Chapter 4

Homotopy analysis Sumudu transform method for time–fractional third order dispersive partial differential equation

4.1 Introduction

The Fractional calculus is as old as classical calculus because importance of this theory was marked as soon as the ideas of the classical calculus were born from the discussion of half derivative in epistle of Leibniz and L’Hopital in the year 1695. Further, many mathematicians contributed on this theory and strengthened the notion of generalized order differentials and integrals viz. Liouville, Euler, Fourier, Abel, Riemann, Weyl, Liouville took initial steps for the fractional order integration and published the series of papers (1832-1837). The Riemann–Liouville operator was the most popular among mathematicians who solved fractional order integration problems [193]. Evidently, up to 300 years
mentioned, theory was the asset of only pure mathematicians due to unavailability of geometrical and physical interpretation of fractional order differential and integral operators. Caputo [37] described useful formula for generalized order derivatives. Oldham and Spanier [193] discussed the initial framework of application in diffusion problem and classical calculus with proper explanation. Ross [213] presented the chronological development of this theory after completing his PhD in fractional calculus and also published a monograph [178]. In the consequence, Podlubny [206], Kilbas et al. [135], Diethelm [57], Caponetto et al. [35], Samko et al. [217] introduced the generalized differential and integral operators in more precise form with existence and uniqueness of results in application. Now a days, enormous model and physical phenomena like anomalous diffusion equation theory [238], mechanics of non-Hamiltonian systems [234], theory of long range interaction [155], astrophysics [233], optics [74], mechanics of fractal media [232], plasma physics [68, 69], physical kinetics [205], quantum mechanics [141], chaotic dynamics [19], which cannot meaningfully describe without means of fractional operators. In the reason being dynamical systems, integer order derivatives only evaluate a fixed number of derivatives wherein fractional derivatives can evaluate the value for any arbitrary order of derivative correspond to real numbers. Payable to its incredible scope and relevance in many branches of science and engineering, an extensive attention has been shown to find the solution of differential and integral equations involving the fractional derivatives. Except for the modelling approach of mentioned differential equations and its solution procedure, including the efficiency of convergence, divergence or junctions, the solutions of the model are uniformly important in numerical evaluation. In order to achieve more convenient and highly adorable results, numerous numerical methods have been proposed to solve the differential equations of fractional order. Some of semi-analytic/analytic methods or numerical methods are differential transform method [41, 119, 266], Variational iteration method (VIM)[255, 247], fractional variational iteration method (FVIM)[218], Wavelet Operational ma-
trix method [18], generalized differential transform method [70], Fractional sub equation method [264], Homotopy perturbation method [91, 92, 95, 97], Homotopy analysis method [159, 160, 161, 162], Homotopy analysis transform method [143, 145, 146, 151, 258], Fractional differential transform and Modified Fractional differential transform method [15, 71], Homotopy analysis Sumudu transform method (HASTM)[199].

In order to convert the complex linear and nonlinear form of fractional order partial differential equations into simpler algebraic form many type of fractional integral and differential transforms have been applied to gain the exact and approximate solutions of FPDE’s [242, 89]. Kumar and his co–workers successfully applied homotopy analysis transform method which is the cumulation of Laplace transform and homotopy analysis method for the solution of fractional Fornberg– Whitham equation arising in wave breaking [143], volterra integral equation [146], fractional wave equations [258], coupled Boussinesq–Burger’s equations arise in propagation of shallow water waves [145], unidirectional propagation of long waves in dispersive media [151]. Watugala [243] introduced the Sumudu transform and some properties discussed by Weerakoon [251, 252]. Further, Belgacem [28, 29, 30, 31, 32, 88, 109, 126] provided precise definition of Sumudu transform and also discussed better implementations for the solution of FDE’s, FPDE’s using many results, properties and relations, which enhanced the literature of this transform. It can easily convert many fractional order linear and nonlinear partial differential equations in time domain without loss of generality for different type of included fractional operators viz. Caputo, Riemann–Liouville, Ritzs space, etc.

Multistage HAM is introduced in [87] for solving non–linear Riccati Differential Equations. Since the homotopy analysis method applied to solve in wide variety of linear and nonlinear partial differential equations such as some fractional order smoking model [113], Lorenz system [13], a class of partial differen-
tial equations [63], space– and time–fractional kdv equation [182], foam drainage equation with space– and time–fractional derivatives [72] and so on. The disadvantage of perturbation method is to solve each iteration and convergence region which is very less. ADM, VIM, provide week convergent and not necessarily accurate always to exact solutions. DTM and FDTM, MFDTM require additional information and basic formula to evaluate the results. While HASTM is easily evaluated the nonlinear term with high accuracy due to independence of physical parameters and absolute convergence of series towards the exact solutions.

In this Chapter, the application of Homotopy analysis Sumudu transform method is explained to solve third- order fractional dispersive partial differential equations [59, 125, 156, 176, 237, 246] including fractional derivative in caputo sense. The HASTM obtains semi analytic solutions in the form of series solutions. It is different from other transforms and semi analytic method, which does not require additional information except some initial and boundary conditions. It easily changes the original problem to lucid manner and then, one can evaluate the results with high convergence and accuracy.

The chapter taxonomy is arranged as follows: The rudimental concept of HASTM is explained in section 4.2. To demonstrate the method and advantages, three examples of fractional order dispersive partial differential equations are solved with discussion of convergence in section 4.3. At the end, the concluding remark is presented in section 4.4.
4.2 Solution by homotopy analysis Sumudu transform method

To illustrate the rudimental conception of the HASTM for the fractional partial differential equation, the linear third order dispersive partial differential equations is considered in the following manner:

\[ D_t^{\alpha} \xi(x_1, x_2, ..., x_n, t) + \sum_{i=1}^{n} l_i \frac{\partial^3 \xi(x_1, x_2, ..., x_n, t)}{\partial x_i^3} = G(x_1, x_2, ..., x_n, t) ; \quad (4.2.1) \]

\[ \forall l_i, t > 0, \forall x_i \in R, \text{ } n - 1 < \alpha \leq n, \text{ and the } G(x_1, x_2, ..., x_n, t) \text{ is the source function}. \]

For simplicity, the initial and boundary conditions, which can be treated in a homogeneous way is ignored. Now, the methodology consists of applying the Sumudu transform first on both sides of the equation (4.2.1), we get

\[ \mathbb{S} [D_t^{\alpha} \xi(x_1, x_2, ..., x_n, t)] + \mathbb{S} \left[ \sum_{i=1}^{n} l_i \frac{\partial^3 \xi(x_1, x_2, ..., x_n, t)}{\partial x_i^3} \right] = \mathbb{S} [G(x_1, x_2, ..., x_n, t)] ; \quad (4.2.2) \]

Using the definition of differentiation property of the Sumudu transform

\[ u^{-\alpha} \mathbb{S} [\xi(x_1, x_2, ..., x_n, t)] \bigg[ - \sum_{k=0}^{n-1} \frac{\xi^{(k)}(0)}{u^{(\alpha-k)}} \bigg] + \mathbb{S} \left[ \sum_{i=1}^{n} l_i \frac{\partial^3 \xi(x_1, x_2, ..., x_n, t)}{\partial x_i^3} \right] = \mathbb{S} [G(x_1, x_2, ..., x_n, t)] ; \]

which gives
\[
S [\xi (x_1, x_2, ..., x_n, t)] - \sum_{k=0}^{n-1} \frac{\xi^{(k)} (0)}{u^{-k}} + u^n S \left[ \sum_{i=1}^{n} l_i \frac{\partial^3 \xi (x_1, x_2, ..., x_n, t)}{\partial x_i^3} \right] - G (x_1, x_2, ..., x_n, t) = 0;
\]

(4.2.3)

The nonlinear operator is defined

\[
N [\phi (x_1, x_2, ..., x_n, t; p)] = S [\phi (x_1, x_2, ..., x_n, t; p)] - \sum_{k=0}^{n-1} \frac{\phi^{(k)} (0)}{u^{-k}} + u^n S \left[ \sum_{i=1}^{n} l_i \frac{\partial^3 \phi (x_1, x_2, ..., x_n, t; p)}{\partial x_i^3} - G (x_1, x_2, ..., x_n, t; p) \right],
\]

(4.2.4)

where \( p \in [0, 1] \) be an embedding parameter and \( \phi (x_1, x_2, ..., x_n, t; p) \) is a real function of \( x_1, x_2, ..., x_n, t \) and \( p \).

The homotopy is constructed as follow:

\[
(1 - p) S [\phi (x_1, x_2, ..., x_n, t; p) - \xi_0 (x_1, x_2, ..., x_n, t)] = phH (x_1, x_2, ..., x_n, t) N [\phi (x_1, x_2, ..., x_n, t; p)];
\]

(4.2.5)

where \( h \) is a nonzero auxiliary parameter and \( H (x_1, x_2, ..., x_n, t) \neq 0 \), an auxiliary function \( \xi_0 (x_1, x_2, ..., x_n, t) \) is an initial guess of \( \xi (x_1, x_2, ..., x_n, t) \) and \( \phi (x_1, x_2, ..., x_n, t) \) is an unknown function. It is important that one has greater freedom to choose auxiliary parameter in HASTM. Obviously, when \( p = 0 \) and \( p = 1 \) it holds

\[
\phi (x_1, x_2, ..., x_n, t; 0) = \xi_0 (x_1, x_2, ..., x_n, t), \quad \phi (x_1, x_2, ..., x_n, t; 1) = \xi (x_1, x_2, ..., x_n, t).
\]

(4.2.6)

Thus, as \( p \) increases from 0 to 1, the solution varies from initial guess \( \xi_0 (x_1, x_2, ..., x_n, t) \) to the solution \( \xi (x_1, x_2, ..., x_n, t) \). Now, expanding \( \phi (x_1, x_2, ..., x_n, t; p) \) on Taylor’s series with respect to \( q \), we get
\[ \phi (x_1, x_2, ..., x_n, t; p) = \xi_0 (x_1, x_2, ..., x_n, t) + \sum_{m=1}^{\infty} p^m \xi_m (x_1, x_2, ..., x_n, t) , \quad (4.2.7) \]

where

\[ \xi_m (x_1, x_2, ..., x_n, t) = \frac{1}{\Gamma (m + 1)} \left. \frac{\partial^m \phi (x_1, x_2, ..., x_n, t; p)}{\partial p^m} \right|_{p=0} . \quad (4.2.8) \]

The convergence of the series solution (4.2.7) is controlled by \( \hbar \). If the auxiliary linear operator, the initial guess, the auxiliary parameter \( \hbar \) and the auxiliary function are properly chosen, the series (4.2.7) converges at \( p = 1 \). Hence, we obtain

\[ \xi (x_1, x_2, ..., x_n, t) = \xi_0 (x_1, x_2, ..., x_n, t) + \sum_{m=1}^{\infty} \xi_m (x_1, x_2, ..., x_n, t) , \quad (4.2.9) \]

which must be one of the solutions of original nonlinear equations. The above expression provides us with a relationship between the initial guess \( \xi_0 (x_1, x_2, ..., x_n, t) \) and the exact solution \( \xi (x_1, x_2, ..., x_n, t) \) by means of the terms \( \xi_m (x_1, x_2, ..., x_n, t) \) \( (m = 1, 2, 3, ...) \), which are still to be determined.

Define the vectors:

\[ \vec{\xi} = \{ \xi_0 (x_1, x_2, ..., x_n, t) , \xi_1 (x_1, x_2, ..., x_n, t) , \xi_2 (x_1, x_2, ..., x_n, t) , ..., \xi_m (x_1, x_2, ..., x_n, t) \} . \quad (4.2.10) \]

Differentiating the zero order deformation Eq. (4.2.5) \( m \) times with respect to embedding parameter \( p \) and then setting \( p = 0 \), and finally dividing them by \( \Gamma (m + 1) \), the \( m^{th} \) order deformation equation is obtained as follows:
\[ S[\xi_m(x_1, x_2, \ldots, x_n, t) - \chi_m\xi_{m-1}(x_1, x_2, \ldots, x_n, t)] = \hbar H(x_1, x_2, \ldots, x_n, t) R_m(\xi_{m-1}, x_1, x_2, \ldots, x_n, t). \]  

(4.2.11)

Operating the inverse Sumudu transform on both sides of Eq. (4.2.11),

\[ \xi_m(x_1, x_2, \ldots, x_n, t) = \chi_m\xi_{m-1}(x_1, x_2, \ldots, x_n, t) + \mathcal{S}^{-1}\left[ \hbar H(x_1, x_2, \ldots, x_n, t) R_m(\xi_{m-1}, x_1, x_2, \ldots, x_n, t) \right], \]  

(4.2.12)

where

\[ R_m(\xi_{m-1}, x_1, x_2, \ldots, x_n, t) = \frac{1}{\Gamma(m)} \frac{\partial^{m-1} \varphi(x_1, x_2, \ldots, x_n, t; p)}{\partial p^{m-1}} \bigg|_{p=0}, \]  

(4.2.13)

and

\[ \chi_m = \begin{cases} 0, & m \leq 1, \\ 1 & m > 1. \end{cases} \]  

(4.2.14)

In this case

\[ R_m(\xi_{m-1}, x_1, x_2, \ldots, x_n, t) = D_t^\alpha \iota_m \xi_{m-1}(x_1, x_2, \ldots, x_n, t) + \sum_{i=1}^n t_i \frac{\partial^\alpha \xi_{m-1}(x_1, x_2, \ldots, x_n, t)}{\partial x_i^\alpha} - (1 - \chi_m) G(x_1, x_2, \ldots, x_n, t). \]  

(4.2.15)

In this way, it is easy to obtain \( \xi_m(x_1, x_2, \ldots, x_n, t) \) for \( m \geq 1 \), at \( M \)th order, we have

\[ \xi(x_1, x_2, \ldots, x_n, t) = \sum_{m=0}^M \xi_m(x_1, x_2, \ldots, x_n, t), \]  

(4.2.16)
where $M \to \infty$, we obtain an accurate approximation of the original equation (4.2.1).

**Theorem 4.1 (Convergence Theorem):** If the series (4.2.16) is converging for $M \to \infty$, where $\xi_m (x_1, x_2, ..., x_n, t)$ is obtained by equation (4.2.12) and by using the conditions (4.2.14) and (4.2.15). Then, it must be the exact solution of original discussed partial differential equation (4.2.1).

**Proof:** Let the series (4.2.16) be the convergent series then

$$
\sum_{m=0}^{\infty} \xi_m (x_1, x_2, ..., x_n, t) = \xi_0 (x_1, x_2, ..., x_n, t) + \sum_{m=1}^{\infty} \xi_m (x_1, x_2, ..., x_n, t) = K (x_1, x_2, ..., x_n, t).
$$

(4.2.17)

Now, $\lim_{M \to \infty} \xi_m (x_1, x_2, ..., x_n, t) = 0$. Using definition of (4.2.11) we obtained

$$
\lim_{M \to \infty} \left[ h H (x_1, x_2, ..., x_n, t) \sum_{m=1}^{M} R_m \left( \xi_{m-1}, x_1, x_2, ..., x_n, t \right) \right]
$$

$$
= \lim_{M \to \infty} \left( \sum_{m=1}^{M} S \left[ \xi_m (x_1, x_2, ..., x_n, t) - \chi_m \xi_{m-1} (x_1, x_2, ..., x_n, t) \right] \right)
$$

$$
= \lim_{M \to \infty} \left( \sum_{m=1}^{M} S \left[ \xi_m (x_1, x_2, ..., x_n, t) - \chi_m \xi_{m-1} (x_1, x_2, ..., x_n, t) \right] \right)
$$

$$
= S \left( \lim_{M \to \infty} \sum_{m=1}^{M} \left[ \xi_m (x_1, x_2, ..., x_n, t) - \chi_m \xi_{m-1} (x_1, x_2, ..., x_n, t) \right] \right)
$$

$$
= S \left( \lim_{M \to \infty} \xi_M (x_1, x_2, ..., x_n, t) \right)
$$

$$
= 0.
$$

Since $h \neq 0$, $H (x_1, x_2, ..., x_n, t) \neq 0$, therefore $\sum_{m=1}^{\infty} R_m \left( \xi_{m-1}, x_1, x_2, ..., x_n, t \right) = 0$.

From (4.2.15)
\[
\sum_{m=1}^{\infty} R_m \left( \xi_{m-1}, x_1, x_2, ..., x_n, t \right) = \sum_{m=1}^{\infty} \left( D_t^{\alpha} \xi_{m-1} (x_1, x_2, ..., x_n, t) \right) \\
+ \sum_{i=1}^{n} l_i \frac{\partial^3 \xi_{m-1} (x_1, x_2, ..., x_n, t)}{\partial x_i^3} - (1 - \chi_m) G (x_1, x_2, ..., x_n, t)
\]

\[
\sum_{m=1}^{\infty} R_m \left( \xi_{m-1}, x_1, x_2, ..., x_n, t \right) = \sum_{m=1}^{\infty} D_t^{\alpha} \xi_{m-1} (x_1, x_2, ..., x_n, t) \\
+ \sum_{m=1}^{\infty} \sum_{i=1}^{n} l_i \frac{\partial^3 \xi_{m-1} (x_1, x_2, ..., x_n, t)}{\partial x_i^3} - \sum_{m=1}^{\infty} (1 - \chi_m) G (x_1, x_2, ..., x_n, t)
\]

\[
\sum_{m=1}^{\infty} R_m \left( \xi_{m-1}, x_1, x_2, ..., x_n, t \right) = D_t^{\alpha} \sum_{m=0}^{\infty} \xi_m (x_1, x_2, ..., x_n, t) \\
+ \sum_{i=1}^{n} l_i \frac{\partial^3 \sum_{m=0}^{\infty} \xi_m (x_1, x_2, ..., x_n, t)}{\partial x_i^3} - G (x_1, x_2, ..., x_n, t)
\]

\[
D_t^{\alpha} K (x_1, x_2, ..., x_n, t) + \sum_{i=1}^{n} l_i \frac{\partial^3 K (x_1, x_2, ..., x_n, t)}{\partial x_i^3} - G (x_1, x_2, ..., x_n, t) = 0.
\]

The above equation (4.2.18) shows that, \( K (x_1, x_2, ..., x_n, t) \) satisfies the original problem (4.2.1).

### 4.3 Numerical illustrations

In this section, examples of the time fractional dispersive partial differential equations are considered to authenticate the method discussed in the previous section.

#### 4.3.1 Example

We consider the linear time fractional KDV [125]
\[\xi_t^\alpha (x, t) + 2 \frac{\partial \xi (x, t)}{\partial x} + \frac{\partial^3 \xi (x, t)}{\partial x^3} = 0, \ t > 0, \ 0 < \alpha \leq 1, \quad (4.3.1)\]

subject to the initial condition

\[\xi (x, 0) = \sin x. \quad (4.3.2)\]

The exact solution at \(\alpha = 1\) is given by

\[\xi (x, t) = \sin (x - t). \quad (4.3.3)\]

Applying the Sumudu transform of both sides of Eq. (4.3.1) and after using the definition of Sumudu transform for fractional derivative, we get

\[S [\xi (x, t)] + u^\alpha S \left[2 \frac{\partial \xi (x, t)}{\partial x} + \frac{\partial^3 \xi (x, t)}{\partial x^3}\right] = 0, \ t > 0. \quad (4.3.4)\]

The nonlinear operator is

\[N [\phi (x, t; p)] = S [\phi (x, t; p)] + u^\alpha S \left[2 \frac{\partial \phi (x, t; p)}{\partial x} + \frac{\partial^3 \phi (x, t; p)}{\partial x^3}\right] = 0, \ t > 0, \ 0 \leq p \leq 1, \quad (4.3.5)\]

and thus

\[R_m \left(\overrightarrow{\xi_{m-1}}, x, t\right) = S [\xi_{m-1} (x, t)] + u^\alpha S \left[2 \frac{\partial \xi_{m-1} (x, t)}{\partial x} + \frac{\partial^3 \xi_{m-1} (x, t)}{\partial x^3}\right] = 0, \ t > 0. \quad (4.3.6)\]

The \(m^{th}\)-order deformation equation is given by

\[S [\xi_m (x, t) - \chi_m \xi_{m-1} (x, t)] = hH (x, t) R_m \left(\overrightarrow{\xi_{m-1}}, x, t\right).\]
Applying the inverse Sumudu transform, we have

\[
\xi_m(x,t) = \chi_m \xi_{m-1}(x,t) + S^{-1} \left[ hH(x,t) R_m \left( \xi_{m-1}, x, t \right) \right].
\]  

(4.3.7)

On solving above equation for \( m = 1, 2, \ldots \). For simplicity, we choose \( H(x,t) = 1 \)

\[
\xi_1(x,t) = \frac{t^\alpha h \cos x}{\Gamma(1+\alpha)},
\]

\[
\xi_2(x,t) = \frac{t^\alpha h \cos x}{\Gamma(1+\alpha)} + \frac{t^{2\alpha-1} \alpha h^2 \cos x \Gamma(\alpha)}{\Gamma(2\alpha) \Gamma(1+\alpha)} - \frac{t^{2\alpha} h^2 \sin x}{\Gamma(1+2\alpha)}.
\]

\[
\xi_3(x,t) = \frac{t^\alpha h \cos x}{\Gamma(1+\alpha)} + \frac{t^{2\alpha-1} \alpha h^2 \cos x \Gamma(\alpha)}{\Gamma(2\alpha) \Gamma(1+\alpha)} - \frac{t^{3\alpha-2} \alpha h^3 \cos x \Gamma(\alpha) \Gamma(2\alpha-1)}{\Gamma(3\alpha-1) \Gamma(2\alpha) \Gamma(1+\alpha)} - \frac{t^{3\alpha} h^3 \sin x \Gamma(\alpha)}{\Gamma(1+2\alpha) \Gamma(3\alpha)}
\]

\[
+ \frac{2t^{2\alpha} h^2 \sin x}{\Gamma(1+2\alpha)} - \frac{2t^{3\alpha-1} \alpha h^3 \Gamma(2\alpha) \sin x}{\Gamma(3\alpha) \Gamma(1+2\alpha)} - \frac{2t^\alpha h^2 \sin x}{\Gamma(1+2\alpha)}.
\]

and so on.

Here, we have taken the results upto \( m = 10 \) and rest of the components can be evaluated by iteration formula (4.3.6).

Therefore, the solution of equation (4.3.1) is given by

\[
\xi(x,t) = \xi_0(x,t) + \sum_{m=1}^{\infty} \xi_m(x,t).
\]  

(4.3.8)

91
At $h = -1$ the following approximation is obtained:

$$
\xi(x, t) = \frac{-3t^\alpha \cos x}{\Gamma(1 + \alpha)} + \frac{3t^{2\alpha - 1} \alpha \cos x \Gamma(\alpha)}{\Gamma(2\alpha) \Gamma(1 + \alpha)} + \frac{t^{3\alpha - 2} \alpha \cos x \Gamma(\alpha) \Gamma(2\alpha - 1)}{\Gamma(2\alpha) \Gamma(1 + \alpha) \Gamma(3\alpha - 1)}
- \frac{2t^{3\alpha - 2} \alpha^2 \cos x \Gamma(\alpha) \Gamma(2\alpha - 1)}{\Gamma(2\alpha) \Gamma(1 + \alpha) \Gamma(3\alpha - 1)} + \frac{t^{3\alpha} \cos x}{\Gamma(1 + 3\alpha)} + \sin x + \frac{t^{3\alpha - 1} \alpha \Gamma(\alpha)}{\Gamma(3\alpha) \Gamma(1 + \alpha)} \sin x
- \frac{3t^{2\alpha} \sin x}{\Gamma(1 + 2\alpha)} + \frac{2t^{3\alpha - 1} \alpha \Gamma(2\alpha) \sin x}{\Gamma(3\alpha) \Gamma(1 + 2\alpha)} + \cdots,
$$

(4.3.9)

when $\alpha = 1$, equation (4.3.9) shows the similar results as [246] which is the exact solution of (4.3.1)

$$
\xi(x, t) = \sin x - \frac{1}{2} t^2 \sin x - t \cos x + \frac{1}{6} t^3 \cos x + \cdots
$$

After simplification, we get equation (4.3.3).

Figure 4.1: Plot of $\xi(x, t)$ w.r.t $x$ and $t$ at $\alpha = 0.9$. 

92
Figure 4.2: Plot of $\xi(x, t)$ w.r.t $x$ and $t$ at $\alpha = 0.95$.

Figure 4.3: Plot of $\xi(x, t)$ w.r.t $x$ and $t$ at $\alpha = 1$. 

93
Figures 4.1, 4.2, 4.3 and 4.4 show that the nature of fractional derivative and fluctuation changes from $\alpha = 0.9, 0.95, 1$ and exact solution at $\alpha = 1$.

### 4.3.2 Example

Consider the linear time fractional KDV equation for one dimensional space

\[
\xi^\alpha_t (x, t) + 3 \frac{\partial^3 \xi (x, t)}{\partial x^3} = 0, \quad t > 0, \quad 0 < x < 1, \quad 0 < \alpha \leq 1, \tag{4.3.10}
\]

subject to the initial condition

\[
\xi (x, 0) = \cos x, \quad 0 \leq x \leq 1. \tag{4.3.11}
\]

The exact solution at $\alpha = 1$ is given by

\[
\xi (x, t) = \cos (x + 3t). \tag{4.3.12}
\]
Applying the Sumudu transform of both sides of eq. (4.3.10)

\[ S[\xi(x, t)] + u^\alpha S \left[ 3 \frac{\partial^3 \xi(x, t)}{\partial x^3} \right] = 0, \quad t > 0. \quad (4.3.13) \]

The nonlinear operator is

\[ N[\phi(x, t; p)] = S[\phi(x, t; p)] + 3u^\alpha S \left[ \frac{\partial^3 \phi(x, t; p)}{\partial x^3} \right] = 0, \quad t > 0, \quad 0 \leq p \leq 1, \quad (4.3.14) \]

and thus,

\[ R_m (\vec{\xi}_{m-1}, x, t) = S[\xi_{m-1}(x, t)] + 3u^\alpha S \left[ \frac{\partial^3 \xi_{m-1}(x, t)}{\partial x^3} \right] = 0, \quad t > 0. \quad (4.3.15) \]

The \( m^{th} \)-order deformation equation is given by

\[ S[\xi_m(x, t) - \chi_m \xi_{m-1}(x, t)] = \hbar H(x, t) R_m (\vec{\xi}_{m-1}(x, t)). \]

Applying the inverse Sumudu transform, we get

\[ \xi_m(x, t) = \chi_m \xi_{m-1}(x, t) + S^{-1} \left[ \hbar H(x, t) R_m (\vec{\xi}_{m-1}(x, t)) \right]. \quad (4.3.16) \]

On solving above equation for \( m = 1, 2, \ldots \). For simplicity, we choose \( H(x, t) = 1 \),

\[ \xi_1(x, t) = \frac{3t^\alpha \hbar \text{Sin} x}{\Gamma(1 + \alpha)}, \]

\[ \xi_2(x, t) = -\frac{9t^{2\alpha} \hbar^2 \text{Cos} x}{\Gamma(1 + 2\alpha)} + \frac{3t^\alpha \hbar \text{Sin} x}{\Gamma(1 + \alpha)} + \frac{3t^{2\alpha-1} \hbar^2 \text{Sin} x \Gamma(\alpha)}{\Gamma(1 + \alpha) \Gamma(2\alpha)}. \]
\[ \xi_3(x, t) = -\frac{9t^{3\alpha-1}\alpha h^3 \cos x \Gamma(\alpha)}{(1 + \alpha) \Gamma(3\alpha)} - \frac{18t^{2\alpha} h^2 \cos x}{\Gamma(1 + 2\alpha)} + \frac{18t^{3\alpha-1}\alpha h^3 \cos x \Gamma(2\alpha)}{(1 + 2\alpha) \Gamma(3\alpha)} + \frac{3t^\alpha h \sin x}{\Gamma(1 + \alpha)} + \frac{6t^{2\alpha-1}\alpha h^2 \sin x \Gamma(\alpha)}{(1 + \alpha) \Gamma(2\alpha)} - \frac{3t^{3\alpha-2}\alpha^2 h^3 \sin x \Gamma(\alpha) \Gamma(2\alpha - 1)}{(2\alpha) \Gamma(1 + \alpha) \Gamma(3\alpha - 1)} + \frac{6t^{3\alpha-2}\alpha^2 h^3 \sin x \Gamma(\alpha) \Gamma(2\alpha - 1)}{(2\alpha) \Gamma(1 + \alpha) \Gamma(3\alpha - 1)} - \frac{27t^{3\alpha} h^3 \sin x}{\Gamma(3\alpha + 1)}, \]

and so on.

Here, we consider the results up to \( m = 10 \) and rest of the components can be evaluated by iteration formula (4.3.16).

Therefore the solution of equation (4.3.10) is given by

\[ \xi(x, t) = \xi_0(x, t) + \sum_{m=1}^\infty \xi_m(x, t). \]  

(4.3.17)

At \( h = -1 \) the following approximation is obtained:

\[
\xi(x, t) = \frac{-3t^\alpha \cos x}{\Gamma(1 + \alpha)} + \frac{3t^{2\alpha-1}\alpha \cos x \Gamma(\alpha)}{(2\alpha) \Gamma(3\alpha)} + \frac{t^{3\alpha-2}\alpha \cos x \Gamma(\alpha) \Gamma(2\alpha - 1)}{(2\alpha) \Gamma(1 + \alpha) \Gamma(3\alpha - 1)} - \frac{2t^{3\alpha-2}\alpha^2 \cos x \Gamma(\alpha) \Gamma(2\alpha - 1)}{(2\alpha) \Gamma(1 + \alpha) \Gamma(3\alpha - 1)} + \frac{t^{3\alpha} \cos x}{\Gamma(1 + 3\alpha)} + \sin x + \frac{t^{3\alpha-1}\alpha \Gamma(\alpha) \sin x}{\Gamma(3\alpha) \Gamma(1 + \alpha)} - \frac{3t^{2\alpha} \sin x}{\Gamma(1 + 2\alpha)} + \frac{2t^{3\alpha-1}\alpha \Gamma(2\alpha) \sin x}{\Gamma(3\alpha) \Gamma(1 + 2\alpha)} + \ldots.
\]

(4.3.18)

when \( \alpha = 1 \) equation (4.3.18) shows the similar results as [246] which is the exact solution of (4.3.10).
\[ \xi(x,t) = \sin x - \frac{1}{2} t^2 \sin x - t \cos x + \frac{1}{6} t^3 \cos x + ... \]

After simplification, the equation (4.3.12) is obtained.

Figure 4.5: Plot of \( \xi(x,t) \) w.r.t \( x \) and \( t \) at \( \alpha = 0.9 \).

Figure 4.6: Plot of \( \xi(x,t) \) w.r.t \( x \) and \( t \) at \( \alpha = 0.95 \).
As discussed in previous example figures 4.5, 4.6, 4.7 and 4.8 show that the nature of fractional derivative and fluctuation changes for $\alpha = 0.9$, 0.95, 1 and exact solution at $\alpha = 1$. 

Figure 4.7: Plot of $\xi(x, t)$ w.r.t $x$ and $t$ at $\alpha = 1$.

Figure 4.8: Plot of exact solution of $\xi(x, t)$ w.r.t $x$ and $t$. 

98
4.3.3 Example

Now, we consider the linear time fractional KDV equation for two dimensional space [125]

\[ \xi_\alpha (x, y, t) + 2 \frac{\partial^3 \xi (x, y, t)}{\partial x^3} + \frac{\partial^3 \xi (x, y, t)}{\partial y^3} = 0, \quad t > 0, \quad 0 < \alpha \leq 1, \quad (4.3.19) \]

subject to the initial condition

\[ \xi (x, y, 0) = \text{Cos} (x + y). \quad (4.3.20) \]

The exact solution at \( \alpha = 1 \) is given by

\[ \xi (x, y, t) = \text{Sin} (x + y + 2t). \quad (4.3.21) \]

Applying the Sumudu transform of both sides of Eq. (4.3.19), and after using the definition of Sumudu transform for fractional derivative, we get

\[ S [\xi (x, y, t)] + u^\alpha S \left[ 2 \frac{\partial^3 \xi (x, y, t)}{\partial x^3} + \frac{\partial^3 \xi (x, y, t)}{\partial y^3} \right] = 0, \quad t > 0. \quad (4.3.22) \]

The nonlinear operator is

\[ N [\phi (x, y, t; p)] = S [\phi (x, y, t; p)] + u^\alpha S \left[ 2 \frac{\partial^3 \phi (x, y, t; p)}{\partial x^3} + \frac{\partial^3 \phi (x, y, t; p)}{\partial y^3} \right] = 0, \quad t > 0, \quad 0 \leq p \leq 1, \quad (4.3.23) \]

and thus
\[ R_m \left( \overrightarrow{\xi}_{m-1}, x, y, t \right) = \mathbb{S} \left[ \xi_{m-1}(x, y, t) \right] + u^\alpha \mathbb{S} \left[ 2 \frac{\partial^3 \xi_{m-1}(x, y, t)}{\partial x^3} + \frac{\partial^3 \xi_{m-1}(x, y, t)}{\partial y^3} \right] = 0, \]
\[
t > 0. \quad (4.3.24)
\]

The \( m^{th} \) - order deformation equation is given by

\[
\mathbb{S} \left[ \xi_m(x, y, t) - \chi_m \xi_{m-1}(x, y, t) \right] = \hbar H(x, y, t) R_m \left( \overrightarrow{\xi}_{m-1}(x, y, t) \right).
\]

Applying the inverse Sumudu transform, we have

\[
\xi_m(x, y, t) = \chi_m \xi_{m-1}(x, y, t) + \mathbb{S}^{-1} \left[ \hbar H(x, y, t) R_m \left( \overrightarrow{\xi}_{m-1}(x, y, t) \right) \right]. \quad (4.3.25)
\]

On solving above equation for \( m = 1, 2, \ldots \). For simplicity, we choose \( H(x, t) = 1 \),

\[
\xi_1(x, y, t) = \frac{2t^\alpha \hbar \sin (x + y)}{\Gamma (1 + \alpha)}.
\]

\[
\xi_2(x, y, t) = -\frac{4t^{2\alpha} \hbar^2 \cos (x + y)}{\Gamma (1 + 2\alpha)} + \frac{2t^\alpha \hbar \sin (x + y)}{\Gamma (1 + \alpha)} + \frac{2t^{2\alpha - 1} \hbar^2 \Gamma (\alpha) \sin (x + y)}{\Gamma (1 + 2\alpha) \Gamma (1 + \alpha)},
\]

\[
\xi_3(x, t) = -\frac{4t^{3\alpha - 1} \hbar^3 \cos (x + y)}{\Gamma (1 + \alpha) \Gamma (3\alpha)} - \frac{8t^{2\alpha} \hbar^2 \cos (x + y)}{\Gamma (1 + 2\alpha)} + \frac{2t^\alpha \hbar \sin (x + y)}{\Gamma (1 + \alpha)} - \frac{8t^{3\alpha - 2} \hbar^3 \Gamma (\alpha) \Gamma (2\alpha - 1) \sin (x + y)}{\Gamma (1 + \alpha) \Gamma (3\alpha) \Gamma (3\alpha - 1)}
\]

\[
+ \frac{4t^{2\alpha - 1} \hbar^2 \Gamma (\alpha) \sin (x + y)}{\Gamma (1 + \alpha) \Gamma (2\alpha)} + \frac{2t^{3\alpha - 2} \hbar^3 \Gamma (\alpha) \Gamma (2\alpha - 1) \sin (x + y)}{\Gamma (1 + \alpha) \Gamma (3\alpha) \Gamma (3\alpha - 1)} - \frac{8t^{3\alpha} \hbar^3 \sin (x + y)}{\Gamma (1 + 3\alpha)}.
\]
and so on.

Here, we consider the results upto \( m = 10 \) and rest of the components can be evaluated by iteration formula (4.3.25).

Therefore the solution of equation (4.3.1) is given by

\[
\xi (x, y, t) = \xi_0 (x, y, t) + \sum_{m=1}^{\infty} \xi_m (x, y, t) .
\]  (4.3.26)

At \( h = -1 \), we obtained the following approximation:

\[
\xi (x, y, t) = \cos (x + y) + \frac{4 t^{3\alpha - 1} \alpha \cos (x + y) \Gamma (\alpha)}{\Gamma (3\alpha) \Gamma (1 + \alpha)} - \frac{12 t^{2\alpha} \cos (x + y)}{\Gamma (1 + 2\alpha)}
\]
\[
+ \frac{8 t^{3\alpha - 1} \alpha \cos (x + y) \Gamma (2\alpha)}{\Gamma (3\alpha) \Gamma (1 + 2\alpha)} - \frac{6 t^{\alpha} \sin (x + y)}{\Gamma (1 + \alpha)} + \frac{6 t^{2\alpha - 1} \alpha \Gamma (\alpha) \sin (x + y)}{\Gamma (2\alpha) \Gamma (1 + \alpha)}
\]
\[
+ \frac{2 t^{3\alpha - 2} \alpha \Gamma (\alpha) \Gamma (2\alpha - 1) \sin (x + y)}{\Gamma (2\alpha) \Gamma (1 + \alpha) \Gamma (3\alpha - 1)} + \frac{4 t^{3\alpha - 2} \alpha^2 \Gamma (\alpha) \Gamma (2\alpha - 1) \sin (x + y)}{\Gamma (2\alpha) \Gamma (1 + \alpha) \Gamma (3\alpha - 1)}
\]
\[
+ \frac{8 t^{3\alpha} \sin (x + y)}{\Gamma (3\alpha + 1)}
\]  (4.3.27)

when \( \alpha = 1 \) equation (4.3.27) shows the similar results as [246] which is the exact solution of (4.3.19)

\[
\xi (x, y, t) = \cos (x + y) - 2 t^2 \cos (x + y) - 2 t \sin (x + y) + \frac{4}{3} t^3 \sin (x + y) + ...
\]

After simplification, we can get equation (4.3.21).
4.4 Conclusion

Here, we have applied HASTM for solving fractional third order dispersive partial differential equations. It is shown that HASTM is an effective alternate tool for the evaluation of linear and nonlinear partial differential equations, which is not required for any physical perturbation quantity, and easy to understand.