the form:

\[
\frac{d}{dt} \begin{bmatrix} X_{bc} \\ Y_d \end{bmatrix} = \begin{bmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{bmatrix} \begin{bmatrix} X_{bc} \\ Y_d \end{bmatrix} + \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} \begin{bmatrix} X_{br} \\ Y_{cr} \end{bmatrix} + \begin{bmatrix} N_{11} & N_{12} \\ N_{21} & N_{22} \end{bmatrix} \begin{bmatrix} X_{be} \\ Y_{ci} \end{bmatrix} \]

\( \ldots (4.117) \)

where \( X_{br} \) and \( Y_{cr} \) represent certain auxiliary variables which are in turn given by equations of the form:

\[
\begin{bmatrix} X_{br} \\ Y_{cr} \end{bmatrix} = \begin{bmatrix} O_{11} & O_{12} \\ O_{21} & O_{22} \end{bmatrix} \begin{bmatrix} X_{bc} \\ Y_d \end{bmatrix} + \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} \begin{bmatrix} X_{bc} \\ Y_d \end{bmatrix} \]

\( \ldots (4.118) \)

If we assume that these auxiliary variables are controllable, then we can set their values as we like. Using these, and a set of independent drivers/excitations \( X_{be} \) and \( Y_{ci} \), and the values of \( X_{bc} \) and \( Y_{ci} \), we may make measurements of the components of the \( [L] \), \( [M] \) and (217)
[N] coefficient matrices. These measurements are subject to the accessibility of these states which is fortunately "guaranteed" in physical system theoretic state model formulation presented here. The measurements may be partially or wholly in the form of historical data records on the behavior of the system. One can also make suitable measurement schemes to identify the parameters of the [O] and [P] coefficient matrices given in the second equation above (Eqn. 4.118).

By taking a series of "test bench" measurements, one can make an empirical fit on the data and thus identify the system model as an object in its own right in terms of terminal characteristics given by equation (4.118). This model can then be used to formulate the system dynamics flow diagram directly and thus also the accompanying DYNAMO equations, thereby totally obviating the need for any matrix inversions in formulating the system dynamics model for simulation as discussed in section 4.3.

4.6. Conclusions

Although the development of equations discussed in the preceding sections relate to timing simulation applications of circuits and systems, the concepts may be readily extended to handle large and complex physical systems (both real and conceptual) in general to provide valuable insights, particularly those following from exploiting model structure as discussed in section 4.6. Notice that $X_{bi}$ and $Y_{ci}$ might represent external inputs to the system such as scenario and policy variables in a socio-economic system, and these could be judiciously varied to simulate the response of the system for alternative scenarios. One could also consider structural changes (which usually turn out to be rather difficult), or even
changes in component characteristics and input parameters to simulate alternative designs and operating environments.

Apart from this, an advantage of the system dynamics modelling discussed in this chapter is the ready extendability of the concepts to degenerate systems (where the states of the system are fewer than the number of the dynamic (energy storing) components in the system) as shown in section 4.4. Non-linear characteristics in degenerate systems can also be readily incorporated in generalized system dynamics simulation in a manner similar to that for the non-degenerate case as shown in section 4.5.