Chapter 3

Numerical modelling for time fractional nonlinear partial differential equation by homotopy analysis Sumudu transform method

3.1 Introduction

In past few decades, considerable interest is shown by many researcher in the field of fractional calculus specially application of ordinary and partial differential equations of fractional order in the modelling and simulation of problems due to their valuable applications in the field of science and engineering. These applications in interdisciplinary sciences show the importance and necessity of fractional calculus. So far, there has been several fundamental works on the fractional derivative and fractional differential equations, written by Oldham and Spanier
[193], Miller and Ross [178], Podlubny [206], Kilbas, Srivastava and Trujillo [135] and others V. Parthiban and K. Balachandran [203], Samko et al. [217], Caponetto et al. [35], Diethelm [55]. All the above mentioned authors provide systematic understanding of the fractional calculus such as the existence and the uniqueness of solutions, some analytical methods for solving fractional differential equations like Green’s function method, the Mellin transform method, the power series method and etc. Yet presently, no method is available that yields an exact solution for nonlinear fractional partial differential equations. Only approximate solutions can be derived using linearization or perturbation methods. Many mathematical methods such as Adomian decomposition method (ADM) [2, 114, 115, 223, 212], homotopy perturbation method (HPM)[84, 91, 92, 93, 95, 97], variational iteration method (VIM) [3, 26, 94, 96, 98, 102, 103], homotopy analysis method (HAM) [26, 27, 82, 162, 267], Laplace decomposition method (LDM) [131, 134, 240], homotopy perturbation transform method (HPTM) [224], homotopy perturbation Sumudu transform method (HPSTM)[130] and homotopy analysis transform method (HATM)[149, 179, 198] have been proposed to obtain exact and approximate analytical solutions of nonlinear equations. Inspired by all above discussion, we have applied HASTM [166] for the solution of fractional partial differential equation.

The main objective of this chapter is to extend the application of homotopy analysis Sumudu transform method for providing approximate solution of initial value problems of nonlinear partial differential equation of fractional order.
3.2 Solution by homotopy analysis Sumudu transform method

To illustrate the rudimental conception of the HASTM for the fractional partial differential equation, we consider the following fractional partial differential equation:

\[ D_t^{\alpha} U(x,t) + R[x] U(x,t) + N[x] U(x,t) = G(x,t); t > 0, x \in \mathbb{R}, n-1 < \alpha \leq n, \]  
\[ (3.2.1) \]

where \( D_t^{\alpha} = \frac{\partial^{\alpha}}{\partial t^{\alpha}} \), \( R[x] \) is the linear operation in \( x \), \( N[x] \) is the general nonlinear operation in \( x \) and \( G(x,t) \) is a continuous function.

For simplicity, we ignore all initial and boundary conditions, which can be treated in a homogeneous way. Now the methodology consists of applying the Sumudu transform first on both sides of the equation (3.2.1), thus

\[ \mathbb{S} [D_t^{\alpha} u(x,t)] + \mathbb{S} [R[x] U(x,t)] + \mathbb{S} [N[x] U(x,t)] = \mathbb{S} [G(x,t)]; t > 0, x \in \mathbb{R}, n-1 < \alpha \leq n, \]  
\[ (3.2.2) \]

Using the differentiation property of the Sumudu transform

\[ \mathbb{S} \left[ \frac{U(x,t)}{u^\alpha} \right] - \sum_{k=0}^{n-1} \frac{U^{(k)}(0)}{u^{(\alpha-k)}} + \mathbb{S} [R[x] U(x,t)] + \mathbb{S} [N[x] U(x,t)] - \mathbb{S} [G(x,t)] = 0, \]

\[ \mathbb{S} [U(x,t)] - u^\alpha \sum_{k=0}^{n-1} \frac{U^{(k)}(0)}{u^{(\alpha-k)}} + u^\alpha \mathbb{S} [R[x] U(x,t)] + N[x] U(x,t) - G(x,t)] = 0, \]  
\[ (3.2.3) \]

The nonlinear operator is defined as
\[ N [\varphi (x, t; q)] = S [\varphi (x, t; q)] - u^\alpha \sum_{k=0}^{n-1} \frac{U^{(k)} (0)}{q^{(\alpha-k)}} + u^\alpha S \left[ R [x] \varphi (x, t; q) \right] + N [x] \varphi (x, t; q) - G (x, t; q) \] 

(3.2.4)

where \( q \in [0, 1] \) be an embedding parameter and \( \varphi (x, t; q) \) is a real function of \( x, t \) and \( q \). We construct a homotopy as follow:

\[ (1 - q) S [\varphi (x, t; q) - U_0 (x, t)] = \hbar q H (x, t) N [\varphi (x, t; q)] \] 

(3.2.5)

where \( \hbar \) is a nonzero auxiliary parameter and \( H (x, t) \neq 0 \), an auxiliary function, \( U_0 (x, t) \) is an initial guess of \( U (x, t) \) and \( \varphi (x, t; q) \) is an unknown function. It is important that one has great freedom to choose auxiliary parameter in HASTM. Obviously, when \( q = 0 \) and \( q = 1 \), it holds

\[ \varphi (x, t; 0) = U_0 (x, t), \quad \varphi (x, t; 1) = U (x, t) \] 

(3.2.6)

Thus, as \( q \) increases from 0 to 1, the solution varies from initial guess \( U_0 (x, t) \) to the solution \( U (x, t) \). Now, expanding \( \varphi (x, t; q) \) on Taylor’s series with respect to \( q \), we get

\[ \varphi (x, t; q) = U_0 (x, t) + \sum_{m=1}^{\infty} q^m U_m (x, t) , \] 

(3.2.7)

where

\[ U_m (x, t) = \frac{1}{\Gamma (m+1)} \left. \frac{\partial^m \phi (x, t; q)}{\partial q^m} \right|_{q=0} \] 

(3.2.8)

The convergence of the series solution (3.2.7) is controlled by \( \hbar \). If the auxiliary linear operator, the initial guess, the auxiliary parameter \( \hbar \) and the auxiliary
function are properly chosen, the series (3.2.7) converges at \( q = 1 \). Hence, we obtain

\[
U (x, t) = U_0 (x, t) + \sum_{m=1}^{\infty} U_m (x, t) ,
\]

(3.2.9)

which must be one of the solutions of original nonlinear equations. The above expression provides us with a relationship between the initial guess \( U_0 (x, t) \) and the exact solution \( U (x, t) \) by means of the terms \( U_m (x, t) \) \((m = 1, 2, 3, \ldots)\), which are still to be determined.

The vectors are defined as

\[
\vec{U} = \{ U_0 (x, t), U_1 (x, t), U_2 (x, t), \ldots, U_m (x, t) \}. \tag{3.2.10}
\]

Differentiating the zero order deformation Eq. (3.2.5) \( m \) times with respect to embedding parameter \( q \) and then setting \( q = 0 \), and finally dividing them by \( m! \), we obtain the \( m^{th} \) order deformation equation as follows:

\[
\mathcal{S} \{ U_m (x, t) - \chi_m U_{m-1} (x, t) \} = \hbar H (x, t) R_m \left( \vec{U}_{m-1}, x, t \right).
\]

(3.2.11)

Operating the inverse Sumudu transform of both sides, we get

\[
U_m (x, t) = \chi_m U_{m-1} (x, t) + \hbar \mathcal{S}^{-1} \left[ H (x, t) R_m \left( \vec{U}_{m-1}, x, t \right) \right],
\]

(3.2.12)

where

\[
R_m \left( \vec{U}_{m-1}, x, t \right) = \frac{1}{(m - 1)!} \left. \frac{\partial^{m-1} \varphi (x, t; q)}{\partial q^{m-1}} \right|_{q=0},
\]

(3.2.13)

and

57
\[
\chi_m = \begin{cases} 
0, & m \leq 1, \\
1, & m > 1.
\end{cases}
\]

In this way, it is easy to obtain \( U_m (x, t) \) for \( m \geq 1 \), at \( M^{th} \) order,

\[
U (x, t) = \sum_{m=0}^{M} U_m (x, t),
\]  

(3.2.14)

where \( M \to \infty \), we obtain an accurate approximation of the original equation (3.2.1).

### 3.3 Illustrative examples

In this section, the illustration of the technique is done by using three examples. These examples are somewhat artificial in the sense to the exact answer, for the special cases, it is known in advance and the initial and boundary conditions are directly taken from this answer. Nonetheless, such an approach is needed to evaluate the accuracy of the analytical techniques and to examine the effect of varying the order of the time-fractional derivative on the behaviour of the solution.

#### 3.3.1 Example

Consider the nonlinear time–fractional advection partial differential equation [192]:

\[
D_t^\alpha U (x, t) + U (x, t) U (x, t) = x + xt^2, \quad t > 0, \ x \in \mathbb{R}, \ 0 \leq \alpha \leq 1,
\]  

(3.3.1)

subject to the initial condition
\[ U(x, 0) = 0. \] (3.3.2)

Operating the Sumudu transform of both sides of Eq. (3.3.1) and after using the differentiation property of Sumudu transform for fractional derivative, we get

\[ \mathbb{S}[U(x, t)] + u^\alpha \mathbb{S}[U(x, t) U_x(x, t)] = x \left(1 + 2u^2\right), \] (3.3.3)

The nonlinear operator is

\[ N[\varphi(x, t; q)] = \mathbb{S}[\varphi(x, t; q)] + u^\alpha \mathbb{S}\left[\varphi(x, t; q) \partial_x \varphi(x, t; q)\right] - x \left(1 + 2u^2\right), \] (3.3.4)

and thus

\[
R_m \left( \vec{U}_{m-1}(x, t) \right) = \mathbb{S}[U_{m-1}(x, t)] - x \left(1 + 2t^2\right) (1 - \chi_m) \\
+ u^\alpha \mathbb{S}\left[ \sum_{j=0}^{m-1} U_j(x, t) (U_j(x, t))_x \right].
\] (3.3.5)

The \( m^{th} \)-order deformation equation is given by

\[ \mathbb{S}[U_m(x, t) - \chi_m U_{m-1}(x, t)] = \hbar H(x, t) R_m \left( \vec{U}_{m-1}(x, t) \right). \]

Applying the inverse Sumudu transform, we have

\[ U_m(x, t) = \chi_m U_{m-1}(x, t) + \mathbb{S}^{-1} \left[ \hbar H(x, t) R_m \left( \vec{U}_{m-1}(x, t) \right) \right]. \] (3.3.6)

On solving above equation for \( m = 1, 2, \ldots \), we get
\[ U_1(x,t) = -\hbar x \left( \frac{t^\alpha}{\Gamma(\alpha + 1)} + \frac{2t^{\alpha+2}}{\Gamma(\alpha + 3)} \right), \]

\[ U_2(x,t) = -\hbar x \left( \frac{t^\alpha}{\Gamma(\alpha + 1)} + \frac{2t^{\alpha+2}}{\Gamma(\alpha + 3)} \right) - \hbar^2 x \left( \frac{t^\alpha}{\Gamma(\alpha + 1)} + \frac{2t^{\alpha+2}}{\Gamma(\alpha + 3)} \right), \]

\[ U_3(x,t) = -\hbar x \left( \frac{t^\alpha}{\Gamma(\alpha + 1)} + \frac{2t^{\alpha+2}}{\Gamma(\alpha + 3)} \right) - \hbar^2 x \left( \frac{t^\alpha}{\Gamma(\alpha + 1)} + \frac{2t^{\alpha+2}}{\Gamma(\alpha + 3)} \right) + \h \left[ -\hbar x \left( \frac{t^\alpha}{\Gamma(\alpha + 1)} + \frac{2t^{\alpha+2}}{\Gamma(\alpha + 3)} \right) - \hbar^2 x \left( \frac{t^\alpha}{\Gamma(\alpha + 1)} + \frac{2t^{\alpha+2}}{\Gamma(\alpha + 3)} \right) \right] + \h^3 x \left[ \frac{\Gamma(2\alpha + 1)}{\Gamma(\alpha + 1)^2} \frac{t^{3\alpha}}{\Gamma(3\alpha + 1)} + \frac{4\Gamma(2\alpha + 3)}{\Gamma(\alpha + 1)\Gamma(\alpha + 3)\Gamma(3\alpha + 3)} \frac{t^{3\alpha+4}}{\Gamma(\alpha + 3)^2} \right] + \h \left[ -\hbar x \left( \frac{t^\alpha}{\Gamma(\alpha + 1)} + \frac{2t^{\alpha+2}}{\Gamma(\alpha + 3)} \right) \right] - \h^2 x \left( \frac{t^\alpha}{\Gamma(\alpha + 1)} + \frac{2t^{\alpha+2}}{\Gamma(\alpha + 3)} \right) + \h \left[ -\hbar x \left( \frac{t^\alpha}{\Gamma(\alpha + 1)} + \frac{2t^{\alpha+2}}{\Gamma(\alpha + 3)} \right) \right] - \h^2 x \left( \frac{t^\alpha}{\Gamma(\alpha + 1)} + \frac{2t^{\alpha+2}}{\Gamma(\alpha + 3)} \right) ] + \h^3 x \left[ \frac{\Gamma(2\alpha + 1)}{\Gamma(\alpha + 1)^2} \frac{t^{3\alpha}}{\Gamma(3\alpha + 1)} + \frac{4\Gamma(2\alpha + 3)}{\Gamma(\alpha + 1)\Gamma(\alpha + 3)\Gamma(3\alpha + 3)} \frac{t^{3\alpha+4}}{\Gamma(\alpha + 3)^2} \right] + \frac{4\Gamma(2\alpha + 3)}{\Gamma(\alpha + 1)\Gamma(\alpha + 3)\Gamma(3\alpha + 3)} \frac{t^{3\alpha+4}}{\Gamma(\alpha + 3)^2} \frac{\Gamma(3\alpha + 5)}{\Gamma(3\alpha + 3)} \]
\[ U_5(x,t) = -\hbar x \left( \frac{t^\alpha}{\Gamma(\alpha + 1)} + \frac{2t^{\alpha+2}}{\Gamma(\alpha + 3)} \right) - \hbar^2 x \left( \frac{t^\alpha}{\Gamma(\alpha + 1)} + \frac{2t^{\alpha+2}}{\Gamma(\alpha + 3)} \right) \]

\[ + \hbar \left[ -\hbar x \left( \frac{t^\alpha}{\Gamma(\alpha + 1)} + \frac{2t^{\alpha+2}}{\Gamma(\alpha + 3)} \right) - \hbar^2 x \left( \frac{t^\alpha}{\Gamma(\alpha + 1)} + \frac{2t^{\alpha+2}}{\Gamma(\alpha + 3)} \right) \right] \]

\[ + \hbar^3 \left( \frac{\Gamma(2\alpha + 1)}{\Gamma(\alpha + 1)^2 \Gamma(3\alpha + 1)} + \frac{4\Gamma(2\alpha + 3)}{\Gamma(\alpha + 1) \Gamma(\alpha + 3) \Gamma(3\alpha + 3)} \right) \]

\[ + \frac{4\Gamma(2\alpha + 5)}{\Gamma(\alpha + 3)^2 \Gamma(3\alpha + 5)} + \hbar \left[ -\hbar x \left( \frac{t^\alpha}{\Gamma(\alpha + 1)} + \frac{2t^{\alpha+2}}{\Gamma(\alpha + 3)} \right) \right] \]

\[ + \hbar^2 \left( \frac{t^\alpha}{\Gamma(\alpha + 1)} + \frac{2t^{\alpha+2}}{\Gamma(\alpha + 3)} \right) \]

\[ + \hbar^3 \left[ \frac{\Gamma(2\alpha + 1)}{\Gamma(\alpha + 1)^2 \Gamma(3\alpha + 1)} + \frac{4\Gamma(2\alpha + 3)}{\Gamma(\alpha + 1) \Gamma(\alpha + 3) \Gamma(3\alpha + 3)} \right] \]

\[ + \frac{4\Gamma(2\alpha + 5)}{\Gamma(\alpha + 3)^2 \Gamma(3\alpha + 5)} + \hbar^4 \left[ \frac{1}{\Gamma(\alpha + 1)^2 \Gamma(3\alpha + 1)} \right] \]

\[ + \frac{4}{\Gamma(\alpha + 3)^2 \Gamma(3\alpha + 5)} \]

\[ + \frac{4}{\Gamma(\alpha + 1)^2 \Gamma(3\alpha + 1)} \]

\[ + \frac{4}{\Gamma(\alpha + 3)^2 \Gamma(3\alpha + 5)} + \hbar^4 \left[ \frac{1}{\Gamma(\alpha + 1)^2 \Gamma(3\alpha + 1)} \right] \]

\[ + \frac{4}{\Gamma(\alpha + 3)^2 \Gamma(3\alpha + 5)} \]

\[ + \frac{4}{\Gamma(\alpha + 1)^2 \Gamma(3\alpha + 1)} + \hbar^4 \left[ \frac{1}{\Gamma(\alpha + 1)^2 \Gamma(3\alpha + 1)} \right] \]

\[ + \frac{4}{\Gamma(\alpha + 3)^2 \Gamma(3\alpha + 5)} + \hbar^4 \left[ \frac{1}{\Gamma(\alpha + 1)^2 \Gamma(3\alpha + 1)} \right] \]
\[ + \hbar^3 x \left[ \frac{\Gamma(2\alpha + 1) t^{3\alpha}}{\Gamma(\alpha + 1)^2 \Gamma(3\alpha + 1)} + \frac{4 \Gamma(2\alpha + 3)}{\Gamma(\alpha + 1) \Gamma(\alpha + 3) \Gamma(3\alpha + 3)} \frac{t^{3\alpha+2}}{\Gamma(\alpha + 2) \Gamma(3\alpha + 5)} + \frac{4 \Gamma(2\alpha + 5)}{\Gamma(\alpha + 3)^2 \Gamma(3\alpha + 5)} \frac{t^{3\alpha+4}}{\Gamma(3\alpha + 7)} \right] + 2x \left( \hbar^3 \left[ \frac{1}{\Gamma(\alpha + 1)^2 \Gamma(3\alpha + 1)} + \frac{4 \Gamma(\alpha + 1)}{\Gamma(\alpha + 3)^2 \Gamma(3\alpha + 5)} \right] \right) \] 

(3.3.7)

etc.

Proceed by same manner to the rest of components of the iteration can be obtained. Setting the \( \hbar = -1 \), in eq. (3.3.7), the above expressions are exactly the same as given by ADM [192].

![Figure 3.1: \( \hbar \) curve for different values of \( \alpha \).](image)
Figure 3.2: Plot of approximate solution for value $\alpha = 0.5$.

Figure 3.3: Plot of approximate solution for value $\alpha = 0.75$.

Figure 3.4: Plot of approximate solution for value $\alpha = 1$. 
Figure 3.5: Plot of exact solution for value $\alpha = 1$.

Figure 3.6: The behaviour of solution for different values $\alpha$ at $x = 1$, $h = -1$.

Figure 3.7: The behaviour of solution for different values $\alpha$ at $t = 1$, $h = -1$. 

64
Fig. 3.1 shows that the \( h \) values admissible between \(-1.6 \leq h \leq -0.4\) obtained from the fifth order solution \( U(x,t) \) for different fractional Brownian motion \( \alpha = 0.5, 0.75 \) and for standard motion, at \( \alpha = 1 \). Figs. 3.2, 3.3, 3.4, 3.5 show the behaviour of approximate solution of \( U(x,t) \) for different fractional Brownian motions \( \alpha = 0.5, 0.75 \) and at standard motions, i.e. exact solution at \( \alpha = 1 \).

Figures 3.6, 3.7 show the behaviour of approximate solutions at \( t = 1 \) and \( x = 1 \) respectively. It can be seen that \( U(x,t) \) increases very rapidly after point \( t \geq 1 \) and constant in nature for \( t < 1.3 \). Also linear behaviour is observed in different fractional Brownian motion \( \alpha = 0.5, 0.75 \) and for standard motion, at \( \alpha = 1 \).
Table 3.1: Numerical values when $\alpha = 0.5$, 0.75 and 1.0 and comparison with [192]

<table>
<thead>
<tr>
<th>t</th>
<th>x</th>
<th>$\alpha = 0.5$</th>
<th>$\alpha = 0.75$</th>
<th>$\alpha = 1.0$</th>
<th>Exact</th>
<th>[u_0 - u]</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.25</td>
<td>0.12844</td>
<td>0.10375</td>
<td>0.111585</td>
<td>0.078787</td>
<td>0.077933</td>
<td>0.078766</td>
</tr>
<tr>
<td>0.5</td>
<td>0.225688</td>
<td>0.2075</td>
<td>0.22317</td>
<td>0.157574</td>
<td>0.155865</td>
<td>0.157532</td>
</tr>
<tr>
<td>0.75</td>
<td>0.311249</td>
<td>0.31125</td>
<td>0.334755</td>
<td>0.236361</td>
<td>0.233798</td>
<td>0.236299</td>
</tr>
<tr>
<td>1</td>
<td>0.451375</td>
<td>0.415</td>
<td>0.446341</td>
<td>0.315148</td>
<td>0.31173</td>
<td>0.315065</td>
</tr>
<tr>
<td>0.25</td>
<td>0.164004</td>
<td>0.17201</td>
<td>0.155966</td>
<td>0.128941</td>
<td>0.134855</td>
<td>0.128203</td>
</tr>
<tr>
<td>0.5</td>
<td>0.328008</td>
<td>0.34403</td>
<td>0.311932</td>
<td>0.257881</td>
<td>0.26971</td>
<td>0.256406</td>
</tr>
<tr>
<td>0.75</td>
<td>0.492011</td>
<td>0.51604</td>
<td>0.467898</td>
<td>0.386821</td>
<td>0.404565</td>
<td>0.384608</td>
</tr>
<tr>
<td>1</td>
<td>0.656015</td>
<td>0.68805</td>
<td>0.623864</td>
<td>0.515762</td>
<td>0.53942</td>
<td>0.512811</td>
</tr>
<tr>
<td>0.25</td>
<td>0.243862</td>
<td>0.21564</td>
<td>0.250596</td>
<td>0.177238</td>
<td>0.17999</td>
<td>0.171831</td>
</tr>
<tr>
<td>0.5</td>
<td>0.487721</td>
<td>0.43128</td>
<td>0.501189</td>
<td>0.354477</td>
<td>0.35979</td>
<td>0.343663</td>
</tr>
<tr>
<td>0.75</td>
<td>0.731581</td>
<td>0.64692</td>
<td>0.751784</td>
<td>0.531715</td>
<td>0.539969</td>
<td>0.515494</td>
</tr>
<tr>
<td>1</td>
<td>0.975441</td>
<td>0.86257</td>
<td>1.00238</td>
<td>0.7089541</td>
<td>0.7089541</td>
<td>0.687326</td>
</tr>
</tbody>
</table>
3.3.2 Example

Consider the nonlinear time–fractional hyperbolic equation [192]:

\[ D_\alpha^t U(x, t) = \frac{\partial}{\partial x} \left( U(x, t) \frac{\partial U(x, t)}{\partial x} \right), \quad t > 0, \; x \in \mathbb{R}, \; 1 < \alpha \leq 2, \]  

(3.3.8)

subject to the initial condition

\[ U(x, 0) = x^2, \quad U_t(x, 0) = -2x^2. \]  

(3.3.9)

Operating the Sumudu transform of both sides of equation (3.3.8) and after using the differentiation property of Sumudu transform for fractional derivative, we get

\[ \mathcal{S}\left[U(x, t)\right] = u^\alpha \mathcal{S}\left[ \frac{\partial}{\partial x} \left( U(x, t) \frac{\partial U(x, t)}{\partial x} \right) \right], \]

The nonlinear operator is

\[ N[\varphi(x, t; q)] = \mathcal{S}[\varphi(x, t; q)] - u^\alpha \mathcal{S}\left[ \frac{\partial}{\partial x} \left( \varphi(x, t; q) \frac{\partial \varphi(x, t; q)}{\partial x} \right) \right], \]  

(3.3.10)

and thus

\[ R_m(\overrightarrow{U}_{m-1}, x, t) = \mathcal{S}[U_{m-1}(x, t)] - x^2 (1 - 2t)(1 - \chi_m) \]

\[ + u^\alpha \mathcal{S}\left[ \sum_{j=0}^{m-1} \left( U_j(x, t) (U_{m-1-j}(x, t))_{xx} + (U_j(x, t))_x U_{m-1-j}(x, t) \right) \right]. \]  

(3.3.11)

The \( m^{th} \)–order deformation equation is given by

\[ \mathcal{S}[U_m(x, t) - \chi_m U_{m-1}(x, t)] = \hbar H(x, t) R_m(\overrightarrow{U}_{m-1}(x, t)). \]  

67
Applying the inverse Sumudu transform, we have

\[ U_m(x,t) = \chi_m U_{m-1}(x,t) + S^{-1} \left[ \hbar H(x,t) R_m \left( \overrightarrow{U}_{m-1}(x,t) \right) \right]. \tag{3.3.12} \]

On solving above equation for \( m = 1, 2, \ldots \), we get

\[ U_0(x,t) = x^2 (1 - 2t), \]

\[ U_1(x,t) = -6\hbar x^2 \left( \frac{t^\alpha}{\Gamma (\alpha + 1)} + \frac{4t^\alpha}{\Gamma (\alpha + 2)} - \frac{8t^\alpha}{\Gamma (\alpha + 3)} \right), \]

\[ U_2(x,t) = -6\hbar (1 + \hbar) x^2 \left( \frac{t^\alpha}{\Gamma (\alpha + 1)} - \frac{4t^\alpha}{\Gamma (\alpha + 2)} + \frac{8t^\alpha}{\Gamma (\alpha + 3)} \right) \]
\[ + \frac{72t^{2\alpha} x^2 \hbar^2}{\Gamma (2\alpha + 1)} - \frac{288t^{2\alpha + 1} x^2 \hbar^2 \Gamma (\alpha + 2)}{\Gamma (2\alpha + 2) \Gamma (\alpha + 1)} \]
\[ + \frac{576t^{2\alpha + 2} x^2 \hbar^2}{\Gamma (2\alpha + 3)} - \frac{576t^{2\alpha + 2} x^2 \hbar^2 \Gamma (\alpha + 3)}{\Gamma (2\alpha + 2) \Gamma (2\alpha + 3)} - \frac{1152t^{2\alpha + 3} x^2 \hbar^2 \Gamma (\alpha + 4)}{\Gamma (2\alpha + 4) \Gamma (\alpha + 3) \Gamma (2\alpha + 3)} \tag{3.3.13} \]

etc.

It can be shown by the same manner to the rest of components of the iteration can be obtained. Setting the \( \hbar = -1 \), in eq. (3.3.13), the above expressions are exactly the same as given by ADM [192].

68
Figure 3.8: $\hbar$ curve for different values of $\alpha$.

Figure 3.9: Plot of approximate solution for value $\alpha = 0.5$. 
Figure 3.10: Plot of approximate solution for value $\alpha = 0.75$.

Figure 3.11: Plot of approximate solution for value $\alpha = 1$.

Figure 3.12: Plot of exact solution for value $\alpha = 1$.
Fig. 3.8 shows that the $h$ values admissible between $-1.5 \leq h \leq -0.5$ obtained from the fifth order solution $U(x, t)$ for different fractional Brownian motion $\alpha = 0.5, 0.75$ and for standard motion, that is, at $\alpha = 1$. Figures 3.9, 3.10, 3.11, 3.12 show the behaviour of approximate solution of $U(x, t)$ for different fractional Brownian motions $\alpha = 0.5, 0.75$ and at standard motions $\alpha = 1$.

Figures 3.13 and 3.14 show the behaviour of approximate solutions at $t = 1$ and $x = 1$ respectively. It is seen that $U(x, t)$ increases very rapidly after point $t \geq 0$ and constant nature $t < 0$. Also, the quadratic behaviour is observed in different fractional Brownian motion $\alpha = 0.5, 0.75$ and for standard motion, that is, at $\alpha = 1$, around origin.
Table 3.2: Numerical values when $\alpha = 1.5, 1.75$ and 2.0 and comparison with [192]

| $t$  | $x$   | $\alpha = 0.5$ | $\alpha = 0.75$ | $\alpha = 1.0$ | Exact | $|u_0 - u|$ |
|------|-------|----------------|----------------|----------------|-------|-------------|
| 0.2  | 0.25  | 0.059283      | 0.047502       | 0.060225       | 0.0497012 | 0.043403  | 0.048787   | 0.0433951| 0.0434     | 0.043403   | 0.043403| 0.043403   | 2.109E-10  |
|      | 0.5   | 0.237133      | 0.190007       | 0.2409         | 0.194805  | 0.18417   | 0.195146   | 0.17358  | 0.1736     | 0.173611   | 0.173611| 8.436E-10  |
|      | 0.75  | 0.533549      | 0.427517       | 0.542025       | 0.438311  | 0.414383  | 0.439078   | 0.390556 | 0.3906     | 0.390625   | 0.390625| 1.898E-9   |
|      | 1     | 0.948532      | 0.760029       | 0.9636         | 0.77922   | 0.73668   | 0.780584   | 0.694321 | 0.6944     | 0.694444   | 0.694444| 3.337E-9   |
| 0.4  | 0.25  | 0.065412      | 0.041853       | 0.081026       | 0.037742  | 0.037742  | 0.045918   | 0.031567 | 0.031779  | 0.031887   | 0.031888| 4.082E-7   |
|      | 0.5   | 0.261647      | 0.167412       | 0.324099       | 0.174992  | 0.150968  | 0.183674   | 0.126268 | 0.127118  | 0.127549   | 0.127551| 1.632E-6   |
|      | 0.75  | 0.588707      | 0.376676       | 0.729222       | 0.393732  | 0.339679  | 0.413266   | 0.284103 | 0.286015  | 0.286986   | 0.28699 | 3.674E-6   |
|      | 1     | 1.04659       | 0.669647       | 1.29639        | 0.699696  | 0.603873  | 0.734695   | 0.505072 | 0.508471  | 0.508471   | 0.508471| 6.531E-6   |
| 0.6  | 0.25  | 0.063177      | 0.037722       | 0.128961       | 0.381836  | 0.304157  | 0.502626   | 0.022005 | 0.023665  | 0.02449    | 0.024414| 2.417E-5   |
|      | 0.5   | 0.25271       | 0.150888       | 0.515844       | 0.152735  | 0.125829  | 0.20105    | 0.088018 | 0.09466   | 0.09756    | 0.097656| 9.671E-5   |
|      | 0.75  | 0.568598      | 0.339499       | 1.16065        | 0.343653  | 0.283114  | 0.452362   | 0.19804  | 0.212984  | 0.219509   | 0.219727| 2.176E-4   |
|      | 1     | 1.01084       | 0.603553       | 2.06337        | 0.610938  | 0.503314  | 0.804199   | 0.352071 | 0.378638  | 0.390238   | 0.390625| 3.868E-4   |
3.3.3 Example

Consider the nonlinear time–fractional Fisher’s equation [192]:

\[ D^\alpha_t U(x, t) = U_{xx}(x, t) + 6U(x, t)(1 - U(x, t)), \quad t > 0, \quad x \in \mathbb{R}, \quad 0 < \alpha \leq 1, \] (3.3.14)

subject to the initial condition

\[ U(x, 0) = \frac{1}{(1 + e^x)^2}, \] (3.3.15)

Operating the Sumudu transform of both sides of Eq. (3.3.14) and after using the differentiation property of Sumudu transform for fractional derivative, we get

\[ \mathcal{S}[U(x, t)] - u^\alpha \mathcal{S}[U_{xx}(x, t) + 6U(x, t) - 6(U(x, t))^2] = 0, \]

The nonlinear operator is

\[ N[\varphi(x, t; q)] = \mathcal{S}[\varphi(x, t; q)] - u^\alpha \mathcal{S}[\varphi_{xx}(x, t; q) + 6\varphi(x, t; q) - 6(\varphi(x, t; q))^2], \] (3.3.16)

and thus

\[ R_m \left( \overline{U}_{m-1}, x, t \right) = \mathcal{S}[U_{m-1}(x, t)] - u^\alpha \mathcal{S}[U_{m-1}(x, t)]_{xx} + 6U_{m-1}(x, t) - 6 \sum_{j=0}^{m-1} U_j(x, t) U_{m-1-j}(x, t), \] (3.3.17)

The \( m^{th} \)– order deformation equation is given by
$S \left[ U_m(x,t) - \chi_m U_{m-1}(x,t) \right] = \hbar H(x,t) R_m \left( \mathcal{U}_{m-1}(x,t) \right)$.

Applying the inverse Sumudu transform, we have

$$U_m(x,t) = \chi_m U_{m-1}(x,t) + S^{-1} \left[ \hbar H(x,t) R_m \left( \mathcal{U}_{m-1}(x,t) \right) \right].$$  (3.3.18)

On solving above equation for $m = 1, 2, \ldots$, we get

$$U_0(x,t) = \frac{1}{(1 + e^x)^2},$$

$$U_1(x,t) = \frac{1}{(1 + e^x)^2} + \hbar \left( \frac{1}{(1 + e^x)^2} - \frac{t^\alpha}{\Gamma(\alpha + 1)} + \frac{10}{(1 + e^x)^3} \right),$$

$$U_2(x,t) = \frac{1}{(1 + e^x)^2} + \hbar \left( \frac{1}{(1 + e^x)^2} - \frac{t^\alpha}{\Gamma(\alpha + 1)} + \frac{10}{(1 + e^x)^3} \right)$$

$$+ \hbar \left( \frac{1}{(1 + e^x)^2} + \hbar \left( \frac{1}{(1 + e^x)^2} - \frac{t^\alpha}{\Gamma(\alpha + 1)} + \frac{10}{(1 + e^x)^3} \right) \right)$$

$$- \frac{6\hbar e^{2x} t^\alpha}{(1 + e^x)^4 \Gamma(\alpha + 1)} + \frac{2\hbar e^{2x} t^\alpha}{(1 + e^x)^3 \Gamma(\alpha + 1)} - \frac{6\hbar^2 e^{2x} t^\alpha}{(1 + e^x)^4 \Gamma(\alpha + 1)}$$

$$- \frac{30\hbar^2 e^{2x} t^\alpha}{(1 + e^x)^4 \Gamma(\alpha + 1)} - \frac{12\hbar t^\alpha}{(1 + e^x)^3 \Gamma(\alpha + 1)} - \frac{6\hbar^2 t^\alpha}{(1 + e^x)^4 \Gamma(\alpha + 1)}$$

$$- \frac{12\hbar t^\alpha}{(1 + e^x)^3 \Gamma(\alpha + 1)} - \frac{12\hbar^2 t^\alpha}{(1 + e^x)^4 \Gamma(\alpha + 1)} - \frac{120\hbar^3 t^2 \alpha}{(1 + e^x)^5 \Gamma(2\alpha + 1)}.$$  (3.3.19)
etc.

Proceed by same manner to the rest of components of the iteration can be obtained. Setting the $\hbar = -1$, in eq. (3.3.19) the above expressions are exactly the same as given by ADM [42].

Figure 3.15: $\hbar$ curve for different values of $\alpha$.

Figure 3.16: Plot of approximate solution for value $\alpha = 0.5$.  

75
Figure 3.17: Plot of approximate solution for value $\alpha = 0.75$.

Figure 3.18: Plot of approximate solution for value $\alpha = 1$.

Figure 3.19: Plot of exact solution for value $\alpha = 1$. 
Fig. 3.20: The behaviour of solution for different values $\alpha$ at $x = 1$, $\hbar = -1$.

Figure 3.21: The behaviour of solution for different values $\alpha$ at $t = 1$, $\hbar = -1$.

Fig. 3.15 shows that the $\hbar$ values admissible between $-1 \leq \hbar \leq 0$ obtained from the fifth order solution $U(x, t)$ for different fractional Brownian motion $\alpha = 0.5, 0.75$ and for standard motion, at $\alpha = 1$. Figs. 3.16, 3.17, 3.18, 3.19 show the behaviour of approximate solution of $U(x, t)$ for different fractional Brownian motions $\alpha = 0.5, 0.75$ and at standard motions $\alpha = 1$.

Figs. 3.20 and 3.21 show the behaviour of approximate solutions at $t = 1$ and $x = 1$ respectively.
| x  | 0.1 | 0.5 | 0.75 | 1.0 | | |
|---|---|---|---|---|---|
| 0.25 | 0.946129 | 0.482361 | 0.483450 | 0.488195 | 0.412450 | 0.458618 | 0.317948 | 0.315940 | 0.316018 | 0.316042 | 2.40905E-05 |
| 0.5 | 0.843908 | 0.394446 | 0.356433 | 0.405740 | 0.334514 | 0.250500 | 0.249926 | 0.249982 | 0.250000 | 1.77145E-05 |
| 0.75 | 0.715013 | 0.311106 | 0.367574 | 0.324457 | 0.262103 | 0.325749 | 0.190964 | 0.191606 | 0.191683 | 0.191689 | 6.11468E-06 |
| 1 | 0.576466 | 0.236710 | 0.490698 | 0.249683 | 0.198407 | 0.265455 | 0.140979 | 0.142411 | 0.142541 | 0.142537 | 3.83664E-06 |
| 0.2 | 1.475320 | 0.746994 | -0.326863 | 0.791250 | 0.617790 | 0.581424 | 0.481199 | 0.459320 | 0.459795 | 0.461284 | 1.48902E-03 |
| 0.5 | 1.359830 | 0.653476 | -1.131290 | 0.690142 | 0.536231 | 0.519219 | 0.396941 | 0.386450 | 0.386202 | 0.387456 | 1.25324E-03 |
| 0.75 | 1.180980 | 0.548977 | -0.309751 | 0.574404 | 0.448264 | 0.483538 | 0.315266 | 0.315433 | 0.316042 | 6.09277E-04 |
| 1 | 0.970076 | 0.441936 | -0.309751 | 0.456647 | 0.359905 | 0.461939 | 0.241175 | 0.249092 | 0.250066 | 0.250000 | 6.55487E-05 |
| 0.3 | 1.967450 | 0.935741 | -2.047010 | 1.124230 | 0.774999 | 0.445118 | 0.681440 | 0.591179 | 0.588679 | 0.604150 | 1.55156E-02 |
| 0.5 | 1.845231 | 0.878473 | -4.603020 | 1.009480 | 0.720112 | 0.322053 | 0.581861 | 0.527635 | 0.519763 | 0.534447 | 1.46838E-02 |
| 0.75 | 1.622910 | 0.788974 | -4.878570 | 0.859509 | 0.643697 | 0.355934 | 0.475833 | 0.459719 | 0.452525 | 0.461284 | 8.75903E-03 |
| 1 | 1.345510 | 0.673844 | -2.902450 | 0.695479 | 0.372917 | 0.495115 | 0.372917 | 0.387025 | 0.386067 | 0.387456 | 1.38825E-03 |
3.4 Conclusion

The new modification of homotopy analysis method is a powerful tool to search the solution of various linear and nonlinear problems arising in science and engineering. The main aim of this chapter is to provide the approximate analytic solution of the time-fractional partial differential equation by using the HASTM. The proposed method is very efficient and easily computable. Three examples were investigated to demonstrate the ease and versatility of the new approach.