Chapter 1

INTRODUCTION
In this chapter we present definitions of the basic concepts which are necessary for the understanding of the results discussed in this thesis. A brief survey of the work done by Bruck, Kleinfeld, Albert, Cutoalt and Sitaram in alternative, $n$-associative and $(3, 2k+1)$-associative rings is also given.

1.1 Algebraic Preliminaries

In this section we give definitions of some algebraic concepts which are indispensable for our work.

Non-associative ring

Definition 1.1.1 A non-associative ring $R$ is an additive abelian group in which a multiplication is defined, which is distributive over addition on the left as well as on the right, that is,

$$(x+y)z = xz + yz, \quad z(x+y) = zx + zy$$

for all $x, y, z$ in $R$

Non-associative algebra

Definition 1.1.2 A non-associative algebra $U$ over a field $F$ is a non-associative ring which is a vector
space over $F$ with

$$\alpha(xy) = (\alpha x)y = x(\alpha y)$$

for all $\alpha$ in $F$, $x, y$ in $U$

A non-associative ring (or algebra) differs from an associative ring (or algebra) in that the multiplication is no longer assumed to be associative. By a ring $R$ being non-associative, we mean that $R$ is not necessarily associative, that is, we do not assume

$$(xy)z = x(yz)$$

for all $x, y, z$ in $R$, as an axiom. However, it does not mean there always exist elements $x, y, z$ in $R$ such that $(xy)z \neq x(yz)$. The well-known examples of non-associative rings are alternative rings, Lie rings, and Jordan rings, which are defined below.

**Alternative ring**

**Definition 1.1.3** An alternative ring $R$ is a ring in which

$$(xx)y = x(xy), \quad y(xx) = (yx)x,$$

for all $x, y$ in $R$. These equations are known as the left and right alternative laws respectively.
Lie ring

Definition 1.1.4. A Lie ring $R$ is a ring in which the multiplication is anticommutative, that is,

$$x^2 = 0 \text{ (implying } xy = -yx)$$

and the Jacobi identity

$$(xy)z + (yz)x + (zx)y = 0$$

for all $x, y, z$ in $R$, is satisfied

Jordan ring

Definition 1.1.5. A Jordan ring $R$ is a ring in which products are commutative, that is,

$$xy = yx,$$

and satisfy the Jordan identity

$$(xy)x^2 = x(yx^2)$$

for all $x, y$ in $R$

Zorn [40] introduced the so-called associator in a non-associative ring, which can in a way be thought of as a measure of the non-associativity of a ring.
**Associator**

**Definition 1.1.6.** The associator \((x,y,z)\) is defined by

\[(x,y,z) = (xy)z - x(yz)\]

for all \(x,y,z\) in a ring. This plays a central role in the study of non-associative rings. Obviously if it is zero for all \(x,y,z\) in the ring, then the ring is associative. In terms of associators, a ring \(R\) is alternative if

\[(x,x,y) = (y,x,x) = 0\]

for all \(x,y\) in \(R\).

**Commutator**

**Definition 1.1.7.** The commutator \((x,y)\) is defined by

\[(x,y) = xy - yx\]

for all \(x,y\) in a ring. This can be considered to be a measure of non-commutativity of a ring.
**Flexible law**

**Definition 1.1.8** The identity \((x,y,x) = 0\), that is,

\[(xy)x = x(yx)\]

for all \(x,y\) in \(R\) is called the flexible law.

Alternative, commutative, anticommutative and thereby Jordan and Lie rings are flexible

**Nucleus**

**Definition 1.1.9** By the nucleus \(N\) of a ring \(R\) we mean the set of all elements \(n\) in \(R\) such that

\[(n,R,R) = (R,n,R) = (R,R,n) = 0\]

**Center**

**Definition 1.1.10** By the center \(C\) of \(R\) we mean the set of all elements \(c\) in \(N\) such that \((c,R) = 0\)

It is easily verified that \(N\) is a subring of \(R\) and \(C\) is a subring of \(N\). Obviously we note that \(N=R\) if and only if \(R\) is an associative ring and \(C=R\) if and only if \(R\) is associative and commutative
Divisibility

Definition 1.1.11. Following Boers [6] we define a ring $R$ to be $m$-divisible (m, a natural number) if $mx = 0$ implies $x = 0$ for all $x$ in $R$. An equivalent definition is: $R$ is $m$-divisible if the order of an arbitrary element ($\neq 0$) of the additive structure $R^+$ does not divide $m$. In many earlier publications such a ring was called a ring with characteristic not equal to a prime $\leq m$. But this is wrong, since the characteristic of a ring is to be defined as the smallest natural number $n$ for which $nx = 0$ for all $x$ in $R$. If $R$ has characteristic $\neq n$, there is no guarantee that $mx = 0$ implies $x = 0$ for all $x$ in $R$. Indeed, if $R$ has characteristic $\neq 2$, say 4 (e.g., in the case that $R^+$ is isomorphic to a cyclic group of order 4), then it is not true that $2x = 0$ implies $x = 0$ for all $x$ in $R$.

Division ring

Definition 1.1.12. A ring is said to be a division ring if its non-zero elements form a group with respect to multiplication.

Assosymmetric ring

Definition 1.1.13. An assosymmetric ring $R$ is one in which
\[(x, y, z) = (P(x), P(y), P(z)),\]

where \(P\) is any permutation of \(x, y, z\) in \(R\)

**Power-associative ring**

**Definition 1.1.14.** A ring \(R\) is power-associative if every subring of \(R\) generated by a single element is associative.

2-divisible alternative, Jordan and Lie rings are power-associative. Obviously asymmetric rings are not power-associative.

**Simple ring**

**Definition 1.1.15.** A ring \(R\) is said to be simple if whenever \(A\) is an ideal of \(R\), then either \(A=R\) or \(A=0\).

**Prime ring**

**Definition 1.1.16.** A ring \(R\) is prime if whenever \(A\) and \(B\) are ideals of \(R\) such that \(AB=0\), then either \(A=0\) or \(B=0\).

**Semi-prime ring**

**Definition 1.1.17.** A ring \(R\) is semi-prime if for any ideal \(A\) of \(R\), \(A^2=0\) implies \(A=0\). These rings are also referred to as rings free from trivial ideals.
Semi-simple ring

Definition 1.1.16 A ring is semi-simple in case the radical (that is, the maximal ideal consisting of all nilpotent elements) is the zero ideal.

Obviously a simple ring is prime, which in turn is free from trivial ideals.

Primitive ring

Definition 1.1.19 A ring is defined as primitive in case it possesses a regular maximal right ideal, which contains no ideal of the ring other than the zero ideal.

Solvable ring

Definition 1.1.20 A ring is solvable if the chain of subrings \( A \supseteq A^2 \supseteq (A^2)^2 \supseteq \) reaches zero in a finite number of steps.

Nilpotent ring

Definition 1.1.21 A ring is called nilpotent if there is a fixed positive integer \( t \) such that every product involving \( t \) elements is zero.

The concepts of the \( n \)-associator and \( n \)-associative rings were defined by Boers [3]. The \( n \)-associator is
is defined as an extension of the associator \((x, y, z) = (xy)z - x(yz)\)

**n-associator**

**Definition 1.1.22** If \(a_1, a_2, \ldots, a_n\) are elements of a ring \(R\), then the \(n\)-associator \((a_1, a_2, \ldots, a_n)\) is defined as follows:

\[
(a_1, a_2) = a_1 a_2
\]

and

\[
(a_1, a_2, \ldots, a_n) = \sum_{k=1}^{n-1} (-1)^{k-1} (a_1, a_2, \ldots, a_k a_{k+1}, \ldots, a_n), n \geq 3
\]

We are interested mainly in \(n\)-associators for \(n \geq 3\)

**n-associative ring**

**Definition 1.1.23** By \(S(n, n)\) we mean the additive subgroup of \(R\) generated by \(n\)-associators. If every \(n\)-associator is zero, that is, if \(S(n, n) = 0\), we say that \(R\) is \(n\)-associative. A 3-associative ring is merely the usual associative ring.

**\((2j+1, 2k+1)\)-associative ring**

**Definition 1.1.24** By \(S(2j+1, 2k+1), k > j \geq 1\) we mean the subgroup of the additive group of \(R\) generated by \((2j+1)\)-associators satisfying the following properties:
(i) \((a_1, \ldots, a_{2j+1})\) is in \(S(2j+1, 2k-1)\)

and

(ii) atleast one of \((2k-1)\) entries in \((a_1, \ldots, a_{2j+1})\) is an element of \(S(3,3)\)

\(R\) is said to be \((2j+1)\)-associative of degree \((2k+1)\), or in short, \((2j+1, 2k+1)\)-associative if \(S(2j+1, 2k+1)=0\)

If \(j=k\), the degree is ignored

**Direct sum**

**Definition 1.1.25** If \(B\) and \(C\) are ideals of an algebra \(U\), then \(U\) is the direct sum of \(B\) and \(C\) when \(U = B \oplus C\) and \(B \cap C = 0\). It is denoted by

\[ U = B \oplus C \]

**Separable algebra**

**Definition 1.1.26.** A finite-dimensional algebra \(U\) over \(F\) is called separable in case, for every extension \(K\) of \(F\), the algebra \(U_K\) is a direct sum of simple ideals.

Clearly any associative ring is alternative. But an important class of alternative algebras which are not associative is the class of 8-dimensional Cayley algebras or Cayley-Dickson algebras. Cayley studied the so-called Cayley numbers in 1845 [11]. Dickson later reformed them in [12].
Cayley-Dickson algebra

**Definition 1.1.27.** Cayley-Dickson algebra is an eight-dimensional algebra over a field $F$ such that any element not in $F$ generates a quadratic field over $F$ and any two elements (not in the same quadratic extension) generates a quaternion algebra. We define a Cayley-Dickson algebra in terms of its multiplication table for our purposes. That is, a Cayley-Dickson algebra is an eight-dimensional vector space over $F$, having basis elements $u_0, u_1, \ldots, u_7$, where $u_0$ acts as a unit element and the rest of the multiplication table is as follows:

\[
\begin{array}{cccccccc}
& u_1 & u_2 & u_3 & u_4 & u_5 & u_6 & u_7 \\
\hline
u_1 & au_0 & u_3 & au_2 & u_5 & au_4 & -u_7 & -au_6 \\
u_2 & -u_3 & bu_0 & -bu_1 & u_6 & u_7 & bu_4 & bu_5 \\
u_3 & -au_2 & bu_1 & -au_0 & u_7 & au_6 & -bu_5 & -bu_4 \\
u_4 & -u_3 & -u_6 & -u_7 & au_0 & au_1 & -au_2 & -au_3 \\
u_5 & -au_4 & -au_7 & -au_6 & au_1 & -au_0 & au_3 & au_2 \\
u_6 & u_7 & -bu_4 & bu_5 & au_2 & -au_3 & -bu_0 & -bu_1 \\
u_7 & au_6 & -bu_3 & au_4 & au_3 & -au_2 & -bu_1 & -bu_0 \\
\end{array}
\]
It is assumed that $\alpha$, $\beta$, $\gamma$ are elements of $F$. In general a Cayley-Dickson algebra need not be a division algebra. Exact conditions on $\alpha$, $\beta$, $\gamma$ and the field $F$ that will make the algebra a division algebra are found in [30].

1.2 Geometric Preliminaries

In this section we give some basic definitions of geometric concepts.

**Projective plane**

**Definition 1.2.1** A projective plane is a set $\Pi$ of elements called points, and some subsets of $\Pi$ called lines, together with an incidence relation between the points and lines such that:

(i) any two distinct points are incident with a unique line

(ii) any two distinct lines are incident with a unique point

(iii) there exists four points, no three of them are collinear

Points on the same line are called collinear and the lines through the same point are called concurrent.
Collineation

**Definition 1.2.2.** An isomorphism $\phi$ of points onto themselves of a projective plane $\Pi$ which preserves incidence is called a collineation.

Central collineation

**Definition 1.2.3.** A collineation $\phi$ of a projective plane $\Pi$ is called a central collineation if it fixes every point on a line $l$ of $\Pi$ and every line through the point $C$ of $\Pi$. The point $C$ is called the center and the line $l$ the axis of the central collineation.

In a projective plane $\Pi$, if for a given center $C$ and axis $l$, all possible central collineations exist then the plane $\Pi$ is said to be $(C,l)$-transitive.

Translation plane

**Definition 1.2.4.** If a projective plane $\Pi$ is $(C,l)$-transitive for every point $C$ on a line $l$, then $\Pi$ is called a translation plane with respect to $l$.

Moufang plane

**Definition 1.2.5.** A projective plane $\Pi$ is called a Moufang plane if $\Pi$ is a translation plane with respect to every line of $\Pi$. 
Desarguesian plane

Definition 1.2.6. If the projective plane \( \Pi \) is 
\((0,1)\)-transitive for every point \( O \) and every line \( l \) 
of \( \Pi \), then \( \Pi \) is called a Desarguesian plane

Harmonic Conjugate

Definition 1.2.7. Let \( A, B, C \) be any three distinct 
points incident with a given line \( l \) and \( a,b,c \) any 
three non-concurrent lines through \( A,B,C \) respectively 
Let \( P = b \cap c \), \( Q = c \cap a \), \( R = a \cap b \) and \( S = AP \cap BQ \) 
If \( RS \cap l = D \), then \( D \) is called the harmonic conjugate 
of \( c \) with respect to \( A, B \) and it is denoted by 
\( D = (A,B)/C \). If \( D = (A,B)/C \) then we say that \( A, B \) 
and \( O,D \) are harmonically conjugate pairs, or simply, 
harmonic pairs. The points \( A, B, C, D \) are also called 
a set of harmonic points

1.3 A Brief Survey of Earlier Work

Bruck and Kleinfeld [10] studied alternative 
division rings and proved the classic result that an 
alternative division ring which is 2-divisible is either 
an associative division ring or isomorphic to a Cayley- 
Dickson algebra. Actually the restriction on the 
divisibility is not necessary This was proved later by 
Kleinfeld [38].
Right alternative algebras were first studied by Albert in 1949, who showed that a semi simple, right alternative algebra over a field is alternative [1]. Skorniakov had shown that a 2-divisible right alternative division ring is alternative [37]. Kleinfeld [20] proved that a simple alternative ring is either a Cayley-Dickson algebra or associative. Thus it is clear that the distinction between alternative and associative rings is really very thin.

In 1956, Boers [3] proved that an n-associative division ring is associative with minor restriction on the divisibility. Later, Outoolt [26] had established the following results: (i) Simple 4-associative and simple 5-associative rings are associative. (ii) Simple 2k-associative rings are (2k-1)-associative or have zero center, and furthermore (iii) Simple, commutative, n-associative rings where 6 ≤ n ≤ 9, are associative. Also it is known that simple, commutative rings which are associative of degree 2k+1 are associative. The divisibility of the ring is however slightly restricted.

In [3] it is shown that a 2n-associative prime ring \( R \) whose center \( Z \neq 0 \) is (2n-1)-associative. For \( n=2 \) the same result is proved for a commutative ring.
without restriction that \( Z \neq 0 \). Also a finite ring whose nucleus contains a regular element has a unity element.

Without using the properties of the nucleus of \( R \), Sitaram [34, 35] proved the following main results:

1. Suppose \( R \) contains only the two trivial right ideals. If \( R \) is either \((3,5)\)-associative and 2-divisible or \((3,7)\)-associative and 2- and 3-divisible, then \( R \) is associative and (ii) If \( R \) is \((3,2k+1)\)-associative and \((k+1)\)-divisible, then \( R \) is \((2k-1)\)-associative.

Also Sitaram had established some geometric applications of non-associative rings. It is known that in a Projective Desarguesian plane there cannot exist more than five distinct harmonic pairs [32]. It has also been shown that there cannot exist more than nine distinct harmonic pairs in a Moufang plane [33].

The above studies of Bruck, Kleinfeld, Albert, Outcalt, Boers and Sitaram have opened up many avenues for further work. There have been several interesting studies of other types of non-associative rings. For example, Kleinfeld had introduced the so-called assosymmetric rings. In the next chapter we discuss the structure of these rings.