Chapter 2

ASSOSYMMETRIC RINGS
It is observed in the last chapter that in the studies of non-associative rings, some weakened form of the associative law of multiplication is assumed. In assosymmetric rings also the associative law of multiplication is weakened to the condition, namely,

\[(x, y, z) = (P(x), P(y), P(z))\]

for each permutation \(P \) of \(x, y, z\). This chapter is devoted to the study of certain conditions on the assosymmetric rings whose validity implies the associativity of multiplication.

In section 2.1, we present some identities \([22]\) on assosymmetric rings that are needed in the subsequent discussion. Using these identities, we give a proof of Kleinfeld's \([22]\) theorem on 2- and 3-divisible assosymmetric rings in section 2.2. In this section we also give some examples of assosymmetric rings which are neither flexible nor power-associative.

In section 2.3 we briefly discuss solvable assosymmetric rings \([27]\). Here we see that a 2-divisible solvable ring with each associator in the nucleus is
nilpotent and hence a solvable 2- and 3-divisible assosymmetric ring is nilpotent. Consequently a 2- and 3-divisible assosymmetric nil ring with descending chain condition on right ideals is nilpotent. Using these results, we establish the Wedderburn principal theorem for assosymmetric algebras.

In the last section, we prove that an assosymmetric ring in which \((xy)^2 = x^2y^2\) for \(x, y\) in the ring is commutative and associative.

2.1 Some Identities

Let us now obtain some useful identities of assosymmetric rings. In an arbitrary ring the following identities hold [22]:

\[(wx, y, z) - (w, xy, z) + (w, x, yz) = w(x, y, z) + (w, x, y)z\]

2.1.1

\[f(w, x, y, z) = (wx, y, z) - x(w, y, z) - (x, y, z)w\]

2.1.2

and

\[(xy, z) - x(y, z) - (x, z)y = (x, y, z) - (x, z, y) + (z, x, y)\]

2.1.3

The identity 2.1.1 is called the Teichmüller identity. In any assosymmetric ring 2.1.3 becomes

\[(xy, z) - x(y, z) - (x, z)y = (x, y, z)\]

2.1.4
From now on \( R \) will denote a 2- and 3-divisible
assosymmetric ring and \( r,s,t,u,v,w,x,y \) and \( z \) are arbi-
trary elements of \( R \). The right hand side of 2.1.1 may
be written as

\[
(zw,x,y) - f(z,w,x,y),
\]

so that

\[
f(z,w,x,y) = (zw,x,y) - (wx,y,z) + (xy,z,w) - (yz,w,x)
\]

From this it follows that

\[
f(w,x,y,z) = -f(z,w,x,y)
\]

All these identities hold in alternative rings also.
Now we give an identity which does not hold in alternative
rings. From 2.1.2 it can be seen that

\[
f(w,x,y,z) = f(w,x,z,y).
\]

Then by using these last two identities alternately, we
obtain

\[
f(w,x,y,z) = -f(z,w,x,y) = -f(z,w,y,x)
\]

\[
= f(x,z,w,y) = f(x,z,y,w)
\]

\[
= -f(w,x,z,y) = -f(w,x,y,z).
\]

Thus 2f(w,x,y,z) = 0
Hence \( f(w, x, y, z) = 0 \), since \( R \) is 2-divisible

That is, \( (wx, y, z) = x(w, y, z) + (x, y, z)w \) \( \text{2.1.6} \)

If each of the four terms of the RHS of 2.1.5 is broken up by means of 2.1.6, which leads to the identity

\[
(w, (x, y, z)) - (x, (y, z, w)) + (y, (z, w, x)) - (z, (w, x, y)) = 0.
\]

On the other hand if one breaks up \((x, w), y, z) + ((z, y), w, x)\) by means of 2.1.6, we obtain the LHS of the above equation. Consequently

\[
((x, w), y, z) + ((z, y), w, x) = 0 \hspace{1cm} \text{2.1.7}
\]

Interchanging \( w \) and \( x \) in 2.1.7 and adding that to 2.1.7 we obtain

\[
2 \cdot ((z, y), w, x) = 0.
\]

So that

\[
((z, y), w, x) = 0 \hspace{1cm} \text{2.1.8}
\]

In an arbitrary ring the following identity can be easily verified \([22]\):

\[
(w, x, y) + (x, y, w) + (y, w, x) = (wx, y) + (xy, w) + (yw, x)
\]

By forming the associators of each side of this identity
with \( s \) and \( t \), and using 2.1.8 it follows that

\[ 3 \, ((w,x,y),s,t) = 0. \]

Since \( R \) is 3-divisible, we have

\[ (w,x,y),s,t) = 0 \]

Thus from 2.1.8 and 2.1.9, we see that every commutator and associator is in the nucleus \( N \)

2.2 Semi-prime Assosymmetric Rings

In this section we prove the theorem of Kleinfeld [22]. To prove this we need the following:

**Lemma 2.2.1:** If \( R \) is a 2- and 3-divisible assosymmetric ring with identity \( ((w,x,y),s,t) = 0 \), then the product of two associators is zero

**Proof:** Let us recall the identity 2.1.1 which holds in an arbitrary ring:

\[ (wx,y,z)-(w,xy,z) + (w,x,ys) = w(x,y,z) + (w,x,y)z \]

By forming the associators of both sides of this with \( s \) and \( t \) and using 2.1.9, we obtain

\[ (w(x,y,z),s,t) + ((w,x,y)z,s,t) = 0 \]

\[ 2.2.1 \]
By interchanging \(y\) and \(z\) in 2.2.1 and subtracting the result from 2.2.1, we have

\[
((w,x,y)z,s,t) = ((w,x,s)y,s,t)
\]

But by 2.1.8,

\[
((w,x,s)y,s,t) = (y(w,x,s),s,t)
\]

So that

\[
((w,x,y)z,s,t) = (y(w,x,s),s,t)
\]

using 2.2.2.

Also by permuting \(w\) and \(y\) in 2.2.1, we obtain

\[
(y(w,x,z),s,t) + ((w,x,y)z,s,t) = 0
\]

Comparison of this last identity with 2.2.3 gives

\[
2 ((w,x,y)z,s,t) = 0,
\]

which leads us \( ((w,x,y)z,s,t) = 0 \), because the ring is 2-divisible. Expanding this in the manner of 2.1.1, and using 2.1.9, we obtain

\[
(w,x,y)(z,s,t) = 0
\]

and the lemma is proved \(\square\)

Let us now prove Kleinfeld's theorem [22] on 2- and 3-divisible assosymmetric rings.
Theorem 2.2.2  
If $R$ is a $2$- and $3$-divisible ring satisfying the identities

(i) $((w,x,y),z,t) = 0$ and

(ii) $(w,x,y)(z,s,t) = 0$

and has no ideals $J \neq 0$ such that $J^2 = 0$, then $R$ is associative.

Proof: We know that in any ring $R$ if $S$ is the additive subgroup of $R$ generated by associators, then $S + SR$ is an ideal of $R$.

Obviously, $S$ is the smallest ideal modulo which $R$ is associative. To prove the theorem 2.2.2 it is enough if we can show that $S^2 = 0$, that is, the product of every two elements of $S$ is zero. Let us first look at (i) It is clear that in view of (ii) we are done if we establish $(r,s,t)w(x,y,z) = 0$, for all $r,s,t,w,x,y,z$ of $R$. But

$$w(x,y,z) = (wx,y,z) - (w,xy,z) + (w,x,yz) - (w,x,y)z$$

because of 2.1.1.

By multiplying this equation throughout by $(r,s,t)$ on the left, with the use of (i) and (ii), it becomes obvious that

$$(r,s,t)w(x,y,z) = 0$$
Then $S^2 = 0$, so that $S = 0$. Hence $R$ is associative.

It is shown by Kleinfeld [22] that the identities (1) and (ii) hold in any divisible assosymmetric ring.

Therefore the main theorem that a 2- and 3-divisible assosymmetric ring without ideals $J = 0$, such that $J^2 = 0$ is associative, is a direct consequence of the theorem just established.

In the above theorem we have used two necessary identities and the restriction on the divisibility turns out to be necessary, because there exist even simple, but not 2- and 3-divisible assosymmetric rings which are not associative.

We conclude this section with some examples of assosymmetric rings which are neither flexible nor power-associative (and thus not associative).

**Example 1**: Consider the algebra with basic elements $1, x, y$ over a field which is not 2-divisible. Here $1$ is the unit element and $x^2 = y$, $y^2 = x$, $xy = 1$, $yx = 0$.

It is verified that this is a simple, assosymmetric algebra since $(x, y, x) = x$, the algebra is neither flexible nor power-associative.
Example 2. Any alternative ring which is not 2-divisible is obviously assosymmetric. Consequently the Cayley-Dickson algebras, which are not 2-divisible serve as examples of simple assosymmetric but not associative algebras. Also an assosymmetric algebra, which is not 2-divisible need not be alternative. This is illustrated in [22] by the algebra having basis elements $x, y, z$ over a field, which is not 2-divisible. We define $x^2 = y$, $yx = z$, and all other products of basis elements are zero except $(x, x, x) = z$. Thus this algebra is assosymmetric, but neither flexible nor power-associative.

In the next section, using the identity 2.1.9 that is, each associator is in the nucleus, we discuss solvable assosymmetric rings

2.3 Solvable Assosymmetric Rings

There is a well known theorem for alternative algebras called Wedderburn Principal theorem [31]. We now establish an analogous of this theorem for assosymmetric algebras.

Obviously solvable associative rings are nilpotent, but solvable alternative rings are not nilpotent [13].

If $A$ is a 2- and 3-divisible assosymmetric ring,
Pokrase and Rodabaugh [27] proved that $A$ is solvable if and only if $A$ is nilpotent. In the first approach, they considered $A^*$, the ring generated by the right and left multiplication operators [31] $R_x$ and $L_x$, where $x$ is in $A$. Then we say that $A$ is right nilpotent of index $n$, if for some fixed $n$, $R_{x_1} R_{x_2} \cdots R_{x_n} = 0$ for all $x_i$. Similarly, we define $A$ to be left nilpotent. It is easy to show that all nilpotent rings are right nilpotent and all right nilpotent rings are solvable.

The following identities which obtain in $A^*$ are used:

\[ R_y L_x = L_x R_y - R_y R_x + R_{yx}, \quad \ldots \quad 2.3.1 \]

\[ L_y L_x = L_{yx} - R_x R_y + R_{xy}, \quad \ldots \quad 2.3.2 \]

\[ R_x R_y = R_{yx} - R_{xy} + R_y R_x, \quad \ldots \quad 2.3.3 \]

\[ 0 = R_y R_z R_w R_x - R_{yz} R_w R_x - R_{yw} R_w R_x \]
\[ \quad \quad \quad = R_y R_z R_{wx} + R_{yz} R_{wx} \quad \quad \quad 2.3.4 \]

Identities 2.3.1 and 2.3.2 are both equivalent to the law

\[ (x, y, z) = (z, x, y) \]
Identity 2.3.3 is equivalent to

\[(x, y, z) = (x, z, y)\]

Identity 2.3.4 is a restatement of the equation 2.1.9, which says that \[((w, x, y), z, t) = 0\]

With the help of these identities first they proved that if \(A\) is both left and right nilpotent then \(A\) is nilpotent. Also if \(A\) is right nilpotent then \(A\) is nilpotent. Then the main result states that a 2- and 3-divisible solvable assosymmetric ring is nilpotent. Consequently a 2- and 3-divisible assosymmetric nilring with descending chain condition on right ideals is nilpotent.

In the second approach, they used the property of a 2- and 3-divisible assosymmetric ring stated in section 2.1 that is, each associator is in the nucleus of this ring.

**Lemma 2.3.1.** Let \(A\) be a ring and \(S\) a subring. Assume \(S^k \subseteq N\) for some \(k \geq 1\), where \(N\) is the nucleus of \(A\). Then for each \(n, m \geq 1\) there exists \(l \geq 1\) such that \(S^l \subseteq (S^n)^m\).

**Proof:** We prove this by induction on \(n\). For \(n=1\) take \(l = m\). Next we assume that \(S^l \subseteq (S^n)^m\) and let
t = \max \{k, l\} \quad \text{We claim that } g^{2t+2m-3} \text{ is contained in } (S^m)^{n+1}. \quad \text{For let } x \text{ be in } g^{2t+2m-3} \text{ Then } x \text{ is a sum of terms each of the form } b = a_1 a_2 \ldots a_s, \text{ where } s = 2t + 2m - 3 \text{ and each } a_i \text{ in } S. \text{ If } u \text{ is a factor of } b \text{ in this particular assumption, we define } d(u) \text{ as the number of the } a_i's \text{ in } u \text{ (For example, if } b = a_1 ((a_2 a_3)a_4) \text{ then } d(a_2 a_3) = 2 \text{ But in this association } a_2 a_3 \text{ is not even a factor). Next we choose } y \text{ with } d(y) > t \text{ otherwise let } d(y) \text{ be as small as possible. We claim that } d(y) \leq 2t - 2 \quad \text{For if } d(y) > 2t - 1 \text{ then } y = uv \text{ in this association of the } a_i's \text{ and } d(u) + d(v) > 2t - 1. \text{ Thus } d(u) > t \text{ or } d(v) > t. \text{ This is a contradiction to the choice of } y \text{ Since } S^k \subseteq \mathbb{N}, \text{ it is clear that } y \in \mathbb{N} \text{ and the product of } y \text{ with any of the } a_i's \text{ is also in } \mathbb{N}. \text{ Thus the } a_i's \text{ can be reassOCIATED so that } b = wy, yz \text{ or } wyz \text{ under some association \text{ If } b = wyz, \text{ then } d(w) + d(y) \geq (2t + 2m - 3) - (2t - 2) = 2m - 1 \text{ Therefore } d(w) \geq m \text{ or } d(y) \geq m.
We conclude that \( b \) is in \( S^{m}S^{r}S^{n}, S^{r}S^{s}S^{n}, S^{m}S^{t} \) or \( S^{t}S^{n} \). Hence \( b \) is in \( (S^{m})^{n+1} \). So \( x \) is in \( (S^{m})^{n+1} \). This proves the lemma \( \square \)

Now we prove the main

**Theorem 2.3.2** Let \( R \) be a 2-divisible ring with each associator in the nucleus. If \( R \) is solvable, then \( R \) is nilpotent.

**Proof:** We know that in any ring \( R \), \( J = S + SR \) is an ideal. First, we show that \( J \subseteq N \). Using the Teichmuller identity and the fact that \( (R,R,R) \subseteq N \), we get

\[
(a,b,c)(x,y,z) = (a,b,c(x,y,z)) = -(a,b,(c,x,y)z)
\]

\[
= -(a,b(c,x,y),z) = (a,(b,c,x)y,z)
\]

\[
= (a,b,c,x), y,z) = -(a,b,c)x,y,z)
\]

\[
= -(a,b,c)(x,y,z)
\]

By the divisibility assumption all the above expressions become 0. This shows \( (R,R,R)R \subseteq N \). So \( J \subseteq N \). Now if \( R \) is solvable, then \( R/J \) is a solvable associative ring and therefore nilpotent.

Hence, \( R^{k} \subseteq J \subseteq N \) for some \( k \).
Also J is associative, so \( J^n = 0 \) (in section 2.2 we have proved that \( J^2 = 0 \))

Now we apply the lemma by taking \( S = R \) and \( m = k \). Then there is an \( l \) for which \( R^l \subseteq (R^k)^R \subseteq J^l = 0 \). This shows that \( R \) is nilpotent

We define \( R \) to be nil if each subring generated by a single element is nilpotent.

**Corollary 2.3.3.** Let \( R \) be a 2- and 3-divisible assosymmetric nilring with descending chain condition on right ideals. Then \( R \) is nilpotent.

**Proof:** Let \( J \) be the ideal generated by all associators. In 2.2 we have proved that \( J^2 = 0 \). Since \( R/J \) is an associative nilring with descending chain condition on right ideals, \( R/J \) is solvable. The solvability of \( J \) and \( R/J \) shows that \( R \) is solvable. By the above theorem, \( R \) is nilpotent

Using these results, let us now prove Wedderburn Principal theorem for assosymmetric algebras, which is analogous to alternative algebras (Th.3 18 of [31])

**Theorem 2.3.4.** (Wedderburn Principal theorem for assosymmetric algebras): Let \( U \) be a finite dimensional assosymmetric algebra over a 2- and 3-divisible field \( F \) with
radical \( J \). If \( U/J \) is separable, then

\[
U = V \oplus J
\]

where \( V \) is a subalgebra of \( U \) and \( V \) is isomorphic to \( U/J \).

**Proof:** It is sufficient to prove the existence of \( V \) isomorphic to \( U/J \). Since the theorem is trivial unless \( J \neq 0 \), and since \( J \) is solvable, we have proper inclusions in the series

\[
J = J^{(1)} \supset J^{(2)} \supset \cdots \supset J^{(r)} = 0
\]

Also \( J^2 = J^{(2)} \) is an ideal of \( U \). For \( a \) in \( U \) and \( x, y \) in \( J \) imply

\[
a(xy) = (ax)y - (a, x, y) = (ax)y - (y, a, x)
\]

\[
= (ax)y - (ya)x + y(ax)
\]

is in \( J^2 \) since \( J \) is an ideal. Hence \( J^2 \) is a left ideal of \( U \). Reciprocally, \( J^2 \) is a right ideal of \( U \). The same inductive argument based on the dimension of \( U \) which is used for associative algebras suffices to reduce the proof of the theorem to the case \( J^2 = 0 \). The remaining steps are those of the proof of alternative algebras \([31]\). \( \Box \)

2.4 **Assosymmetric Rings with** \((xy)^2 = x^2y^2\)

Many sufficient conditions are known under which a given ring becomes commutative. Notable among them are
some given by Jacobson, Kaplansky and Herstein. In all these results, they take the ring to be associative. In 1968 Johnson, Outoalt and Yaqub [17] proved that if $R$ is a non-associative ring with unity, in which $(xy)^2 = x^2 y^2$ holds for all $x, y$ in $R$, then $R$ is commutative. In the hypothesis the existence of the unity is indeed essential. Moreover, they have proved that this result does not hold if $(xy)^2 = x^2 y^2$ is replaced by $(xy)^k = x^k y^k$ for any $k > 2$.

Further, Gupta [14] proved that if $R$ is a non-associative ring with unity satisfying the condition $(xy)^2 = (yx)^2$ for all $x, y$ in $R$ and additive group of $R$ has no element of order 2, then $R$ is commutative. Ram Awtar [28] generalized the above mentioned results.

Boere [5] extended the proof of Outoalt and others [17] to show that such a ring is also associative provided it is of 2- and 3-divisible.

Without any restriction on the divisibility, we now prove that any assosymmetric ring in which $(xy)^2 = x^2 y^2$ for all $x, y$ in the ring is commutative and associative.

Theorem 2.4.1. Let $R$ be an assosymmetric ring with unity $e$ such that $(xy)^2 = x^2 y^2$ for all $x, y$ in $R$. Then $R$ is commutative and associative.

Proof: Suppose $x, y$ in $R$. Then $(xy)^2 = x^2 y^2$. Moreover, $(x(e+y))^2 = x^2 (e+y)^2 = x^2 + 2x^2 y + x^2 y^2$.
But also \( (x(y+e)) = (x+y)^2 = x^2 + x(xy) + (xy)x + (xy)^2 \)

Hence

\[
x(xy) + (xy)x = 2x^2y \quad \ldots \quad 2.4.1.
\]

Now replacing \( x \) by \( e+x \) in 2.4.1, we obtain

\[
(e+x) \{ (e+x)y \} + \{(e+x)y \} \ (e+x) = 2(e+x)^2y
\]

That is, \( (e+x)(y+xy) + (y+xy)(e+x) = 2(e+2x+x^2)y \)

That is, \( y+xy+xy+x(xy)+y+xy+yx+(xy)x = 2y+4xy+2x^2y \) This reduces to

\[
x(xy) + yx + (xy)x = xy + 2x^2y
\]

Using 2.4.1 in this, we get

\[
yx = xy
\]

Thus \( R \) is commutative

In an assosymmetric ring the following identity holds:

\[
(xy,z) = x(y,z) + (x,z)y + (x,y,z)
\]

Now this identity together with commutativity imply that \( (x,y,z) = 0 \) Hence \( R \) is associative. \( \square \)

We close this chapter by giving some examples which show that the existence of the unity is essential in the theorem 2 4.1
Example 1. We consider the algebra having basis elements $x, y, z$ over an arbitrary field. We define $x^2 = y$, $yx = z$, and all other products of basis elements equal to zero. This is an asymmetric ring satisfying $(xy)^2 = x^2 y^2$ for all $x$ and $y$ and it is neither commutative nor associative.

In [5] and [17], we have the following examples:

Example 2. Consider the ring consisting of four elements $0, a, b$ and $c$, additively isomorphic to the Klein four group and with multiplication $a^2 = ca = a$, $ab = cb = b$, $ac = c^2 = c$, other products zero.

This is an associative ring satisfying $(xy)^2 = x^2 y^2$ for all $x$ and $y$, and yet not commutative.

Example 3. Let

$$R = \left\{ \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} \mid a, b \text{ integers} \right\}$$

It is readily verified that $(xy)^k = x^k y^k$ for all $x, y$ in $R$ and all $k \geq 1$. However, $R$ is not commutative.

Example 4. Let

$$R = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \right\}$$

be a subring of $2 \times 2$ matrices. Again, it is readily
verified that \((xy)^k = x^k y^k\) for all \(x, y\) in \(R\) and all \(k \geq 1\).

However, \(R\) is not commutative.

Now in the following example we see that the theorem 2 4 1 does not hold if \((xy)^2 = x^2 y^2\) is replaced
by \((xy)^k = x^k y^k\) for any \(k > 2\), even if \(R\) is assumed
to have a unity.

Example 5 Let \(k \geq 3\) be fixed. Let \(p\) be any arbitrary
but fixed prime such that \(p\) divides \(k\) if \(k\) is odd
and \(p\) divides \(k/2\) if \(k\) is even. This is possible
since \(k > 2\). Let \(R\) be a subring of \(3 \times 3\) matrices over
\(GF(p)\) defined by

\[
R = \left\{ \begin{pmatrix} a & b & c \\ 0 & a & d \\ 0 & 0 & a \end{pmatrix} \right/ \left\{ a, b, c, d \text{ are in } GF(p) \right\}
\]

It is readily verified that \((xy)^k = x^k y^k\) for all \(x, y\)
in \(R\). Since \(R\) is not commutative, the equation
\((xy)^2 = x^2 y^2\) in the theorem 2 4 1 cannot be replaced by
\((xy)^k = x^k y^k\) for any \(k \geq 3\).