Chapter 7

Homotopy analysis Sumudu transform method for time fractional Kuramoto–Sivashinsky partial differential equations

7.1 Introduction

In this chapter, we establish numerically approximate solution to the time fractional Kuramoto-Sivashinsky (KS) equations [45, 50, 51, 133, 153, 210, 214, 244]

\[
D_t^\alpha U(x,t) + U(x,t)U_x(x,t) + aU_{xx}(x,t) + bU_{xxx}(x,t) + cU_{xxxx}(x,t) = 0,
\]

(7.1.1)

With the initial and boundary conditions are \( U(l,t) = g_1(t) \), \( U(m,t) = g_2(t) \), \( U_{xx}(l,t) = g_3(t) \), \( U_{xx}(m,t) = g_4(t) \) and \( U(x,0) = f(x) \), \( l \leq x \leq m \), \( a, b, c \in \mathbb{R} \) respectively, where \( 0 < \alpha \leq 1 \), \( t > 0 \) and \( a, b, c \) are some arbitrary constant
The studies of canonical models for chaotic dynamical systems are used for the various types of pattern formation in scientific and engineering applications [46]. The KS equations are originally derived for plasma instabilities associated with the trapped particles, propagation of flame front, and reaction diffusions phase turbulence [244]. The KS equations transfer the energy using the linear term to the nonlinear term with compensate to each other. Most of its solution is travelling wave type form which moves in spatial domain without change of nature; also reveal chaotic nature [239].

The exact solution of time fractional KS equations using complex transform is discussed in [214]. The invariant subspace method is applied to obtain the exact solution of time fractional modified KS equation [195]. Many numerical methods have been proposed for the solution of integer order KS equations, implicit–explicit BDF methods[10], cubic spline collocation method [174], Radial basis function based method [239], He’s variational iteration method [207], Homotopy Analysis Method [153], Homotopy Perturbation Method [207].

In the past few decades, for expounding dynamical systems, the integer–order system of differential equations are paramount implement up to recent era. Modern studies have depicted that integer- order derivatives are not plausibly explicating the multifaceted and typical nature of many types of non dynamical system. Currently, differential equations of fractional order are popularly utilized by many researchers all over world to compose many scientific models. Importantly, fractional derivatives initiate for understanding of genuine life phenomena to reduce shortcoming of classical calculus and withal for the explication Brownian nature of particle in non dynamic system [37, 135, 206].

The time variable derivatives better portrayed the time changed phenomena instead of variable order coefficients [17, 83, 227]. Further, many authors provide mathematical and physical analysis of these operators [40, 43, 168, 175]. In order to convert the intricate linear and nonlinear form of fractional order partial
differential equations into simpler time domain, many types of fractional integral and differential transforms have been applied to gain the exact and approximate solutions of FPDE’s [89, 242]. Watugala [243] introduced the Sumudu transform which can facilely convert many fractional order linear and nonlinear partial differential equations in time domain without loss of generality for variant of included fractional operators viz. Caputo, Riemann–Liouville, Ritz space, etc. Many other properties of Sumudu transform were intended by Weerakoon [251, 252].

Sumudu transform is different with other well known transforms viz. Laplace transform due to its unity property, which means the unity constant function is same in both domain. This special nature composes the Sumudu transform a good way to solve fractional differential equations. We can just rewrite the equation in u–domain without much complex steps. However, dealing with algebra is more easier than the dealing with various kinds of functions, which is the main motive of transformation of domain change for more, see [158].

In this chapter, the aim of the researcher is to apply the homotopy analysis Sumudu transform method for solution of time fractional KS equations [198, 199, 200, 201, 202], which is a cumulation of Sumudu transform and homotopy analysis method [159, 160, 161, 162]. The HASTM obtains semi analytic solutions in the form of series solutions. It is different from other transforms and semi analytic method, which does not require additional information except some initial and boundary conditions. It easily changes the original problem to lucid manner and then, one can evaluate the result with high convergence and accuracy.
7.2 The numerical solution of time fractional Kuramoto–Sivashinsky equations arises in plasma instabilities associated with the trapped particles

We consider the Kuramoto–Sivashinsky equation and taking the Sumudu transform of Eq. (7.1.1) on both sides, we get

\[
\mathbb{S} \left[ \frac{U(x, t)}{u^\alpha} \right] - \sum_{k=0}^{n-1} \frac{U^{(k)}(0)}{u^{(n-k)}} = -\mathbb{S} \left[ \frac{U(x, t) U_x(x, t) + aU_{xx}(x, t) + bU_{xxx}(x, t) + cU_{xxxx}(x, t)}{u^\alpha} \right]
\]

(7.2.1)

The nonlinear operator is given as follows:

\[
N \left[ \varphi(x, t; q) \right] = \mathbb{S} \left[ \varphi(x, t; q) \right] - \sum_{k=0}^{n-1} \frac{\varphi^{(k)}(0)}{u^{(n-k)}}
+ u^{\alpha} \mathbb{S} \left[ \varphi(x, t; q) \varphi_x(x, t; q) + a\varphi_{xx}(x, t; q) + b\varphi_{xxx}(x, t; q) + c\varphi_{xxxx}(x, t; q) \right],
\]

(7.2.2)

where \( q \in [0, 1] \) be an embedding parameter and \( \varphi(x, t; q) \) is a real function of \( x, t \) and \( q \). we construct a homotopy as follow:

\[
(1 - q) \mathbb{S} \left[ \varphi(x, t; q) - U_0(x, t) \right] = hqH(x, t) \ N \left[ \varphi(x, t; q) \right]
\]

(7.2.3)

where \( h \) is a nonzero auxiliary parameter and \( H(x, t) \neq 0 \) an auxiliary function, \( U_0(x, t) \) is an initial guess of \( U(x, t) \) and \( \varphi(x, t; q) \) is an unknown function. It is important that one has great freedom to choose auxiliary parameter in HASTM. Obviously, when \( q = 0 \) and \( q = 1 \) it holds.
\[ \varphi(x, t; 0) = U_0(x, t), \quad \varphi(x, t; 1) = U(x, t) \quad (7.2.4) \]

Thus, as \( q \) increases from 0 to 1, the solution varies from initial guess \( U_0(x, t) \) to the solution \( U(x, t) \). Now, expanding \( \varphi(x, t; q) \) on Taylor’s series with respect to \( q \), we get

\[ \varphi(x, t; q) = U_0(x, t) + \sum_{m=1}^{\infty} q^m U_m(x, t) \quad (7.2.5) \]

where

\[ U_m(x, t) = \frac{1}{\Gamma(m+1)} \left. \frac{\partial^m \phi(x, t; q)}{\partial q^m} \right|_{q=0} \quad (7.2.6) \]

The convergence of the series solution (7.2.5) is controlled by \( h \). If the auxiliary linear operator, the initial guess, the auxiliary parameter \( h \) and the auxiliary function are properly chosen, the series (7.2.5) converges at \( q = 1 \). Hence, we obtain

\[ U(x, t) = U_0(x, t) + \sum_{m=1}^{\infty} U_m(x, t) \quad (7.2.7) \]

which must be one of the solutions of original nonlinear equations. The above expression provides us with a relationship between the initial guess \( U_0(x, t) \) and the exact solution \( U(x, t) \) by means of the terms \( U_m(x, t) \) \((m = 1, 2, 3, \ldots)\), which are still to be determined.

Define the vectors

\[ \vec{U} = \{U_0(x, t), U_1(x, t), U_2(x, t), \ldots, U_m(x, t)\} \quad (7.2.8) \]

Differentiating the zero order deformation Eq. (7.2.3) \( m \) times with respect to embedding parameter \( q \) and then setting \( q = 0 \), and finally dividing them by \( m! \),
we obtain the \( m^{th} \) order deformation equation as follows:

\[
\mathcal{S} [U_m (x, t) - \chi_m U_{m-1} (x, t)] = \hbar H (x, t) R_m \left( \hat{U}_{m-1}, x, t \right).
\] (7.2.9)

Operating the inverse Sumudu transform of both sides, we get

\[
U_m (x, t) = \chi_m U_{m-1} (x, t) + \hbar \mathcal{S}^{-1} \left[ H (x, t) R_m \left( \hat{U}_{m-1}, x, t \right) \right],
\] (7.2.10)

where

\[
R_m \left( \hat{U}_{m-1}, x, t \right) = \frac{1}{(m - 1)!} \left. \frac{\partial^{m-1} \varphi (x, t; q)}{\partial q^{m-1}} \right|_{q=0}
\] (7.2.11)

and

\[
\chi_m = \begin{cases} 
0, & m \leq 1, \\
1, & m > 1.
\end{cases}
\] (7.2.12)

In the Kuramoto–Sivashinsky equation, the formulation of \( R_m \left( \hat{U}_{m-1}, x, t \right) \) for the iterative solution of Eq. (7.2.1) is as follows:

\[
R_m \left( \hat{U}_{m-1}, x, t \right) = D_t^\alpha U_{m-1} (x, t) + \sum_{i=0}^{m-1} U_i (x, t) (U_{m-1-i})_x (x, t) + a (U_{m-1})_{xx} (x, t)
\]
\[
+ b (U_{m-1})_{xxx} (x, t) + c (U_{m-1})_{xxxx} (x, t),
\] (7.2.13)

In this way, it is easy to obtain \( U_m (x, t) \) for \( m \geq 1 \), at \( N^{th} \) order, as

\[
U (x, t) = \sum_{m=0}^{N} U_m (x, t),
\] (7.2.14)
where \( N \to \infty \), we obtain an accurate approximation of the original equation (7.1.1).

**Theorem 7.1** (*Convergence Discussion*): The series solution obtained by equation (7.2.14) must converge for \( N \to \infty \), evaluated by algorithmic equation (7.2.10) using the conditions (7.2.13). Therefore, it must be the solution of original K-S equation (7.1.1).

**Proof:** Let Eq. (7.2.14) be convergent, then it holds the following:

\[
L (x, t) = \sum_{m=0}^{\infty} U_m (x, t) \tag{7.2.15}
\]

The validity of convergent series is obtained by necessary condition

\[
\lim_{N \to \infty} U_m (x, t) = 0 \tag{7.2.16}
\]

From Eq. (7.2.9) and Eq. (7.2.16), we obtain

\[
\lim_{N \to \infty} \left[ \hbar H (x, t) \sum_{m=1}^{N} R_m \left( \overrightarrow{U}, x, t \right) \right] = 0
\]

Using the Eq. (7.2.10), the expansion becomes zero i.e.

\[
\lim_{N \to \infty} \left[ \hbar H (x, t) \sum_{m=1}^{N} R_m \left( \overrightarrow{U}, x, t \right) \right] = 0 \tag{7.2.17}
\]
According to the assumption $\hbar \neq 0$, $H(x, t) = 1$, therefore $\sum_{m=1}^{\infty} R_m(\mathbf{U}, x, t) = 0$. From Eq. (7.2.13)

$$\sum_{m=1}^{\infty} R_m(\mathbf{U}, x, t) = \sum_{m=1}^{\infty} \left( D_i^o U_{m-1} (x, t) + \sum_{i=0}^{m-1} U_i (x, t) (U_{m-1-i})_x (x, t) \right)$$

$$+ a (U_{m-1})_{xx} (x, t) + b (U_{m-1})_{xxx} (x, t) + c (U_{m-1})_{xxxx} (x, t),$$

$$\sum_{m=1}^{\infty} R_m(\mathbf{U}, x, t) = \sum_{m=1}^{\infty} \left( D_i^o U_{m-1} (x, t) + \sum_{i=0}^{m-1} U_i (x, t) (U_{m-1-i})_x (x, t) \right)$$

$$+ \sum_{m=1}^{\infty} (aU_{m-1})_{xx} (x, t) + \sum_{m=1}^{\infty} (bU_{m-1})_{xxx} (x, t) + \sum_{m=1}^{\infty} (cU_{m-1})_{xxxx} (x, t),$$

(7.2.18)

Since, the series obtained by $m^{th}$ order deformation equation is a Cauchy, therefore the product of Cauchy series holds,

$$\Rightarrow \sum_{m=1}^{\infty} \left( \sum_{i=0}^{m-1} U_i (x, t) (U_{m-1-i})_x (x, t) \right) = \sum_{i=0}^{\infty} \sum_{m=i+1}^{\infty} U_i (x, t) (U_{m-1-i})_x (x, t)$$

$$= \sum_{i=0}^{\infty} U_i (x, t) \sum_{m=1}^{\infty} (U_{m-1})_x (x, t)$$

$$= \left( \sum_{m=1}^{\infty} U_{m-1} (x, t) \right) \left( \sum_{m=1}^{\infty} (U_{m-1})_x (x, t) \right)$$

Also, it satisfies $\sum_{m=0}^{\infty} a U_m (x, t) = a \sum_{m=0}^{\infty} U_m (x, t)$ and so on.

Now, the Eq. (7.2.18) becomes

$$\sum_{m=1}^{\infty} R_m(\mathbf{U}, x, t) = D_i^o \sum_{m=1}^{\infty} U_{m-1} (x, t) + \left( \sum_{m=1}^{\infty} U_{m-1} (x, t) \right) \left( \sum_{m=1}^{\infty} (U_{m-1})_x (x, t) \right)$$

$$+ a \sum_{m=1}^{\infty} (U_{m-1})_{xx} (x, t) + b \sum_{m=1}^{\infty} (U_{m-1})_{xxx} (x, t) + c \sum_{m=1}^{\infty} (U_{m-1})_{xxxx} (x, t) = 0,$$

(7.2.19)
\[ \sum_{m=1}^{\infty} R_m \left( \overline{U}_{m-1} (x, t) \right) = D_t^\alpha \sum_{m=0}^{\infty} U_m (x, t) + \left( \sum_{m=0}^{\infty} U_m (x, t) \right) \left( \sum_{m=0}^{\infty} (U_m)_x (x, t) \right) + a \sum_{m=0}^{\infty} (U_m)_{xx} (x, t) + b \sum_{m=0}^{\infty} (U_m)_{xxx} (x, t) + c \sum_{m=0}^{\infty} (U_m)_{xxxx} (x, t) = 0, \]

\[ D_t^\alpha L (x, t) + L (x, t) L_x (x, t) + aL_{xx} (x, t) + bL_{xxx} (x, t) + cL_{xxxx} (x, t) = 0, \] (7.2.20)

The Eq. (7.2.20) shows that the K–S equations satisfies (7.1.1).

### 7.3 Numerical examples

In this section, we apply the application of the technique discussed in the previous section to find numerical solution of Kuramoto–Sivashinsky equations.

#### 7.3.1 Example

Consider the following Kuramoto–Sivashinsky equation:

\[ U_t^\alpha (x, t) + U (x, t) U_x (x, t) + U_{xx} (x, t) + U_{xxxx} (x, t) = 0 \] (7.3.1)

with initial condition

\[ U (x, 0) = C + \frac{15}{19} \sqrt{\frac{11}{19}} \left[ 11 \tanh^3 (k (x - x_0)) - 9 \tanh (k (x - x_0)) \right]. \] (7.3.2)

The exact solution of the above problem is given by
\[ U(x, t) = C + \frac{15}{19} \sqrt{\frac{11}{19}} \left(11 \tanh^3 (k(x - Ct - x_0)) - 9 \tanh (k(x - Ct - x_0)) \right). \]  
\hspace{1cm} (7.3.3)

The boundary conditions for (7.3.1) is given by

\[ U(l, t) = C + \frac{15}{19} \sqrt{\frac{11}{19}} \left(11 \tanh^3 (k(l - Ct - x_0)) - 9 \tanh (k(l - Ct - x_0)) \right). \]  
\hspace{1cm} (7.3.4)

\[ U(m, t) = C + \frac{15}{19} \sqrt{\frac{11}{19}} \left(11 \tanh^3 (k(m - Ct - x_0)) - 9 \tanh (k(m - Ct - x_0)) \right). \]  
\hspace{1cm} (7.3.5)

\[ U_{xx}(l, t) = \frac{-180}{19} \sqrt{\frac{11}{19}} k^2 \sec h[k(l - Ct - x_0)]^5 \left[-10 \sinh (k(l - Ct - x_0)) + \sinh (3k(l - Ct - x_0)) \right]. \]  
\hspace{1cm} (7.3.6)

\[ U_{xx}(m, t) = \frac{-180}{19} \sqrt{\frac{11}{19}} k^2 \sec h[k(m - Ct - x_0)]^5 \left[-10 \sinh (k(m - Ct - x_0)) + \sinh (3k(m - Ct - x_0)) \right]. \]  
\hspace{1cm} (7.3.7)

Taking the Sumudu transform of (7.3.1) on both sides, we get

\[ \mathcal{S}[U(x, t)] - \sum_{k=0}^{n-1} \frac{U^{(k)}(0)}{u^{(-k)}} + u^\alpha \mathcal{S}[U(x, t) U_x(x, t) + U_{xx}(x, t) + U_{xxxx}(x, t)] = 0. \]  
\hspace{1cm} (7.3.8)

The nonlinear operator is defined by

\[ N[\varphi(x, t; q)] = \mathcal{S}[\varphi(x, t; q)] - \sum_{k=0}^{n-1} \frac{\varphi^{(k)}(0)}{u^{(-k)}} + u^\alpha \mathcal{S}[\varphi(x, t; q) \varphi_x(x, t; q) + \varphi_{xx}(x, t; q) + \varphi_{xxxx}(x, t; q)]. \]  
\hspace{1cm} (7.3.9)
and thus

\[
R_m \left( \mathcal{U}_{m-1}, x, t \right) = S \left[ U_{m-1} (x, t) \right] + u^\alpha S \left[ \sum_{i=0}^{m-1} U_{m-1} (x, t) \left( U_{m-1} \right)_x (x, t) \right] + \left( U_{m-1} \right)_{xx} (x, t) + \left( U_{m-1} \right)_{xxx} (x, t) \right].
\]  

(7.3.10)

The \( m^{th} \)– order deformation equation is given by

\[
U_m (x, t) = \chi_m U_{m-1} (x, t) + S^{-1} \left[ \hbar H (x, t) R_m \left( \mathcal{U}_{m-1}, x, t \right) \right].
\]  

(7.3.11)

On solving above equation for \( m = 1, 2, \ldots \), we get

\[
U_1 (x, t) = C \hbar + \frac{30 \sqrt{11/19} C k t^\alpha \hbar \sec h \left[ k (x - x_0) \right]^2}{19 \Gamma (1 + \alpha)} - \frac{165 \sqrt{11/19} C k t^\alpha \hbar \sec h \left[ k (x - x_0) \right]^4}{19 \Gamma (1 + \alpha)}
\]

\[
+ \frac{30 \sqrt{11/19} \hbar \tan h \left[ k (x - x_0) \right] - 165 \sqrt{11/19} \hbar \sec h \left[ k (x - x_0) \right]^2 \tanh \left[ k (x - x_0) \right]}{6859 \Gamma (1 + \alpha)}
\]

\[
+ \frac{9900 k t^\alpha \hbar \sec h \left[ k (x - x_0) \right]^2 \tanh \left[ k (x - x_0) \right]}{6859 \Gamma (1 + \alpha)}
\]

\[
- \frac{60 \sqrt{11/19} k^2 t^\alpha \hbar \sec h \left[ k (x - x_0) \right]^4 \tanh \left[ k (x - x_0) \right]}{19 \Gamma (1 + \alpha)}
\]

\[
- \frac{108900 k t^\alpha \hbar \sec h \left[ k (x - x_0) \right]^4 \tanh \left[ k (x - x_0) \right]}{19 \Gamma (1 + \alpha)}
\]

\[
+ \frac{1320 \sqrt{11/19} k^2 t^\alpha \hbar \sec h \left[ k (x - x_0) \right]^4 \tanh \left[ k (x - x_0) \right]}{19 \Gamma (1 + \alpha)}
\]

\[
+ \frac{480 \sqrt{11/19} k^2 t^\alpha \hbar \sec h \left[ k (x - x_0) \right]^4 \tanh \left[ k (x - x_0) \right]}{19 \Gamma (1 + \alpha)}
\]

\[
+ \frac{299475 k t^\alpha \hbar \sec h \left[ k (x - x_0) \right]^6 \tanh \left[ k (x - x_0) \right]}{6859 \Gamma (1 + \alpha)}
\]

\[
- \frac{22440 \sqrt{11/19} k^4 t^\alpha \hbar \sec h \left[ k (x - x_0) \right]^6 \tanh \left[ k (x - x_0) \right]}{19 \Gamma (1 + \alpha)}
\]
\[
\frac{330\sqrt{\frac{11}{19}} Ckt^a \hbar \sec h [k (x - x_0)]^2 \tanh [k (x - x_0)]^2}{19\Gamma (1 + \alpha)} + \frac{108900kt^a \hbar \sec h [k (x - x_0)]^2 \tanh [k (x - x_0)]^3}{6859\Gamma (1 + \alpha)} - \frac{660\sqrt{\frac{11}{19}} k^2 t^a \hbar \sec h [k (x - x_0)]^2 \tanh [k (x - x_0)]^3}{19\Gamma (1 + \alpha)} - \frac{240\sqrt{\frac{11}{19}} k^4 t^a \hbar \sec h [k (x - x_0)]^2 \tanh [k (x - x_0)]^3}{19\Gamma (1 + \alpha)} - \frac{598950kt^a \hbar \sec h [k (x - x_0)]^4 \tanh [k (x - x_0)]^3}{6859\Gamma (1 + \alpha)} - \frac{34320\sqrt{\frac{11}{19}} k^4 t^a \hbar \sec h [k (x - x_0)]^4 \tanh [k (x - x_0)]^5}{19\Gamma (1 + \alpha)} - \frac{2640\sqrt{\frac{11}{19}} k^2 t^a \hbar \sec h [k (x - x_0)]^4 \tanh [k (x - x_0)]^5}{19\Gamma (1 + \alpha)}
\]

\ldots

\ldots

etc. In the same manner, the rest of the components of (7.3.11) as a series \(m \geq 2\) can be obtained.

Due to very large values of successive results, we have shown only \(U_1 (x, t)\) but considered in analysis upto \(m = 3\).

The solution of (7.3.1) is given by

\[
U (x, t) = U_0 (x, t) + \sum_{m=1}^{\infty} U_m (x, t). \quad (7.3.12)
\]

The accuracy and convergence of the HASTM series solution depend on the careful selection of the auxiliary parameter \(\hbar\), which can be chosen using the curve parallel to the horizontal axis region as shown in fig. 7.1.

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Figure 7.1: Plot of $U(x, t)$ with respect to $\hbar$ at $t = 0.55$ and $x = 5$.

Figure 7.2: Plot of $U(x, t)$ with respect to $x$, $t$ at $\alpha = 0.5$
Figure 7.3: Plot of $U(x,t)$ with respect to $x$, $t$ at $\alpha = 0.75$

Figure 7.4: Plot of $U(x,t)$ with respect to $x$, $t$ at $\alpha = 1$. 
The plots of $U(x,t)$ are shown in figs. 7.2, 7.3, 7.4 and 7.5 for the brownian motion of $\alpha = 0.5, 0.75, 1$ and exact solution at $C = 0.2$, $\hbar = 1$, $x_0 = -10$ and $K = \frac{1}{2\sqrt{19}}$.

At $\alpha = 1$ example shows the standard solution of KS equations discussed in [239]

### 7.3.2 Example

Consider the following Kuramoto –Sivashinsky equation:

$$U_t^\alpha (x,t) + U (x,t) U_x (x,t) - U_{xx} (x,t) + U_{xxxx} (x,t) = 0$$

(7.3.13)

with initial condition

$$U (x,0) = C + \frac{15}{19\sqrt{19}} \left[ \tanh^3 (k (x - x_0)) - 3 \tanh (k (x - x_0)) \right].$$

(7.3.14)
The exact solution of the above problem is given by,

\[ U(x, t) = C + \frac{15}{19\sqrt{19}} \left[ \tanh^3(k(x - Ct - x_0)) - 3 \tanh(k(x - Ct - x_0)) \right]. \]  

(7.3.15)

The boundary conditions for (7.3.13) is given by

\[ U(l, t) = C + \frac{15}{19\sqrt{19}} \left[ \tanh^3(k(l - Ct - x_0)) - 3 \tanh(k(l - Ct - x_0)) \right]. \]  

(7.3.16)

\[ U(m, t) = C + \frac{15}{19\sqrt{19}} \left[ \tanh^3(k(m - Ct - x_0)) - 3 \tanh(k(m - Ct - x_0)) \right]. \]  

(7.3.17)

\[ U_{xx}(l, t) = \frac{180k^2}{19\sqrt{19}} \left[ \text{sech}^4(k(l - Ct - x_0)) \tanh(k(l - Ct - x_0)) \right], \]  

(7.3.18)

\[ U_{xx}(m, t) = \frac{180k^2}{19\sqrt{19}} \left[ \text{sech}^4(k(m - Ct - x_0)) \tanh(k(m - Ct - x_0)) \right]. \]  

(7.3.19)

Taking the Sumudu transform of (7.3.13) on both sides, we get

\[ \mathcal{S}[U(x, t)] - \sum_{k=0}^{n-1} \frac{U^{(k)}(0)}{u^{(-k)}} + u^a \mathcal{S}[U(x, t)U_x(x, t) - U_{xx}(x, t) + U_{xxxx}(x, t)] = 0. \]  

(7.3.20)

The nonlinear operator is defined by

\[ N[\varphi(x, t; q)] = \mathcal{S}[\varphi(x, t; q)] - \sum_{k=0}^{n-1} \frac{\varphi^{(k)}(0)}{u^{(-k)}} + u^a \mathcal{S}[\varphi(x, t; q) \varphi_x(x, t; q) - \varphi_{xx}(x, t; q) + \varphi_{xxxx}(x, t; q)], \]  

(7.3.21)
\[ R_m \left( \overrightarrow{U}_{m-1}, x, t \right) = S \left[ U_{m-1} (x,t) \right] + u^0 S \left[ \sum_{i=0}^{m-1} U_{m-1} (x,t) (U_{m-1-i})_x (x,t) \right] \]
\[ \quad - (U_{m-1})_{xx} (x,t) + (U_{m-1})_{xxx} (x,t) \] .

(7.3.22)

The \( m^{th} \) order deformation equation is given by

\[ U_m (x,t) = \chi_m U_{m-1} (x,t) + S^{-1} \left[ \hbar H (x, t) R_m \left( \overrightarrow{U}_{m-1}, x, t \right) \right] . \]  

(7.3.23)

On solving above equation for \( m = 1, 2, \ldots \), we get

\[ U_1 (x, t) = C \hbar + \frac{30 \sqrt{\frac{11}{19}} C k^0 \hbar \sec h \left[ k (x - x_0) \right]^2 - 165 \sqrt{\frac{11}{19}} C k^0 \hbar \sec h \left[ k (x - x_0) \right]^4}{19 \Gamma (1 + \alpha)} \]
\[ + \frac{30 \sqrt{\frac{11}{19}} \hbar \tan h \left[ k (x - x_0) \right] - 165 \sqrt{\frac{11}{19}} \hbar \sec h \left[ k (x - x_0) \right]^2 \tanh \left[ k (x - x_0) \right]}{19 \Gamma (1 + \alpha)} \]
\[ + \frac{9900 k^0 \hbar \sec h \left[ k (x - x_0) \right]^2 \tan \left[ k (x - x_0) \right]}{6859 \Gamma (1 + \alpha)} \]
\[ - \frac{60 \sqrt{\frac{11}{19}} k^2 t^0 \hbar \sec h \left[ k (x - x_0) \right]^2 \tan \left[ k (x - x_0) \right]}{19 \Gamma (1 + \alpha)} \]
\[ - \frac{108900 k^0 \hbar \sec h \left[ k (x - x_0) \right]^4 \tan \left[ k (x - x_0) \right]}{19 \Gamma (1 + \alpha)} \]
\[ + \frac{1320 \sqrt{\frac{11}{19}} k^2 t^0 \hbar \sec h \left[ k (x - x_0) \right]^4 \tan \left[ k (x - x_0) \right]}{19 \Gamma (1 + \alpha)} \]
\[ + \frac{480 \sqrt{\frac{11}{19}} k^2 t^0 \hbar \sec h \left[ k (x - x_0) \right]^4 \tan \left[ k (x - x_0) \right]}{19 \Gamma (1 + \alpha)} \]
\[ + \frac{299475 k^0 \hbar \sec h \left[ k (x - x_0) \right]^6 \tan \left[ k (x - x_0) \right]}{6859 \Gamma (1 + \alpha)} \]

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\[
\begin{align*}
\frac{22440\sqrt{\frac{11}{19}} k^4 t^\alpha h \sec h \left[k(x-x_0)\right]^6 \tanh \left[k(x-x_0)\right]}{19\Gamma(1+\alpha)} \\
+ \frac{330\sqrt{\frac{11}{19}} C k t^\alpha h \sec h \left[k(x-x_0)\right]^2 \tanh \left[k(x-x_0)\right]^2}{19\Gamma(1+\alpha)} \\
+ \frac{108900 k t^\alpha h \sec h \left[k(x-x_0)\right]^2 \tanh \left[k(x-x_0)\right]^3}{6859\Gamma(1+\alpha)} \\
- \frac{660\sqrt{\frac{11}{19}} k^2 t^\alpha h \sec h \left[k(x-x_0)\right]^2 \tanh \left[k(x-x_0)\right]^3}{19\Gamma(1+\alpha)} \\
- \frac{240\sqrt{\frac{11}{19}} k^4 t^\alpha h \sec h \left[k(x-x_0)\right]^2 \tanh \left[k(x-x_0)\right]^3}{19\Gamma(1+\alpha)} \\
- \frac{598950 k t^\alpha h \sec h \left[k(x-x_0)\right]^4 \tanh \left[k(x-x_0)\right]^3}{6859\Gamma(1+\alpha)} \\
- \frac{34320\sqrt{\frac{11}{19}} k^4 t^\alpha h \sec h \left[k(x-x_0)\right]^4 \tanh \left[k(x-x_0)\right]^5}{19\Gamma(1+\alpha)} \\
- \frac{2640\sqrt{\frac{11}{19}} k^2 t^\alpha h \sec h \left[k(x-x_0)\right]^2 \tanh \left[k(x-x_0)\right]^5}{19\Gamma(1+\alpha)} \\
\end{align*}
\]

etc.

In the same manner, the rest of the components of (7.3.23) as a series \(m \geq 2\) can be obtained.

Due to very large values of successive results, we have shown only \(U_1(x,t)\) but considered in analysis upto \(m = 3\).

The solution of (7.3.13) is given by

\[
U(x,t) = U_0(x,t) + \sum_{m=1}^{\infty} U_m(x,t) .
\] (7.3.24)
The accuracy and convergence of the HASTM series solution depend on the careful selection of the auxiliary parameter $\hbar$, which can be chosen using the curve parallel to the horizontal axis region as shown in fig. 7.6.

Figure 7.6: Plot of $U(x,t)$ with respect to $\hbar$ at $t = 0.55$ and $x = 5$.

Figure 7.7: Plot of $U(x,t)$ with respect to $x$, $t$ at $\alpha = 0.5$. 

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Figure 7.8: Plot of $U(x, t)$ with respect to $x$, $t$ at $\alpha = 0.75$

Figure 7.9: Plot of $U(x, t)$ with respect to $x$, $t$ at $\alpha = 1$.  

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The plots of $U(x,t)$ are shown in figs. 7.7, 7.8, 7.9 and 7.10 for the brownian motion of $\alpha = 0.5, 0.75, 1$ and exact solution at $C = 2.5, \ h = -1, x_0 = -10$ and $K = \frac{1}{2\sqrt{19}}$.

At $\alpha = 1$, the example shows the standard solution of KS equations discussed in [239].

### 7.3.3 Example

Consider the following Kuramoto–Sivashinsky equation:

\[
U_t^\alpha (x, t) + U(x, t) U_x(x, t) + U_{xx}(x, t) + U_{xxx}(x, t) + U_{xxxx}(x, t) = 0 \quad (7.3.25)
\]

with initial condition
\[ U(x,0) = C + 9 - 15 \left[ \tanh(k(x-x_0)) + \tanh^2(k(x-x_0)) - \tanh^3(k(x-x_0)) \right]. \tag{7.3.26} \]

The exact solution of the above problem is given by

\[ U(x,t) = C + 9 - 15 \left[ \tanh(k(x-Ct-x_0)) + \tanh^2(k(x-Ct-x_0)) - \tanh^3(k(x-Ct-x_0)) \right]. \tag{7.3.27} \]

The boundary conditions for (7.3.25) is given by

\[ U(l,t) = C + 9 - 15 \left[ \tanh(k(l-Ct-x_0)) + \tanh^2(k(l-Ct-x_0)) - \tanh^3(k(l-Ct-x_0)) \right]. \]
\[ U(m,t) = C + 9 - 15 \left[ \tanh(k(m-Ct-x_0)) + \tanh^2(k(m-Ct-x_0)) - \tanh^3(k(m-Ct-x_0)) \right]. \]

\[ U_{xx}(l,t) = -15k^2 \sec h(k(l-Ct-x_0)) \left[-7 + 5 \cosh(2k(l-Ct-x_0)) + 3 \sinh(2k(l-Ct-x_0)) \right] \left[ -1 + \tanh(k(l-ct-x_0)) \right]. \]
\[ U_{xx}(m,t) = -15k^2 \sec h(k(m-Ct-x_0)) \left[-7 + 5 \cosh(2k(m-Ct-x_0)) + 3 \sinh(2k(m-Ct-x_0)) \right] \left[ -1 + \tanh(k(m-ct-x_0)) \right]. \]

Taking the Sumudu transform of (7.3.25) on both sides, we get

\[ \mathcal{S} [U(x,t)] - \sum_{k=0}^{n-1} \frac{U^{(k)}(0)}{u^{(-k)}} + u^n \mathcal{S}[U(x,t)U_x(x,t) + U_{xx}(x,t) + U_{xxx}(x,t)] = 0. \tag{7.3.28} \]

The nonlinear operator is defined by
\[ N \[ \phi \left( x, t; q \right) \] = S \[ \phi \left( x, t; q \right) \] - \sum_{k=0}^{n-1} \frac{\phi^{(k)}(0)}{u^{(k)}} \]
\[ + u^\alpha S \left[ \phi \left( x, t; q \right) \varphi_x \left( x, t; q \right) + \varphi_{xx} \left( x, t; q \right) + \varphi_{xxx} \left( x, t; q \right) + \varphi_{xxxx} \left( x, t; q \right) \right], \quad (7.3.29) \]

and thus,
\[ R_m \left( \hat{U}_{m-1} \right) = S \left[ U_{m-1} \right] + u^\alpha S \left[ \sum_{i=0}^{m-1} U_{m-1} \left( x, t \right) \left( U_{m-1-i} \right) \left( x, t \right) + \left( U_{m-1} \right)_{xx} \left( x, t \right) \right. \]
\[ + \left. \left( U_{m-1} \right)_{xxx} \left( x, t \right) + \left( U_{m-1} \right)_{xxxx} \left( x, t \right) \right]. \quad (7.3.30) \]

The \( m^{th} \)-order deformation equation is given by
\[ U_m \left( x, t \right) = \chi_m U_{m-1} \left( x, t \right) + S^{-1} \left[ \hbar H \left( x, t \right) R_m \left( \hat{U}_{m-1}, x, t \right) \right]. \quad (7.3.31) \]

On solving above equation for \( m = 1, 2, \ldots \), we get
\[ U_1 (x, t) = -6\hbar + C\hbar + 15\hbar \sec h [k (x - x_0)]^2 + \frac{90k t^\alpha \hbar \sec h [k (x - x_0)]^4}{\Gamma (1 + \alpha)} \]

\[- \frac{15Ckt^\alpha \hbar \sec h [k (x - x_0)]^4}{\Gamma (1 + \alpha)} - \frac{30k^2 t^\alpha \hbar \sec h [k (x - x_0)]^4}{\Gamma (1 + \alpha)} + \frac{225kt^\alpha \hbar \sec h [k (x - x_0)]^6}{\Gamma (1 + \alpha)} + \frac{120k^3 t^\alpha \hbar \sec h [k (x - x_0)]^6}{\Gamma (1 + \alpha)} + \frac{240k^4 t^\alpha \hbar \sec h [k (x - x_0)]^6}{\Gamma (1 + \alpha)} - 15\hbar \sec h [k (x - x_0)]^2 \tanh [k (x - x_0)] \]

\[+ \frac{180kt^\alpha \hbar \sec h [k (x - x_0)]^2 \tan h [k (x - x_0)]}{\Gamma (1 + \alpha)} - \frac{30Ckt^\alpha \hbar \sec h [k (-l + x)]^2 \tan h [k (-l + x)]}{\Gamma (1 + \alpha)} - \frac{450k t^\alpha \hbar \sec h [k (x - x_0)]^4 \tan h [k (x - x_0)]}{\Gamma (1 + \alpha)} + \frac{120k^2 t^\alpha \hbar \sec h [k (x - x_0)]^4 \tan h [k (x - x_0)]}{\Gamma (1 + \alpha)} \]
\[-240k^3\alpha \hbar \sec h [k(x - x_0)]^4 \tan h [k(x - x_0)] \frac{\Gamma (1 + \alpha)}{} + 225k^4\alpha \hbar \sec h [k(x - x_0)]^6 \tan h [k(x - x_0)] \frac{\Gamma (1 + \alpha)}{} + 2040k^4\alpha \hbar \sec h [k(x - x_0)]^6 \tan h [k(-l + x)] \frac{\Gamma (1 + \alpha)}{} - 180k^4\alpha \hbar \sec h [k(x - x_0)]^6 \tan h [k(-l + x)]^2 \frac{\Gamma (1 + \alpha)}{} + ... - 450k^4\alpha \hbar \sec h [k(x - x_0)]^4 \tan h [k(x - x_0)]^3 \frac{\Gamma (1 + \alpha)}{} + 3120k^4\alpha \hbar \sec h [k(x - x_0)]^4 \tan h [k(x - x_0)]^3 \frac{\Gamma (1 + \alpha)}{} + ... + 205\]
etc. In the same manner, the rest of the components of (7.3.31) as a series \( m \geq 2 \) can be obtained.

Due to very large values of successive results, we have only shown \( U_1 (x,t) \) but considered in analysis upto \( m = 3 \).

The solution of (7.3.25) is given by

\[
U (x,t) = U_0 (x,t) + \sum_{m=1}^{\infty} U_m (x,t). \tag{7.3.32}
\]

The accuracy and convergence of the HASTM series solution depend on the careful selection of the auxiliary parameter \( \hbar \), which can be chosen using the curve parallel to the horizontal axis region as shown in fig. 7.11.

![Figure 7.11: Plot of \( U (x,t) \) with respect to \( \hbar \) at \( t = 0.55 \) and \( x = 5 \).](image)

Figure 7.11: Plot of \( U (x,t) \) with respect to \( \hbar \) at \( t = 0.55 \) and \( x = 5 \).
Figure 7.12: Plot of $U(x, t)$ with respect to $x, t$ at $\alpha = 0.5$

Figure 7.13: Plot of $U(x, t)$ with respect to $x, t$ at $\alpha = 0.75$
The plots of $U(x,t)$ are shown in figs. 7.12, 7.13, 7.14 and 7.15 for the brownian motion of $\alpha = 0.5, 0.75, 1$ and exact solution at $C = 2.5$, $h = -1$, $x_0 = -10$ and $K = \frac{1}{2\sqrt{19}}$. At $\alpha = 1$ example shows the standard solution of KS equations discussed in [239].
7.4 Conclusion

In this chapter, the homotopy analysis Sumudu transform method is applied to solve time fractional Kuramoto–Shivashinsky equations. We discussed the application of the method with three different examples and compared the results with exact solution to reveal the accuracy and effectiveness for integer order and fractional order derivative. In addition, the little iteration is required to obtain an adequate result. The examples discussion supports this argument.