Trajectory of quarks in Cornell Potential

Chapter 6

Trajectory of quarks in Cornell Potential

6.1 Introduction

In the initial stages of QGP, immediately after $q\bar{q}$ pair production, quarks and gluons may be moving in the Cornell potential and hence the study of their trajectory may be important. So in this chapter we obtained a physical picture of movements of quarks and anti-quarks confined in the Cornell potential. We obtained the trajectory of quarks in Cornell potential using Hamilton Jacobi method [4]. Hamilton-Jacobi equation [HJE] may be derived from Hamiltonian mechanics by treating $S$ the action as the generating function for a canonical transformation of the classical Hamiltonian. The HJE is a single, first-order partial differential equation for the function $S$ of the $N$ generalized coordinates and the time $t$. The Hamiltonian of the quarks in
Cornell potential is given by

\[ H = \frac{p^2}{2m} + \frac{4a_s}{3} \frac{1}{r} - \sigma r \]  

(6.1)

This can be written as

\[ H = \frac{p^2}{2m} + \frac{a}{r} + br \]  

(6.2)

### 6.2 Hamilton-Jacobi Method

The Hamilton-Jacobi equation is given by

\[ H + \frac{\partial S}{\partial t} = 0 \]  

(6.3)

where \( S \) is the action which contains the complete information regarding the dynamics of the particle. To evaluate the action, we start from the Hamilton-Jacobi equation.

\[ \frac{p^2}{2m} + \frac{a}{r} + br + \frac{\partial S}{\partial t} = 0 \]  

(6.4)

\[ p^2 = p_r^2 + \frac{p_\theta^2}{r^2} = \frac{\partial S}{\partial r}^2 + \frac{1}{r^2} \left( \frac{\partial S}{\partial \theta} \right)^2 \]  

(6.5)

Hamiltonian represents the total energy \( E \) of the system.

Then the action \( S \) is given by

\[ S = -Et + \ell \theta + f(r) \]
6.3 Trajectory of Quarks

The trajectory of quarks can be found out from the action. The equation of the trajectory is given by the condition \( \frac{\partial S}{\partial t} = \text{a constant} \)

\[
\frac{\partial S}{\partial t} = \theta - \int \frac{\frac{l}{r^2}}{\sqrt{2mE - \frac{l^2}{r^2} - \frac{2ma}{r} - 2mbr'}} \, dr'
\]

By proper choice of the pericentre of the trajectory \( \frac{\partial S}{\partial t} \) is chosen to be equal to zero with this we get

\[
\theta = \int \frac{\frac{l}{r^2}}{\sqrt{2mE - \frac{l^2}{r^2} - \frac{2ma}{r} - 2mbr'}} \, dr'
\]

Put \( r' = \frac{pc}{\omega_0}, l = \frac{\lambda mc^2}{\omega_0} \) and \( \varepsilon = \frac{E}{mc^2} \) we get

\[
\theta = \int \frac{d\rho}{\rho \sqrt{\frac{2mE\rho^2}{\varepsilon^2} - 1 - \frac{2ma}{\varepsilon} - \frac{2mb}{\varepsilon} \rho^3}}
\]

Put \( \frac{\omega_0}{mc} = \alpha \) and \( \frac{b}{mc\omega_0} = \beta \), we get

\[
\theta = \int \frac{d\rho}{\rho \sqrt{\frac{2\varepsilon \rho^2}{\lambda^2} - 1 - \frac{2\alpha}{\lambda^2} - \frac{2\beta}{\lambda^2} \rho^3}}
\]
Making the change of variable \( \rho = \frac{1}{x} \), we get

\[
\theta = - \int \frac{dx}{x \sqrt{\frac{2\varepsilon x}{\lambda^3} - 1 - \frac{2\alpha}{\lambda^3 x} - \frac{2\beta}{\lambda^3 x^3}}}
\]

\[
\theta = - \int \frac{dx}{\sqrt{\frac{2\varepsilon x}{\lambda^3} - x^2 - \frac{2\alpha x}{\lambda^3} - \frac{2\beta}{\lambda^3}}}
\]

\[
\theta = - \int \frac{dx \sqrt{x}}{\sqrt{\frac{2\varepsilon x}{\lambda^3} - x^3 - \frac{2\alpha x^2}{\lambda^3} - \frac{2\beta}{\lambda^3}}} \quad (6.6)
\]

In order to calculate this integral, the polynomial occurring in the denominator of the radicant will be decomposed into factors. For this we will find its roots. They are given by the algebraic equation

\[
x^3 + \frac{2\alpha x^2}{\lambda^2} - \frac{2\varepsilon x}{\lambda^2} + \frac{2\beta}{\lambda^2} = 0
\]

\[
x^3 + 3a_1 x^2 + 3a_2 x + a_3 = 0 \quad (6.7)
\]

Written in normal form, having coefficients \( a_0 = 1 \), \( a_1 = \frac{2\alpha}{3\lambda^2} \), \( a_2 = -\frac{2\varepsilon}{3\lambda^2} \) and \( a_3 = \frac{2\beta}{\lambda^3} \). To solve the cubic equation, put \( z = x - a_1 \), we get

\[
z^3 + 3z(a_2 - a_1^2) + a_3 - 3a_1a_2 + 2a_1^3 = 0
\]

\[
z^3 + 3zH + G = 0 \quad (6.8)
\]

Where the quadratic term is absent with the coefficients \( H = a_2 - a_1^2 \) and \( G = a_3 - 3a_1a_2 + 2a_1^3 \).
Now put $z = u + v$, we get

$$z^3 - 3uvz - u^3 - v^3 = 0$$

Comparing this with eqn(6.8) we get

$$H = -uv \Rightarrow u^3v^3 = -H^3$$

and

$$G = -(u^3 + v^3)$$

If $u^3$ and $v^3$ are the roots, then

$$(t - u^3)(t - v^3) = 0$$

$$t^2 - (u^3 + v^3)t + u^3v^3 = 0$$

$$t^2 + Gt - H^3 = 0$$

$$t_{1,2} = \frac{-G \pm \sqrt{G^2 + 4H^3}}{2}$$

The nature of the roots of the eqn(6.8) depends on the values of the discriminant

$$\Delta = G^2 + 4H^3$$

Evaluating this, we get

$$\Delta = \frac{8}{27\lambda^8} \left( \frac{27\beta^2\lambda^4}{2} + \lambda^2\epsilon^3 + 8\alpha^3\beta - 2\alpha^2\epsilon^2 + 18\alpha\epsilon\beta \right) \quad (6.9)$$

107
\[
\lambda_0 = \sqrt{-\frac{\varepsilon^3}{\beta^2} + \frac{\varepsilon^6}{\beta^4} - 54\left[\frac{8\alpha^3}{\beta} - \frac{2\alpha^2\varepsilon^2}{\beta^2} + \frac{18\alpha\varepsilon^3}{\beta}\right]^{1/2}}
\]

The positive root (the only meaningful solution) of the polynomial on the right hand side of eqn(6.9), the quantity \(\Delta\) is negative, null or positive as \(0 < \lambda < \lambda_0\), \(\lambda = \lambda_0\) or \(\lambda > \lambda_0\) respectively.

For \(\Delta < 0\) eqn(6.7) has three simple roots, say \(x_1\), \(x_2\) and \(x_3\). Then eqn(1) of the trajectory becomes

\[
\theta = \int \frac{x}{(x_1 - x)(x_2 - x)(x_3 - x)} \, dx
\]

Then the implicit equation of the trajectory has the form

\[
\frac{x_1}{\sqrt{(x_1 - x_3)x_2}} \Pi \left( \arcsin \sqrt{\frac{x_2(x_1 - \frac{1}{p^2})}{x_2^2}}, \frac{x_1 - x_2}{x_2}, \sqrt{\frac{(x_1 - x_2)x_3}{(x_1 - x_3)x_2}} \right) = \theta
\]

(6.10)

Where \(\Pi(\psi, n, k)\) is the elliptic integral of the third kind

\[
\Pi(\psi, n, k) = \int \frac{d\psi'}{(1 + n \sin^2 \psi') \sqrt{1 - k^2 \sin^2 \psi'}}
\]

6.4 Conclusion

From the above mathematical analysis we conclude the following things. Eq.(6.10) describes a finite motion within a ring bounded by circles of radii corresponding to the roots \(x_1\) and \(x_2\). During a complete rotation of the particle on trajectory, the pericenter suffers an angular shift. This trajectory is an open curve which is called a Rosette. This shows that the trajectory of
quarks in Cornell potential is in a rosette shape. In the non-relativistic case, when $z_3 = 0$ the equation of the trajectory can be written as

$$\frac{X^2}{A^2} + \frac{Y^2}{B^2} = 1$$

which represents the equation of an ellipse with the centre at the field centre and of semi axis $A$ and $B$. That is if consider quarks as non-relativistic particles and allow to move in Cornell potential they move in elliptical orbit.

6.5 References