Chapter 5

Numerical simulation of diffusion type multi–term time fractional partial differential equation

5.1 Introduction

Fractional calculus is as old as classical calculus. Today, it plays significant role in various field of science and engineering including mathematical modeling astrophysics, biology etc. Recently, many researchers and mathematicians give valuable contributions to enhance the knowledge in this field [36, 135, 206].

For explaining dynamical systems, the integer–order system of differential equations are significant tool up to recent era. Unfortunately, modern study depicts that integer–order derivatives are not reasonably explaining the multifaceted and typical nature of various type of non dynamical system. Currently, differential equations of fractional order are popularly used by many researchers in all over world to form various scientific models. Importantly, fractional derivatives introduce for understanding of real life phenomena to reduce shortcoming of classical
calculus and also for the explanation of Brownian nature of particle in non-dynamical system. Some classical examples can be observed in the study of groundwater system in heterogeneous media.

The multi-term time fractional order partial differential equations played significant role to explain many physical nonlinear phenomenon such as the non-Markovian diffusion process with memory, propagation of mechanical waves in viscoelastic media, the effects of thermo-diffusion, transport in amorphous semiconductors [172, 177, 209, 220, 221]. The variable order differential operators better describes the behavior of various time varying processes instead of time varying coefficients[17, 83, 227]. Variable order and distributed order fractional operators are also discussed by Lorenzo and Hartley [165]. Many authors proposed the physical meaning of variable operators, see references and therein [40, 43, 168].

Various methods applied to solve system of fractional partial differential equations namely Adomian decomposition method (ADM) [54, 56, 58, 61, 78, 79], homotopy perturbation method (HPM) [225], homotopy analysis method (HAM)[116, 117], Predict, Evaluate, Correct, Evaluate (PECE)[64], Chebyshev spectral methods [60], Variational Iteration Method [47], Spectral method [167]. These methods have been proposed to obtain exact and approximate analytical solutions of multi-term fractional partial differential equations.

In this chapter, the multi-term time fractional nonlinear partial differential equation is solved. Using applicability of HASTM, [198, 199, 201, 202] we transform it into system of fractional order partial differential equations [58]. Some numerical experiments of linear and nonlinear systems of fractional PDE's will be presented.
5.2 Multi–Term FPDE

Here, we consider the following time fractional diffusion wave equations of multi–term [265]

\[
\sum_{i=1}^{n} c_i \, {}^\circ D_t^{\alpha_i} U(x,t) = K_{\alpha_1} U_{xx}(x,t) + f(x,t),
\]

(5.2.1)

where \(0 < \alpha_1 < ... < \alpha_n < 1\) or \(0 < \alpha_1 < ... < \alpha_n < 2\) and \(K_{\alpha_1}, c_i\) are constants, \(\, {}^\circ D_t^{\alpha_i}\) denotes the caputo derivative of arbitrary order \(\forall \alpha_i \in \mathbb{Q}, \alpha_i - \alpha_{i-1} \leq 1, \forall i\) and \(0 \leq \alpha_i \leq 1\).

We write equation (5.2.1) as a system of FPDE, using the algorithm proposed by Zheng et al.[265].

5.3 Analysis of the homotopy analysis Sumudu transform method

Consider the fractional multi–term diffusion equations in the following form:

\[
\begin{align*}
D_t^{\alpha_i} U_i(x,t) &= U_{i+1}, \quad i = n - 1, n - 2, ..., 1, \\
D_t^{\alpha_n} U_1(x,t) &= f(x,t, U_1, U_2, ..., U_n), \\
U(k)(x,0) &= C_k^j, \quad 0 \leq k \leq m_j, \quad m_j < \alpha_i \leq m_{j+1}, \quad 1 \leq j \leq n.
\end{align*}
\]

(5.3.1)

Now, applying the Sumudu transform on both sides of equation (5.3.1), we get

\[
\begin{align*}
\mathbb{S}[D_t^{\alpha_i} U_i(x,t)] &= \mathbb{S}[U_{i+1}], \quad i = n - 1, n - 2, ..., 1, \\
\mathbb{S}[D_t^{\alpha_n} U_1(x,t)] &= \mathbb{S}[f(x,t, U_1, U_2, ..., U_n)], \quad 0 \leq k \leq m_j, \quad m_j < \alpha_i \leq m_{j+1}, \quad 1 \leq j \leq n.
\end{align*}
\]

Using the differentiation property of the Sumudu transform, we get
Now, the nonlinear operator is defined as

\[ N_i [\varphi_i (x, t; q)] = \mathcal{S} [\varphi_i (x, t; q)] - \sum_{k=0}^{n-1} \frac{U_i^{(k)}(0)}{u^{(n-k)}} - u^{\alpha_i} \mathcal{S} [\varphi_{i+1} (x, t; q)], \quad i = 1, 2, ..., n - 1, \]

\[ N_n [\varphi_n (x, t; q)] = \mathcal{S} [\varphi_n (x, t; q)] - \sum_{k=0}^{n-1} \frac{U_n^{(k)}(0)}{u^{(n-k)}} - u^{\alpha_n} \mathcal{S} [f (x, t, \varphi_1, \varphi_2, ..., \varphi_n)], \quad (5.3.3) \]

where \( q \in [0, 1] \) be an embedding parameter and \( \varphi (x, t; q) \) is a real function of \( x, t \) and \( q \).

The following homotopies are constructed:

\[ (1 - q) \mathcal{S} [\varphi_i (x, t; q) - U_{i0} (x, t)] = h_i q H_i (x, t) N [\varphi_i (x, t; q)], \quad (5.3.4) \]

\[ (1 - q) \mathcal{S} [\varphi_n (x, t; q) - U_{n0} (x, t)] = h_n q H_n (x, t) N [\varphi_n (x, t; q)]. \]

\( h_i \neq 0 \) and \( H_i (x, t) \neq 0, i = 1, 2, 3, ..., n \) are nonzero auxiliary functions, \( U_{i0} (x, t) \) are initial guesses of \( U_i (x, t) \) and \( \varphi_i (x, t; q) \) are unknown functions. It is important that one has great freedom to choose auxiliary parameter in HASTM. Obviously, when \( q = 0 \) and \( q = 1 \) it holds

\[ \varphi_i (x, t; 0) = U_{i0} (x, t), \quad \varphi_i (x, t; 1) = U_i (x, t), \quad i = 1, 2, 3, ..., n. \quad (5.3.5) \]

Thus as \( q \) increases from 0 to 1, then the solution varies from initial guess \( U_{i0} (x, t) \) to \( U_i (x, t) \) Now, expanding \( \varphi_i (x, t; q) \) on Taylor’s series with respect to \( q \), we get

\[ \varphi_i (x, t; q) = U_{i0} (x, t) + \sum_{m=1}^{\infty} q^m U_{im} (x, t), \quad (5.3.6) \]
where

\[ U_{im}(x, t) = \frac{1}{m!} \left. \frac{\partial^m \varphi_i(x, t; q)}{\partial q^m} \right|_{q=0} \quad (5.3.7) \]

The convergence of the series solution (5.3.6) is controlled by \( h \). If the auxiliary linear operator, the initial guess, the auxiliary parameter \( h \) and the auxiliary function are properly chosen, the series (5.3.6) converges at \( q = 1 \). Hence, we obtain

\[ U(x, t) = U_{i0}(x, t) + \sum_{m=1}^{\infty} U_{im}(x, t) , \quad (5.3.8) \]

which must be one of the solutions of original nonlinear equations. The above expression provides us with a relationship between the initial guess \( U_{i0}(x, t) \) and the exact solution \( U(x, t) \) by means of the terms \( U_{im}(x, t) \) \((m = 1, 2, 3, \ldots)\), which are still to be determined.

Define the vectors

\[ \vec{U} = \{U_{i0}(x, t), U_{i1}(x, t), U_{i2}(x, t), \ldots, U_{im}(x, t)\} . \quad (5.3.9) \]

Differentiating the eq. (5.3.4) \( m \) times with respect to embedding parameter \( q \) and then setting \( q = 0 \), and finally dividing them by \( m! \), we obtain the \( m^{th} \) order deformation equations as follows:

\[ \mathcal{S} \left[ U_{im}(x, t) - \chi_{im} U_{i(m-1)}(x, t) \right] = h_i H_i(x, t) N_i \left[ U_i(x, t) \right] , \]
\[ \mathcal{S} \left[ U_{nm}(x, t) - U_{n(m-1)}(x, t) \right] = h_n H_n(x, t) N_m \left[ U_n(x, t) \right] . \quad (5.3.10) \]

Operating the inverse Sumudu transform of both sides of above equation, we get
\[ U_{im} (x, t) = \chi_m U_{i(m-1)} (x, t) + \hbar \mathcal{S}^{-1} \left[ H_i (x, t) R_{im} (\mathcal{U}_{i(m-1)} ; x, t) \right] , \]
\[ U_{nm} (x, t) = \chi_m U_{n(m-1)} (x, t) + \hbar \mathcal{S}^{-1} \left[ H_n (x, t) R_{nm} (\mathcal{U}_{n(m-1)} ; x, t) \right] , \]

where

\[ R_{im} (\mathcal{U}_{i(m-1)} , x ; t) = \frac{1}{(m - 1)!} \frac{\partial^{m-1} \phi_i (x, t; q)}{\partial q^{m-1}} \bigg|_{q=0} \]  

and

\[ \chi_m = \begin{cases} 
0, & m \leq 1, \\
1, & m > 1.
\end{cases} \]

In this way, it is easy to obtain \( U_{im} (x, t) \) for \( m \geq 1 \), at \( M^{th} \) order, then \( U_{im} (x, t) \) is given by

\[ U_i (x, t) = \sum_{m=0}^{M} U_{im} (x, t) , \]  

where \( M \to \infty \), we obtain an accurate approximation of the original equation (5.3.1).

### 5.4 Illustrative examples

To illustrate the efficiency and accuracy of above discussed method, some multi–term time fractional diffusion equations are considered. The transformation of MTTFDE as a system of FPDE is evaluated using the HASTM.
5.4.1 Example

Consider the following two-term time fractional diffusion equation [265]

\[
\begin{aligned}
&\mathcal{D}_0^\alpha_1 U(x,t) + \mathcal{D}_0^\alpha_2 U(x,t) = \partial_{xx} U(x,t) + F(x,t), \\
&U(x,0) = 0, \quad x \in (0,1), \\
&U(0,t) = U(1,t) = 0, \quad t \in (0,1],
\end{aligned}
\]

(5.4.1)

where

\[
F(x,t) = \frac{6}{\Gamma(4 - \alpha_1)} t^{3-\alpha_1} \sin\pi x + \frac{6}{\Gamma(4 - \alpha_2)} t^{3-\alpha_2} \sin\pi x + \pi^2 t^3 \sin\pi x.
\]

The exact solution of Eq. (5.4.1) is \( U(x,t) = t^3 \sin\pi x \).

We can convert the equation (5.4.1) into following system of time fractional partial differential equations

\[
\begin{aligned}
&D_t^{\alpha_2} U(x,t) = V(x,t), \quad U(x,0) = 0, \\
&D_t^{\alpha_1-\alpha_2} V(x,t) = -V(x,t) + \partial_{xx} U(x,t) + F(x,t)
\end{aligned}
\]

(5.4.2)

Applying the Sumudu transform of Eq. (5.4.2)

\[
\begin{aligned}
&\mathcal{S}[U(x,t)] - \frac{n-1}{\alpha_2} \mathcal{S}[V(x,t)] = 0, \\
&\mathcal{S}[V(x,t)] - \frac{n-1}{\alpha_1-\alpha_2} \mathcal{S}[V(x,t)] - \mathcal{S}[\partial_{xx} U(x,t) - F(x,t)] = 0,
\end{aligned}
\]

\[
\begin{aligned}
&\mathcal{S}[U(x,t)] - \sum_{k=0}^{n-1} \frac{U^{(k)}}{\alpha_2^{k+1}} - u^{\alpha_2} \mathcal{S}[V(x,t)] = 0, \\
&\mathcal{S}[V(x,t)] - \sum_{l=0}^{n-1} \frac{V^{(l)}}{\alpha_1-\alpha_2^{l+1}} + u^{\alpha_1-\alpha_2} \mathcal{S}[\partial_{xx} U(x,t) - F(x,t)] = 0
\end{aligned}
\]

(5.4.3)
Now, the nonlinear operators are defined as

\[ N [\phi_1 (x, t; q)] = S [\phi_1 (x, t; q)] - \sum_{k=0}^{n-1} \frac{\phi_1^{(k)} (0)}{u^{(-k)}} - u^{a_2} S [\phi_1 (x, t; q)] \]

\[ N [\phi_2 (x, t; q)] = S [\phi_2 (x, t; q)] - \sum_{l=0}^{n-1} \frac{\phi_2^{(l)} (0)}{u^{(-l)}} + u^{a_1-a_2} S [\phi_2 (x, t; q) - \partial_{xx} \phi_1 (x, t; q) - F (x, t)] , \]

(5.4.4)

In the view of discussion, the zeroth–order deformation equations are constructed as follows:

\[ (1 - q) S [\varphi_1 (x, t; q) - U_0 (x, t)] = h_1 q H_1 (x, t) N [\varphi_1 (x, t; q)] , \]

\[ (1 - q) S [\varphi_2 (x, t; q) - V_0 (x, t)] = h_2 q H_2 (x, t) N [\varphi_2 (x, t; q)] . \]

(5.4.5)

The \( m^{th} \)-order deformation equations are given by

\[ U_m (x, t) = \chi_m U_{m-1} (x, t) + h_1 S^{-1} \left[ H_1 (x, t) R_{1m} \left( \frac{\nabla U}{m-1} , x, t \right) \right] , \]

\[ V_m (x, t) = \chi_m V_{m-1} (x, t) + h_2 S^{-1} \left[ H_2 (x, t) R_{2m} \left( \frac{\nabla V}{m-1} , x, t \right) \right] , \]

(5.4.6)

where

\[ R_{1m} \left( \frac{\nabla U}{m-1} , x, t \right) = S [U_{m-1} (x, t)] - u^{a_2} S [U_{m-1} (x, t)] , \]

\[ R_{2m} \left( \frac{\nabla V}{m-1} , x, t \right) = S [V_{m-1} (x, t)] + u^{a_1-a_2} S [V_{m-1} (x, t) - \partial_{xx} U_{m-1} (x, t) - F (x, t)] . \]

(5.4.7)

On solving above equations for \( m = 1, 2, \ldots \), we get

\[ U_1 (x, t) = 0 , \]
\[ V_1(x, t) = -\hbar^2 6t^{3-2\alpha_1} \sin \pi x \left( \frac{t^{\alpha_1}}{\Gamma(4 - \alpha_1)} + \frac{t^{\alpha_2}}{\Gamma(4 - 2\alpha_1 + \alpha_2)} + \frac{\pi^2 t^{\alpha_1+\alpha_2}}{\Gamma(4 - 2\alpha_1 + 2\alpha_2)} \right), \]

\[ U_2(x, t) = \hbar^1 \hbar^2 6t^{3-2\alpha_1+\alpha_2} \sin \pi x \left( \frac{t^{\alpha_1}}{\Gamma(4 - \alpha_1 + \alpha_2)} + \frac{t^{\alpha_2}}{\Gamma(4 - 2\alpha_1 + 2\alpha_2)} \right. \]
\[ + \left. \frac{\pi^2 t^{\alpha_1+\alpha_2}}{\Gamma(4 - 2\alpha_1 + 2\alpha_2)} \right), \]

\[ V_2(x, t) = -\hbar^2 6t^{3-2\alpha_1} (1 + \hbar^2) \sin \pi x \left( \frac{t^{\alpha_1}}{\Gamma(4 - \alpha_1)} + \frac{t^{\alpha_2}}{\Gamma(4 - 2\alpha_1 + 2\alpha_2)} \right. \]
\[ + \left. \frac{\pi^2 t^{\alpha_1+\alpha_2}}{\Gamma(4 - 2\alpha_1 + 2\alpha_2)} \right) - \hbar^2 6t^{3-2\alpha_1+\alpha_2} \sin \pi x \left( \frac{t^{\alpha_1}}{\Gamma(4 - 2\alpha_1)} \right. \]
\[ + \left. \frac{t^{\alpha_2}}{\Gamma(4 - 3\alpha_1 + 2\alpha_2)} + \frac{\pi^2 t^{\alpha_1+\alpha_2}}{\Gamma(4 - 2\alpha_1 + 2\alpha_2)} \right), \]

\[ U_3(x, t) = \frac{12\hbar^1 \hbar^2 t^{3-\alpha_1+\alpha_2} \sin \pi x}{\Gamma(4 - \alpha_1 + \alpha_2)} + \frac{12\hbar^1 \hbar^2 t^{3-\alpha_1+\alpha_2} \sin \pi x}{\Gamma(4 - \alpha_1 + \alpha_2)} \]
\[ + \frac{12\pi^2 \hbar^1 \hbar^2 t^{3-\alpha_1+\alpha_2} \sin \pi x}{\Gamma(4 - \alpha_1 + \alpha_2)} + \frac{6\hbar^2 \hbar^2 t^{3-\alpha_1+\alpha_2} \sin \pi x}{\Gamma(4 - \alpha_1 + 2\alpha_2)} \]
\[ + \frac{6\hbar^2 \hbar^1 t^{3-\alpha_1+2\alpha_2} \sin \pi x}{\Gamma(4 - 2\alpha_1 + 2\alpha_2)} + \frac{6\pi^2 \hbar^2 \hbar^1 t^{3-\alpha_1+2\alpha_2} \sin \pi x}{\Gamma(4 - \alpha_1 + 2\alpha_2)} \]
\[ + \frac{6\hbar^2 \hbar^1 t^{3-\alpha_1+\alpha_2} \sin \pi x}{\Gamma(4 - \alpha_1 + \alpha_2)} + \frac{12\hbar^2 \hbar^1 t^{3-\alpha_1+2\alpha_2} \sin \pi x}{\Gamma(4 - 2\alpha_1 + 2\alpha_2)} \]
\[ + \frac{6\pi^2 \hbar^2 \hbar^1 t^{3-\alpha_1+2\alpha_2} \sin \pi x}{\Gamma(4 - \alpha_1 + 2\alpha_2)} + \frac{6\pi^2 \hbar^2 \hbar^1 t^{3-\alpha_1+3\alpha_2} \sin \pi x}{\Gamma(4 - 2\alpha_1 + 3\alpha_2)} \].
\[ V_3(x, t) = \frac{-6\hbar_2 t^{3-\alpha_1} \sin \pi x}{\Gamma (4 - \alpha_1)} - \frac{6\pi^2 \hbar_2 t^{3-\alpha_1+\alpha_2} \sin \pi x}{\Gamma (4 - \alpha_1 + \alpha_2)} - \frac{24 \hbar_2^2 t^{3-2\alpha_1+\alpha_2} \sin \pi x}{\Gamma (4 - 2\alpha_1 + \alpha_2)} - \frac{12 \hbar_2^2 t^{3-3\alpha_1+2\alpha_2} \sin \pi x}{\Gamma (4 - 3\alpha_1 + 2\alpha_2)} - \frac{6 \pi^2 h_1 h_2^3 t^{3-2\alpha_1+2\alpha_2} \sin \pi x}{\Gamma (4 - 2\alpha_1 + 2\alpha_2)} + \frac{6 \pi^2 h_1 h_2^3 t^{3-3\alpha_1+3\alpha_2} \sin \pi x}{\Gamma (4 - 3\alpha_1 + 3\alpha_2)} + \frac{18 \hbar_2^4 t^{3-2\alpha_1+3\alpha_2} \sin \pi x}{\Gamma (4 - 2\alpha_1 + 3\alpha_2)} + \frac{6 \pi^4 h_1 h_2^3 t^{3-3\alpha_1+3\alpha_2} \sin \pi x}{\Gamma (4 - 2\alpha_1 + 3\alpha_2)} + \frac{18 \hbar_2^3 t^{3-3\alpha_1+2\alpha_2} \sin \pi x}{\Gamma (4 - 3\alpha_1 + 2\alpha_2)} + \frac{6 \hbar_2^3 t^{3-4\alpha_1+3\alpha_2} \sin \pi x}{\Gamma (4 - 4\alpha_1 + 3\alpha_2)} - \frac{12 \pi^2 h_2 t^{3-3\alpha_1+2\alpha_2} \sin \pi x}{\Gamma (4 - 3\alpha_1 + 3\alpha_2)} - \frac{12 \pi^2 h_2^2 t^{3-3\alpha_1+2\alpha_2} \sin \pi x}{\Gamma (4 - 3\alpha_1 + 3\alpha_2)} - \frac{6 \pi^2 h_2^3 t^{3-3\alpha_1+3\alpha_2} \sin \pi x}{\Gamma (4 - 3\alpha_1 + 3\alpha_2)} - \frac{12 \pi^2 h_2^3 t^{3-3\alpha_1+2\alpha_2} \sin \pi x}{\Gamma (4 - 3\alpha_1 + 3\alpha_2)} - \frac{6 \pi^2 h_2^3 t^{3-3\alpha_1+2\alpha_2} \sin \pi x}{\Gamma (4 - 3\alpha_1 + 3\alpha_2)}, \]

etc.
Figures 5.1, 5.2 reflect the association of Brownian motion of variable order multi–term time fractional advection differential equations at $h_1 = h_2 = -1$, for
$x = 0.5$ with respect to time variable $t$.

## 5.4.2 Example

Consider the following two-term time fractional diffusion equation [265]

$$
\begin{aligned}
\begin{cases}
\partial_{t}^{\alpha_{1}}U(x,t) + \partial_{t}^{\alpha_{2}}U(x,t) + \partial_{t}^{\alpha_{3}}U(x,t) = \partial_{xx}U(x,t) + F(x,t), \\
U(x,0) = 0, \quad x \in (0,1), \\
U(0,t) = U(1,t) = 0, \quad t \in (0,1],
\end{cases}
\end{aligned}
$$

(5.4.8)

where

$$
F(x,t) = \pi^2 t^3 \sin \pi x + \frac{6}{\Gamma(4-\alpha_1)} t^{3-\alpha_1} \sin \pi x + \frac{6}{\Gamma(4-\alpha_2)} t^{3-\alpha_2} \sin \pi x
$$

$$
+ \frac{6}{\Gamma(4-\alpha_3)} t^{3-\alpha_3} \sin \pi x.
$$

The exact solution of Eq. (5.4.8) is $U(x,t) = t^3 \sin \pi x$.

We can convert the Eq. (5.4.8) into the following system of time fractional partial differential equations

$$
\begin{aligned}
D_{t}^{\alpha_{1}}U(x,t) &= V(x,t), \quad U(x,0) = 0, \\
D_{t}^{\alpha_{2}-\alpha_{3}}V(x,t) &= W(x,t), \quad V(x,0) = 0, \\
D_{t}^{\alpha_{1}-\alpha_{2}}W(x,t) &= -W(x,t) - V(x,t) + \partial_{xx}U(x,t) + F(x,t).
\end{aligned}
$$

(5.4.9)

Applying the Sumudu transform on both sides of Eq. (5.4.9)
\[
\begin{align*}
S[U(x,t)] - \sum_{k=0}^{n-1} \frac{U^{(k)}(0)}{u^{(\alpha_3-k)}} - S[V(x,t)] &= 0, \\
S[V(x,t)] - \sum_{l=0}^{n-1} \frac{V^{(l)}(0)}{u^{(\alpha_2-\alpha_3-l)}} - S[W(x,t)] &= 0, \\
S[W(x,t)] - \sum_{m=0}^{n-1} \frac{W^{(m)}(0)}{u^{(\alpha_1-\alpha_2-m)}} + S[V(x,t) + W(x,t) - \partial_{xx}U(x,t) - F(x,t)] &= 0, \\
\end{align*}
\]

Now, the nonlinear operators are defined as

\[
\begin{align*}
N[\phi_1(x,t;q)] &= S[\phi_1(x,t;q)] - \sum_{k=0}^{n-1} \frac{\phi_1^{(k)}(0)}{u^{(-k)}} - u^{\alpha_3} S[\phi_1(x,t;q)], \\
N[\phi_2(x,t;q)] &= S[\phi_2(x,t;q)] - \sum_{l=0}^{n-1} \frac{\phi_2^{(l)}(0)}{u^{(-l)}} - u^{\alpha_2-\alpha_3} S[\phi_2(x,t;q)], \\
N[\phi_3(x,t;q)] &= S[\phi_3(x,t;q)] - \sum_{m=0}^{n-1} \frac{\phi_3^{(m)}(0)}{u^{(-m)}} + u^{\alpha_1-\alpha_2} S[\phi_3(x,t;q) + \phi_2(x,t;q)] \\
&\quad - \partial_{xx} \phi_1(x,t;q) - F(x,t). \\
\end{align*}
\]

In the view of discussion, we can construct the zeroth–order deformation equations:
\[(1 - q) S [\varphi_1 (x, t; q) - U_0 (x, t)] = h_1 q H_1 (x, t) N [\varphi_1 (x, t; q)], \]
\[(1 - q) S [\varphi_2 (x, t; q) - V_0 (x, t)] = h_2 q H_2 (x, t) N [\varphi_2 (x, t; q)], \]
\[(1 - q) S [\varphi_3 (x, t; q) - W_0 (x, t)] = h_3 q H_3 (x, t) N [\varphi_3 (x, t; q)]. \]

The \(m^{th}\)-order deformation equations are given by

\[
U_m (x, t) = \chi_m U_{m-1} (x, t) + h_1 S^{-1} \left[ H_1 (x, t) R_{1m} \left( \vec{U}_{m-1}, x, t \right) \right],
\]
\[
V_m (x, t) = \chi_m V_{m-1} (x, t) + h_2 S^{-1} \left[ H_2 (x, t) R_{2m} \left( \vec{V}_{m-1}, x, t \right) \right],
\]
\[
W_m (x, t) = \chi_m W_{m-1} (x, t) + h_3 S^{-1} \left[ H_3 (x, t) R_{3m} \left( \vec{W}_{m-1}, x, t \right) \right],
\]

where

\[
R_{1m} \left( \vec{U}_{m-1}, x, t \right) = S \left[ U_{m-1} (x, t) \right] - u^{\alpha_3} S \left[ U_{m-1} (x, t) \right],
\]
\[
R_{2m} \left( \vec{V}_{m-1}, x, t \right) = S \left[ V_{m-1} (x, t) \right] - u^{\alpha_3 - \alpha_2} S \left[ V_{m-1} (x, t) \right],
\]
\[
R_{3m} \left( \vec{W}_{m-1}, x, t \right) = S \left[ W_{m-1} (x, t) \right] + u^{\alpha_1 - \alpha_2} S \left[ W_{m-1} (x, t) + V_{m-1} (x, t) \right.
\]
\[
\left. - \partial_{xx} U_{m-1} (x, t) - F (x, t) \right].
\]

On solving above equations for \(m = 1, 2, \ldots\), we get

\[
U_1 [x, t] = 0,
\]
\[
V_1 [x, t] = 0,
\]
\[
W_1 [x, t] = \frac{-6h_3 t^{3+\alpha_1 - 2\alpha_2} \sin \pi x}{\Gamma (4 + \alpha_1 - 2\alpha_2)} - \frac{6h_3 t^{3-\alpha_2} \sin \pi x}{\Gamma (4 - \alpha_2)}
\]
\[
- \frac{6h_3 t^{3-\alpha_2} \sin \pi x}{\Gamma (4 - \alpha_2)} - \frac{6h_3 t^{3+\alpha_1 - 2\alpha_2 - \alpha_3} \sin \pi x}{\Gamma (4 + \alpha_1 - \alpha_2 - \alpha_3)},
\]
\[
U_2 \left[ x, t \right] = 0,
\]
\[
V_2 \left[ x, t \right] = \frac{6h_2h_3t^{3+\alpha_1-2\alpha_2} \sin \pi x}{\Gamma \left( 4 + \alpha_1 - 2\alpha_2 \right)} + \frac{6h_2h_3t^{3-\alpha_3} \sin \pi x}{\Gamma \left( 4 - \alpha_3 \right)} + \frac{6\pi^2h_2h_3\pi^2t^{3+\alpha_1-\alpha_3} \sin \pi x}{\Gamma \left( 4 + \alpha_1 - \alpha_3 \right)};
\]
\[
W_2 \left[ x, t \right] = \frac{-6h_3t^{3+\alpha_1-2\alpha_2} \sin \pi x}{\Gamma \left( 4 + \alpha_1 - 2\alpha_2 \right)} - \frac{6h_3t^{3-\alpha_2} \sin \pi x}{\Gamma \left( 4 - \alpha_2 \right)} - \frac{6h_3\pi^2t^{3+\alpha_1-\alpha_2} \sin \pi x}{\Gamma \left( 4 + \alpha_1 - \alpha_2 \right)};
\]
\[
U_3 \left[ x, t \right] = -6h_1h_2h_3 \sin \pi x \left( \frac{t^3}{6} + \frac{\pi^2t^{3+\alpha_1}}{\Gamma \left( 4 + \alpha_1 \right)} + \frac{t^{3+\alpha_1-\alpha_2}}{\Gamma \left( 4 + \alpha_1 - \alpha_2 \right)} + \frac{t^{3+\alpha_1-\alpha_3}}{\Gamma \left( 4 + \alpha_1 - \alpha_3 \right)} \right),
\]
\[
V_3 \left[ x, t \right] = \frac{12h_2h_3t^{3+\alpha_1-2\alpha_3} \sin \pi x}{\Gamma \left( 4 + \alpha_1 - 2\alpha_2 \right)} + \frac{12h_2h_3t^{3-\alpha_3} \sin \pi x}{\Gamma \left( 4 - \alpha_3 \right)} + \frac{12\pi^2h_2h_3\pi^2t^{3+\alpha_1-\alpha_3} \sin \pi x}{\Gamma \left( 4 + \alpha_1 - \alpha_3 \right)};
\]
\[
W_3 \left[ x, t \right] = \frac{-6h_3t^{3+\alpha_1-2\alpha_2} \sin \pi x}{\Gamma \left( 4 + \alpha_1 - 2\alpha_2 \right)} - \frac{6h_3t^{3-\alpha_2} \sin \pi x}{\Gamma \left( 4 - \alpha_2 \right)} - \frac{6h_3\pi^2t^{3+\alpha_1-\alpha_2} \sin \pi x}{\Gamma \left( 4 + \alpha_1 - \alpha_2 \right)};
\]
Figure 5.3: Plot of $U(x,t)$ w.r.t $t$ at $x = 0.5$ for $\alpha = 0.3, 0.6, 0.9, 1.0$ and exact solution.
Figure 5.4: Plot of $V(x, t)$ w.r.t $t$ at $x = 0.5$ for $\alpha = 0.3, 0.6, 0.9, 1.0$ and exact solution.

Figure 5.5: Plot of $W(x, t)$ w.r.t $t$ at $x = 0.5$ for $\alpha = 0.3, 0.6, 0.9, 1.0$ and exact solution.

Figs. 5.3, 5.4 and 5.5 show the association of Brownian motion of variable order multi-term time fractional advection differential equations at $h_1 = h_2 =$
\[ h_3 = -1, \text{ for } x = 0.5 \text{ with respect to time variable } t. \]

### 5.5 Conclusion

In this chapter, the multi–order fractional partial differential equations are considered and transformed it to change the domain to reduced the complexity without loss of generality, provided the more approximate results, using the homotopy analysis Sumudu transform method. Results obtained are lucid in understanding and free from rounding off errors, which are mostly occurring in mess method or perturbation methods. This method can be generalized to solve any kind of multi–order fractional partial differential equations.