Neutral Difference Equations With "Maxima" - II

6.1 Introduction

In the final chapter, we again consider a neutral difference equation with "maxima" of the form

\[ \Delta (a(n) \Delta (y(n) + p(n)y(n - k))) - q(n) \max_{[n-l,n]} y(s) = 0, n \in \mathbb{N}_0, \]

(6.1.1)

where \( k \) and \( \ell \) are integers with \( k \geq 1 \) and \( \ell \geq 0 \), \( \{a(n)\}, \{p(n)\} \) and \( \{q(n)\} \), are real sequences and discuss the oscillatory and asymptotic behavior of solutions of equation (6.1.1).

Let \( \theta = \max \{k, \ell\} \). Then by a solution of equation (6.1.1), we mean a real sequence \( \{y(n)\} \) defined for \( n \geq -\theta \) and which satisfies equation (6.1.1) for \( n \in \mathbb{N}_0 \).

The content of this chapter has been communicated for publication Journal of Mathematical Research and Exposition.
Clearly in this case if we are given real numbers
\[ y(n) = b(n), \quad n = -m_0, -m_0 + 1, \ldots, 0 \]  
(6.1.2)
as a set of initial conditions, then equation (6.1.1) has a unique solution satisfying (6.1.2).

In this chapter, we investigate asymptotic behavior of nonoscillatory solutions of equation (6.1.1). We also establish sufficient conditions to ensure that every bounded/unbounded solutions of equation (6.1.1) to be oscillatory.

Note that since equation (6.1.1) is nonlinear, assuming a solution is of one sign requires that the cases \( y(n) > 0 \) and \( y(n) < 0 \) must both be considered. We shall say that conditions (H) are met if the following conditions hold:

(H1) \( \{a(n)\} \) is a positive sequence of real numbers such that \( \sum_{n=n_0}^{\infty} \frac{1}{a(n)} = \infty \);

(H2) \( \{q(n)\} \) is a sequence of nonnegative real numbers such that \( \sum_{n=n_0}^{\infty} q(n) = \infty \).

We often use the sequence \( \{z(n)\} \) which is defined as follows:
\[ z(n) = y(n) + p(n) y(n - k). \]  
(6.1.3)
Then equation (6.1.1) implies that
\[ \Delta (a(n) \Delta z(n)) = q(n) \max_{[n-\ell,m]} y(s), \]  
(6.1.4)
and
\[ a(n) \Delta z(n) = a(n_0) \Delta z(n_0) + \sum_{s=n_0}^{n-1} q(s) \max_{[s-\ell,s]} y(t). \]  
(6.1.5)
6.2 Preliminary Lemmas

In this section we state and prove some lemmas, which are needed in the sequel to prove our main results.

**Lemma 6.2.1.** Let conditions \((H)\) hold and there exists a constant \(p\) such that \(p \leq p(n) \leq 0\).

(a) If \(\{y(n)\}\) is an eventually positive solution of equation (6.1.1), then the sequences \(\{z(n)\}\) and \(\{a(n)\Delta z(n)\}\) are eventually monotonic and either

\[
z(n) > 0, \ \Delta z(n) > 0, \ \Delta (a(n)\Delta z(n)) \geq 0
\]

and

\[
\lim_{n \to \infty} z(n) = \lim_{n \to \infty} a(n)\Delta z(n) = \infty \tag{6.2.1}
\]

or

\[
z(n) > 0, \ \Delta z(n) < 0, \ \Delta (a(n)\Delta z(n)) \geq 0
\]

and

\[
\lim_{n \to \infty} z(n) = \lim_{n \to \infty} a(n)\Delta z(n) = 0. \tag{6.2.2}
\]

(b) If \(\{y(n)\}\) is an eventually negative solution of equation (6.1.1), then the sequences \(\{z(n)\}\) and \(\{a(n)\Delta z(n)\}\) are eventually monotonic and either

\[
z(n) < 0, \ \Delta z(n) < 0, \ \Delta (a(n)\Delta z(n)) \leq 0
\]
and

\[ \lim_{n \to \infty} z(n) = \lim_{n \to \infty} a(n) \Delta z(n) = -\infty \]  
(6.2.3)

or

\[ z(n) < 0, \ \Delta z(n) > 0, \ \Delta (a(n) \Delta z(n)) \leq 0 \]

and

\[ \lim_{n \to \infty} z(n) = \lim_{n \to \infty} a(n) \Delta z(n) = 0. \]  
(6.2.4)

Proof. (a) Let \( \{y(n)\} \) be an eventually positive solution of equation (6.1.1). From (6.1.4), it follows that

\[ \Delta (a(n) \Delta z(n)) = q(n) \max_{\lfloor n - \ell, n \rfloor} y(s) \geq 0 \]

eventually and \( a(n) \Delta z(n) \) is a nondecreasing sequence. On the other hand \((H_4)\) implies that \( q(n) \neq 0 \) and therefore \( \{a(n) \Delta z(n)\} \) is eventually of one sign and in consequence \( \{z(n)\} \) is eventually monotonic.

First suppose that there exists an integer \( n_1 \geq n_0 \in \mathbb{N} \) such that \( a(n) \Delta z(n) > 0 \) for \( n \geq n_1 \). Then there exists an integer \( n_2 > n_1 \) such that

\[ a(n) \Delta z(n) \geq a(n_2) \Delta z(n_2) = c > 0 \text{ for } n \geq n_2. \]

Summing the last inequality, by \((H_1)\) we have

\[ z(n) \geq z(n_2) + c \sum_{s=n_2}^{n-1} \frac{1}{a(s)} \to \infty, \quad n \to \infty, \]

hence \( z(n) \to \infty \) as \( n \to \infty \). Since \( y(n) \geq z(n) \) so \( y(n) \to \infty \) as \( n \to \infty \). From (6.1.5) and \((H_2)\), we see that \( a(n) \Delta z(n) \to \infty \) as \( n \to \infty \), and thus (6.2.1) holds.
Next if \( a(n) \Delta z(n) < 0 \) for \( n \geq n_0 \), then \( a(n) \Delta z(n) \to L \leq 0 \) as \( n \to \infty \). Suppose that \( L < 0 \). Then \( a(n) \Delta z(n) < L \) and by \((H_1)\), \( \lim_{n \to \infty} z(n) = -\infty \). From (6.1.3) it follows that the inequality
\[
z(n) > p(n) y(n - k) > py(n - k)
\]
is valid and therefore \( \lim_{n \to \infty} y(n) = \infty \). From (6.1.5) we obtain that \( \lim_{n \to \infty} a(n) \Delta z(n) = \infty \). The contradiction obtained shows that \( \lim_{n \to \infty} a(n) \Delta z(n) = 0 \) and since \( \{a(n) \Delta z(n)\} \) is a nondecreasing sequence, we have \( a(n) \Delta z(n) < 0 \) and \( \{z(n)\} \) is a decreasing sequence.

Suppose \( \lim_{n \to \infty} z(n) = L \), where \( L \) is finite. Let \( L > 0 \). The inequality \( y(n) > z(n) \) implies that \( y(n) > L \). From (6.1.5) and \((H_2)\) it follows that the relation \( \lim_{n \to \infty} a(n) \Delta z(n) = \infty \) is valid and we obtain a contradiction. Then \( L \leq 0 \). Let \( L < 0 \). The estimate
\[
\frac{L}{2} > z(n) = y(n) + p(n) y(n - k) > p(n) y(n - k) > py(n - k)
\]
is valid. From the inequality \( y(n - k) > \frac{L}{2p} > 0 \), we obtain
\( \lim_{n \to \infty} a(n) \Delta z(n) = \infty \), which is a contradiction. Thus \( L = 0 \) and since \( \{z(n)\} \) is a decreasing sequence, then \( z(n) > 0 \). Suppose that \( \lim_{n \to \infty} z(n) = -\infty \). As above the inequality \( y(n - k) > \frac{z(n)}{p} \) holds and \( \lim_{n \to \infty} y(n) = \infty \). From (6.1.5), it follows that \( \lim_{n \to \infty} a(n) \Delta z(n) = \infty \), and we again obtain a contradiction. Thus if \( \Delta z(n) < 0 \), then (6.2.2) is valid.

The proof of (b) is similar to that of (a) and hence the details are omitted. \( \square \)

**Lemma 6.2.2.** The sequence \( \{y(n)\} \) is a negative solution of equation
(6.1.1) if and only if \(\{-y(n)\}\) is a positive solution of the equation

\[
\Delta (a(n) \Delta (y(n) + p(n) y(n - k))) - q(n) \min_{[n-n, n]} y(s) = 0, \quad (6.2.5)
\]

**Proof.** The proof is straightforward and hence the details are omitted. \(\square\)

### 6.3 Asymptotic Behavior of Nonoscillatory Solutions

Here we give some oscillatory and asymptotic properties of solutions of equation (6.1.1).

**Theorem 6.3.1.** Let conditions (H) hold. If there exists a constant \(p\) such that \(p \leq p(n) \leq -1\), then every nonoscillatory solution \(\{y(n)\}\) of equation (6.1.1) satisfies \(|y(n)| \to \infty\) as \(n \to \infty\).

**Proof.** Let \(\{y(n)\}\) be an eventually negative solution of (6.1.1). Then Lemma 6.2.1 implies that (6.2.3) or (6.2.4) is valid. Suppose that (6.2.3) holds. Then from the inequality \(y(n) < z(n)\) it follows that \(\lim_{n \to \infty} y(n) = -\infty\) and the assertion of the theorem is proved. Suppose that (6.2.4) is valid and \(c = \limsup_{n \to \infty} y(n)\). If \(c < 0\), then \(y(n) < \frac{c}{2}\) and from (6.1.5) we obtain \(\lim_{n \to \infty} a(n) \Delta z(n) = -\infty\), which contradicts the relation \(\lim_{n \to \infty} a(n) \Delta z(n) = 0\) proved in Lemma 6.2.1. Hence \(c = 0\), that is, \(\limsup_{n \to \infty} y(n) = 0\). Then there is an increasing sequence of positive integers \(\{n_j\}\) such that \(y(n_j) \to 0\) as \(j \to \infty\) and

\[
\max_{[n_1, n_j]} y(s) = y(n_j). \quad (6.3.1)
\]

On the other hand, since \(z(n) < 0\), then \(y(n) < -p(n) y(n - k) \leq y(n - k)\). But the inequality \(y(n_j) < y(n_j - k)\) contradicts (6.3.1).
Thus only relation (6.2.3) holds and \( \lim_{n \to \infty} y(n) = -\infty \). The case when \( \{y(n)\} \) is eventually positive considered analogously. This completes the proof.

From Theorem 6.3.1, we immediately obtain,

**Corollary 6.3.2.** Under the assumptions of Theorem 6.3.1 all bounded solutions of equation (6.1.1) are oscillatory.

**Theorem 6.3.3.** Let conditions (H) hold and let \( \{p(n)\} \) satisfy one of the following conditions

\[
-1 < p \leq p(n) \leq 0 \quad (6.3.2)
\]

or

\[
0 < p(n) \leq p \leq 1 \text{ and } k \leq \ell. \quad (6.3.3)
\]

Then for each nonoscillatory solution \( \{y(n)\} \) of equation (6.1.1) either

\[
\lim_{n \to \infty} y(n) = 0 \text{ and } \lim_{n \to \infty} |y(n)| = \infty.
\]

**Proof.** We shall first consider the case when (6.3.2) is satisfied. Let \( \{y(n)\} \) be an eventually bounded positive solution of equation (6.1.1). Clearly in this case of the relations (6.2.1) and (6.2.2) only (6.2.2) is valid and thus \( \lim_{n \to \infty} z(n) = 0 \). Suppose that \( c = \limsup_{n \to \infty} y(n) > 0 \). Then there is an increasing sequence of integers \( \{n_j\} \) such that \( \lim_{j \to \infty} y(n_j) = c \). Choose a constant \( \alpha \) such that \( 1 < \alpha < -\frac{1}{p} \) (if \( p(n) \equiv 0 \) then \( p \) could be any constant in \((-1,0))\). Then \( y(n) < \alpha c \) for sufficiently large \( n \) and we have

\[
z(n) = y(n) + p(n)y(n-k) \geq y(n) + \alpha c
\]
6.3. ASYMPTOTIC BEHAVIOR OF NONOSCILLATORY SOLUTIONS

Hence
\[ z(n_j) \geq y(n_j) + p\alpha c \]
as \( j \to \infty \) and obtain \( 0 \geq c + p\alpha c = c(1 + p\alpha) > 0 \). This contradiction shows that \( \limsup_{n \to \infty} y(n) = 0 \) and \( \lim_{n \to \infty} y(n) = 0 \). Next let us assume that \( \{y(n)\} \) is an unbounded solution of (6.1.1). We shall show that in this case relation (6.2.1) is valid. Assume this is not true. Since \( \{y(n)\} \) is unbounded there is an increasing sequence of positive integers \( \{n_i\} \) such that \( y(n_i) \to \infty \) as \( i \to \infty \) and \( y(n_i) = \max_{[n_1, n_i]} y(n) \). Then we have
\[ z(n_i) = y(n_i) + p(n_i) y(n_i - k) \]
\[ \geq y(n_i) + p(n_i) y(n_i) \geq y(n_i) (1 + p) . \] (6.3.4)

From (6.3.2), (6.3.4) implies that \( \lim_{i \to \infty} z(n_i) = \infty \) which contradicts the relation \( \lim_{n \to \infty} z(n) = 0 \). Hence (6.2.1) is valid and \( \lim_{n \to \infty} z(n) = \infty \).

From the inequality \( y(n) > z(n) \), it follows that \( \lim_{n \to \infty} y(n) = \infty \). The proof is similar when \( \{y(n)\} \) is an eventually negative solution of equation (6.1.1). Now assume that (6.3.3) holds. Let \( \{y(n)\} \) be an eventually positive solution of equation (6.1.1). From (6.1.4), it follows that \( \Delta (a(n) \Delta z(n)) \geq 0 \) and \( a(n) \Delta z(n) \) and is nondecreasing.

Condition (H2) then implies that either \( a(n)\Delta z(n) > 0 \) or \( a(n)\Delta z(n) < 0 \). Let \( a(n) \Delta z(n) > 0 \). Clearly, \( \lim_{n \to \infty} z(n) = \infty \) and \( \{z(n)\} \) is an increasing sequence. From (6.1.3), we have
\[ y(n) = z(n) - p(n) y(n - k) \]
\[ \geq z(n) - p(n) z(n - k) \geq (1 - p) z(n) , \] (6.3.5)
where we have used the increasing character of \( \{z(n)\} \) and \( z(n) \geq y(n) \). Since \( \lim_{n \to \infty} z(n) = \infty \), from (6.3.5) we have \( \lim_{n \to \infty} y(n) = \infty \).
Next assume that \( \{a(n) \Delta z(n)\} \) is eventually negative. In this case, we obtain \( \{z(n)\} \) is a positive decreasing sequence. If \( \lim_{n \to \infty} a(n) \Delta z(n) = c < 0 \), then by (H1), we have \( \lim_{n \to \infty} z(n) = -\infty \).

Therefore, \( \lim_{n \to \infty} a(n) \Delta z(n) = 0 \). Second, we prove that \( \lim_{n \to \infty} z(n) = 0 \). Since \( \{z(n)\} \) is a positive decreasing sequence, \( \lim_{n \to \infty} z(n) = d \) exists with \( d \geq 0 \). Assume \( d > 0 \). Then \( z(n) > d \) eventually and

\[
d < y(n) + py(n - k) < (1 + p) \max \{y(n), y(n - k)\}.
\]

Thus

\[
\max \{y(n), y(n - k)\} > \frac{d}{1 + p}.
\]

Since \( k \leq \ell \), from the previous inequality it follows that

\[
\max \{y(n - \ell), y(n - \ell + 1), y(n - \ell + 2), \ldots, y(n)\} > \frac{d}{1 + p}.
\]

From (6.1.5) and (H2) we obtain \( \lim_{n \to \infty} a(n) \Delta z(n) = \infty \). This contradiction shows that \( \lim_{n \to \infty} z(n) = 0 \). Then (6.1.3) implies that \( \lim_{n \to \infty} y(n) = 0 \). A similar argument treats the case of negative solution of equation (6.1.1). This completes the proof of the theorem.

\[\square\]

**Theorem 6.3.4.** Let \( p(n) \equiv 1 \) and conditions (H1) and (H2) hold with the condition \( \sum_{n=1}^{\infty} q(n) = \infty \) replaced by

\[
\sum_{n=1}^{\infty} \bar{q}(n) = \infty \text{ when } \bar{q}(n) = \min\{q(n), q(n + k)\}.
\]

Then for each bounded positive solution \( \{y(n)\} \) of equation (6.1.1),

\[
\lim_{n \to \infty} y(n) = 0.
\]
6.3. ASYMPTOTIC BEHAVIOR OF NONOSCILLATORY SOLUTIONS

Proof. Since \( \{ y (n) \} \) is an eventually positive solution of equation (6.1.1), we have \( \Delta (a(n) \Delta z(n)) \geq 0 \) and \( \{ a(n) \Delta z(n) \} \) is a nondecreasing sequence. Condition (6.3.6) implies that \( q(n) \neq 0 \) eventually. Then either \( a(n) \Delta z(n) > 0 \) or \( a(n) \Delta z(n) < 0 \) eventually. If \( a(n) \Delta z(n) > 0 \) then \( \lim_{n \to \infty} z(n) = \infty \) which contradicts the boundedness of \( \{ y(n) \} \). Hence \( a(n) \Delta z(n) < 0 \) and \( \{ z(n) \} \) is a positive decreasing sequence.

Let \( c = \lim_{n \to \infty} z(n) \) and suppose that \( c > 0 \). From (6.1.4) it follows that

\[
\Delta (a(n) \Delta z(n)) + q(n - k) \max_{[n-\ell-k,n-k]} y(s) = q(n) \max_{[n-\ell,n]} y(s) + q(n - k) \max_{[n-\ell-k,n-k]} y(s).
\]

Then using the definition of \( \bar{q}(n) \) and of (6.1.3), we obtain that

\[
\Delta (a(n) \Delta z(n)) + q(n - k) \max_{[n-\ell-k,n-k]} y(s) \geq \bar{q}(n - k) \left[ \max_{[n-\ell,n]} y(s) + \max_{[n-\ell-k,n-k]} y(s) \right] \geq \bar{q}(n - k) \left[ \max_{[n-\ell,n]} y(s) + \max_{[n-\ell,n]} y(s - k) \right] = \bar{q}(n - k) \max_{[n-\ell,n]} (y(s) + y(s - k)) = \bar{q}(n - k) \max_{[n-\ell,n]} z(s) = \bar{q}(n - k) z(n - \ell).
\]

Since \( \{ z(n) \} \) is a decreasing sequence and \( \lim_{n \to \infty} z(n) = c \), then \( z(n) > c \), and the last inequality takes the form

\[
\Delta (a(n) \Delta z(n)) + q(n - k) \max_{[n-\ell-k,n-k]} y(s) \geq c\bar{q}(n - k).
\]
Summing the last inequality from \( n_1 \) to \( n - 1 \), we obtain
\[
a(n) \Delta z(n) - a(n_1) \Delta z(n_1) + \sum_{s=n_1}^{n-1} q(s - k) \max_{[n-\ell-k,n-k]} y(t) > c \sum_{s=n_1}^{n-1} \bar{q}(s - k)
\]
or
\[
a(n) \Delta z(n) - a(n_1) \Delta z(n_1) + \sum_{s=n_1-k}^{n-1-k} q(s) \max_{[s-\ell,s]} y(t) \geq c \sum_{s=n_1-k}^{n-1-k} \bar{q}(s).
\]

Since \( \{a(n) \Delta z(n)\} \) is a negative nondecreasing sequence, then \( \{a(n) \Delta z(n)\} \) is a bounded sequence. On the other hand (6.3.6) implies that the right hand side of (6.3.7) tends to infinity as \( n \to \infty \). Thus from (6.3.7) we obtain
\[
\sum_{n=n_1}^{\infty} q(n) \max_{[n-\ell,n]} y(s) = \infty.
\]

Summing (6.1.4) from \( n_1 \) to \( n - 1 \), we obtain
\[
a(n) \Delta z(n) - a(n_1) \Delta z(n_1) = \sum_{s=n_1}^{n-1} q(s) \max_{[s-\ell,s]} y(t)
\]

Then (6.3.8) implies that \( \lim_{n \to \infty} a(n) \Delta z(n) = \infty \). The contradiction obtained shows that \( c = 0 \), that is, \( \lim_{n \to \infty} z(n) = 0 \). But from the inequality \( y(n) < z(n) \) it follows that \( \lim_{n \to \infty} y(n) = 0 \) and the proof is complete.

**Remark 6.3.1.** In contrast to neutral equations without "maxima", the assertion of Theorem 6.3.4 is not valid for bounded negative solutions.
of equation (6.1.1) even under stronger conditions $q(n) \geq q > 0$ for all $n \in \mathbb{N}_0$. We shall illustrate this fact with the following example.

**Example 6.3.1.** Consider the difference equation

$$
\Delta^2 (y(n) + y(n - 1)) - q(n) \min_{[n-1,n]} y(s) = 0, \quad (6.3.9)
$$

where

$$
q(n) = e^{-n-2}(1 - e)^2(e + 1) \left( \min_{[n-\ell,n]} \left( \phi(s) + e^{-s} \right) \right)^{-1}
$$

and \( \{\phi(n)\}_{n \geq 1} \) is the sequence defined by

$$
\phi(2k + 1) = 0, \quad \phi(2k + 2) = \frac{1}{2}, \quad k \in \mathbb{N}_0.
$$

It is easy to verify that

$$
\{y(n)\} = \{\phi(n) + e^{-n}\}
$$

is a positive solution of equation (6.3.9). Further more, obviously

$$
\lim_{n \to \infty} \inf y(n) = 0 \quad \text{and} \quad \lim_{n \to \infty} \sup y(n) = \frac{1}{2}.
$$

On the other hand, the inequality

$$
e^{-n} \leq \min_{[n-\ell,n]} \{\phi(s) + e^{-s}\} \leq e^{-n+1}
$$

implies that

$$
\frac{(e + 1)(1 - e)^2}{e^2} \leq q(n) \leq \frac{(e + 1)(1 - e)^2}{e^2}.
$$

Thus condition (6.3.6) is valid (in fact, even the stronger conditions $q(n) \geq q > 0$ holds). Clearly the function \( \{z(n)\} = \{-\phi(n) - e^{-n}\} \) is a negative solution of the equation

$$
\Delta^2 (x(n) + x(n - 1)) - q(n) \max_{[n-1,n]} x(s) = 0.
$$
Thus, although the conditions of Theorem 6.3.4 are met, equation (6.1.1) could have negative bounded solution which does not tend to zero.

We conclude this chapter with the following theorem.

**Theorem 6.3.5.** Let conditions (H) hold and there exist constants $p_1$ and $p_2$ such that $1 < p_1 \leq p(n) \leq p_2$. If $\{y(n)\}$ is a bounded nonoscillatory solution of equation (6.1.1), then $y(n) \to 0$ as $n \to \infty$.

**Proof.** Let $\{y(n)\}$ be an eventually positive solution of equation (6.1.1). As in Theorem 6.3.4 it is proved that if $\{y(n)\}$ is bounded positive solution of equation (6.1.1) then $\Delta (a(n) \Delta z(n)) \geq 0$, $a(n) \Delta z(n)$ and $z(n) > 0$ eventually. Suppose $d = \lim\inf_{n \to \infty} y(n) > 0$. Then $y(n) > \frac{d}{2}$. From this inequality and (6.1.5) it follows that $\lim_{n \to \infty} a(n) \Delta z(n) = \infty$ which is a contradiction. Hence $\lim_{n \to \infty} \inf y(n) = 0$. Then there is an increasing sequence of integers $\{n_i\}$ such that $\lim_{i \to \infty} y(n_i - k) = 0$. Suppose that $c = \lim_{n \to \infty} z(n) > 0$. Passing to the limit in the equality $z(n_i) = y(n_i) + p(n_i) y(n_i - k)$, we obtain that $\lim_{n \to \infty} y(n_i) = c$. On the other hand

$$z(n_i + k) = y(n_i + k) p(n_i + k) y(n_i) > p_1 y(n_i).$$

Taking the limit in the last inequality we obtain $c > p_1 c > c$. Hence $\lim_{n \to \infty} z(n) = 0$ and since $z(n) > y(n)$, we have $\lim_{n \to \infty} y(n) = 0$. The case $\{y(n)\}$ eventually negative can be proved analogously and the proof is now complete. \qed