Chapter 2

Homotopy analysis Sumudu transform method for time-fractional fourth order partial differential equations with variable coefficients

2.1 Introduction

An integral transform, referred to as Sumudu transform, was introduced by Watugala [243] to facilitate the process of solving differential and integral equations in the time domain, and for the utilization in sundry applications of system engineering and applied physics. Although the mathematical properties of the Sumudu transform have been explained by various authors [16, 66, 124, 136, 138, 139, 251, 252]. No systematic derivation of the Sumudu transform is available in the open literature.
Fractional calculus is a field of applied mathematics that deals with derivatives and integrals of arbitrary orders. During the last decade, fractional calculus has found applications in numerous ostensibly diverse fields of science and engineering. Fractional differential equations are increasingly used to model quandaries in fluid mechanics, acoustics, biology, electromagnetism, diffusion, signal processing, and many other physical processes.

In various fields of science and engineering, it is consequential to obtain exact or numerical solution of the nonlinear partial differential equations. Probing of exact and numerical solution of nonlinear equations in science and engineering is still quite problematic, that’s needed incipient methods for finding the exact and approximate solutions. Sundry potent mathematical methods such as Adomian decomposition method (ADM) [2, 4, 114, 115, 212, 223], homotopy perturbation method (HPM) [84, 91, 92, 93, 95, 97], homotopy analysis method (HAM) [26, 27, 82, 162, 267], variational iteration method (VIM) [3, 26, 98, 96, 94, 102, 103], Laplace decomposition method (LDM)[131, 134, 240], homotopy perturbation transform method (HPTM)[130, 150], homotopy perturbation Sumudu transform method (HPSTM)[224] and homotopy analysis transform method (HATM)[144, 149, 179] have been proposed to obtain exact and approximate analytical solutions of nonlinear equations.

Inspired by all these perpetual research, the researcher proposes HASTM for the solution of fourth order differential equations with variable coefficient.

2.2 Formulation of fourth order time fractional parabolic partial differential equations:

The fourth-order time fractional parabolic partial differential equation with variable coefficients of the following form is considered
\[ \frac{\partial^\alpha U(x, y, z, t)}{\partial t^\alpha} + \mu(x, y, z) \frac{\partial^4 U(x, y, z, t)}{\partial x^4} + \lambda(x, y, z) \frac{\partial^4 U(x, y, z, t)}{\partial y^4} + \eta(x, y, z) \frac{\partial^4 U(x, y, z, t)}{\partial z^4} = g(x, y, z, t), \quad a < x, y, z < b, \ t > 0, \ 1 < \alpha \leq 2. \]  

(2.2.1)

where \( \mu(x, y, z, t) \), \( \lambda(x, y, z, t) \) and \( \eta(x, y, z, t) \) are positive.

Subject to the initial conditions:

\[ U(x, y, z, 0) = f_0(x, y, z), \quad \frac{\partial U(x, y, z, 0)}{\partial t} = f_1(x, y, z) \]  

(2.2.2)

and the boundary conditions:

\[ U(a, y, z, t) = g_0(y, z, t), \quad U(b, y, z, t) = g_1(y, z, t), \]
\[ U(x, a, z, t) = h_0(y, z, t), \quad U(x, b, z, t) = h_1(y, z, t), \]
\[ \frac{\partial^2 U(a, y, z, t)}{\partial x^2} = \bar{g}_0(y, z, t), \quad \frac{\partial^2 U(b, y, z, t)}{\partial y^2} = \bar{g}_1(y, z, t), \]
\[ \frac{\partial^2 U(x, a, z, t)}{\partial y^2} = \bar{h}_0(x, z, t), \quad \frac{\partial^2 U(b, y, z, t)}{\partial u^2} = \bar{h}_1(x, z, t), \]
\[ \frac{\partial^2 U(x, y, a, t)}{\partial z^2} = \bar{k}_0(x, z, t), \quad \frac{\partial^2 U(x, y, b, t)}{\partial u^2} = \bar{k}_1(x, z, t). \]  

(2.2.3)

where the functions \( f_i, g_i, k_i, h_i, \bar{g}_i, \bar{h}_i, \bar{k}_i, \ i = 0, 1 \) are continuous. Operating the Sumudu transform of both sides in Eq. (2.2.1) and after using the differentiation property of Sumudu transform for fractional derivative, we get

\[ \mathbb{S} \left[ D_t^\alpha U(x, y, z, t) \right] + \mathbb{S} \left[ \mu(x, y, z) \frac{\partial^4 U(x, y, z, t)}{\partial x^4} + \lambda(x, y, z) \frac{\partial^4 U(x, y, z, t)}{\partial y^4} + \eta(x, y, z) \frac{\partial^4 U(x, y, z, t)}{\partial z^4} - g(x, y, z, t) \right] = 0, \]  

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\[ \begin{align*}
\mathbb{S} \left[ \frac{U(x, y, z, t)}{u^\alpha} \right] & - \sum_{k=0}^{n-1} \frac{U^{(k)}(0)}{u^{(\alpha-k)}} + \mathbb{S} \left[ \mu(x, y, z) \frac{\partial^4 U(x, y, z, t)}{\partial x^4} \right. \\
& + \lambda(x, y, z) \frac{\partial^4 U(x, y, z, t)}{\partial y^4} + \eta(x, y, z) \frac{\partial^4 U(x, y, z, t)}{\partial z^4} - g(x, y, z, t) \left. \right] = 0,
\end{align*} \]

Now, the nonlinear operator for fractional partial differential equation is discussed as follows:

\[ \begin{align*}
N[\varphi(x, y, z, t; q)] &= \mathbb{S}[\varphi(x, t; q)] - \sum_{k=0}^{n-1} \frac{\varphi^{(k)}(0)}{u^{(\alpha-k)}} + u^\alpha \mathbb{S} \left[ \mu(x, y, z) \frac{\partial^4 \varphi(x, y, z, t; q)}{\partial x^4} \right. \\
& + \lambda(x, y, z) \frac{\partial^4 \varphi(x, y, z, t; q)}{\partial y^4} + \eta(x, y, z) \frac{\partial^4 \varphi(x, y, z, t; q)}{\partial z^4} - g(x, y, z, t) \left. \right],
\end{align*} \]

where \( q \in [0, 1] \) be an embedding parameter and \( \varphi(x, y, z, t; q) \) is a real function of \( x, y, z, t \) and \( q \). We construct a homotopy as follow:

\[ \begin{align*}
(1 - q) \mathbb{S}[\varphi(x, y, z, t; q) - U_0(x, y, z, t)] &= h q H(x, y, z, t) N[\varphi(x, y, z, t)],
\end{align*} \]

where \( h \) is a nonzero auxiliary parameter and \( H(x, y, z, t) \neq 0 \) an auxiliary function, \( U_0(x, y, z, t) \) is an initial guess of \( U(x, y, z, t) \) and \( \varphi(x, y, z, t; q) \) is an unknown function. It is important that one has great freedom to choose auxiliary parameter in HASTM. Obviously, when \( q = 0 \) and \( q = 1 \) it holds

\[ \begin{align*}
\varphi(x, y, z, t; 0) &= U_0(x, y, z, t), \\
\varphi(x, y, z, t; 1) &= U(x, y, z, t)
\end{align*} \]

Thus, as \( q \) increases from 0 to 1, the solution varies from initial guess \( U_0(x, y, z, t) \) to the solution \( U(x, y, z, t) \). Now, expanding \( \varphi(x, y, z, t; q) \) on Taylor’s series with...
respect to \(q\),

\[
\varphi (x, y, z, t; q) = U_0 (x, y, z, t) + \sum_{m=1}^{\infty} q^m U_m (x, y, z, t) \tag{2.2.8}
\]

where

\[
U_m (x, y, z, t) = \frac{1}{m!} \frac{\partial^m \varphi (x, y, z, t; q)}{\partial q^m} \bigg|_{q=0} \tag{2.2.9}
\]

The convergence of the series solution, Eq. (2.2.8) is controlled by \(\hbar\). If the auxiliary linear operator, the initial guess, the auxiliary parameter \(\hbar\) and the auxiliary function are properly chosen, the series (2.2.8) converges at \(q = 1\). Hence,

\[
U (x, y, z, t) = U_0 (x, y, z, t) + \sum_{m=1}^{\infty} U_m (x, y, z, t) \tag{2.2.10}
\]

which must be one of the solutions of original nonlinear equations. The above expression provides us with a relationship between the initial guess \(U_0 (x, y, z, t)\) and the exact solution \(U (x, y, z, t)\) by means of the terms \(U_m (x, y, z, t)\) \((m = 1, 2, 3, ... )\), which are still to be determined. Define the vectors

\[
\vec{U} = \{U_0 (x, y, z, t), U_1 (x, y, z, t), U_2 (x, y, z, t), ..., U_m (x, y, z, t)\}. \tag{2.2.11}
\]

Differentiating the zero order deformation Eq. (2.2.6) \(m\) times with respect to embedding parameter \(q\) and then setting \(q = 0\), and finally dividing them by \(m!\), the \(m^{th}\) order deformation equation is obtained which is

\[
S [U_m (x, y, z, t) - \chi_m U_{m-1} (x, y, z, t)] = \hbar H (x, y, z, t) R_m \left( \vec{U}_{m-1}, x, y, z, t \right). \tag{2.2.12}
\]

Operating the inverse Sumudu transform of both sides,
\[ U_m(x, y, z, t) = \chi_mU_{m-1}(x, y, z, t) + h^{\delta-1} \left[ H(x, y, z, t) R_m \left( \vec{U}_{m-1}, x, y, z, t \right) \right], \]  

(2.2.13)

where

\[ R_m \left( \vec{U}_{m-1}, x, y, z, t \right) = \frac{1}{(m-1)!} \left. \partial^{m-1} \varphi (x, y, z, t; q) \right|_{q=0} \]  

(2.2.14)

and

\[ \chi_m = \begin{cases} 
0, & m \leq 1, \\
1, & m > 1. 
\end{cases} \]  

(2.2.15)

In the fourth–order time fractional parabolic partial differential equation, the formulation of \( R_m \left( \vec{U}_{m-1}, x, y, z, t \right) \) for the iterative solution of equation Eq. (2.2.1) is as follows:

\[ R_m \left( \vec{U}_{m-1}, x, y, z, t \right) = \frac{\partial^\alpha U(x, y, z, t)}{\partial t^\alpha} + \mu(x, y, z) \frac{\partial^4 U(x, y, z, t)}{\partial x^4} + \lambda(x, y, z) \frac{\partial^4 U(x, y, z, t)}{\partial y^4} + \eta(x, y, z) \frac{\partial^4 U(x, y, z, t)}{\partial z^4} - (1 - \chi_m)g(x, y, z, t). \]  

(2.2.16)

In this way, it is easy to obtain \( U_m(x, y, z, t) \) for \( m \geq 1 \), at \( N^{th} \) order, we have

\[ U(x, y, z, t) = \sum_{m=0}^{N} U_m(x, y, z, t), \]  

(2.2.17)

which is an accurate approximation of the original equation (2.2.1).
2.3 Numerical examples

In this section, the application of the HASTM is developed to solve one and two dimensional initial boundary value quandaries with variable coefficients. The methods may additionally be applicable for higher dimensional spaces. Numerical results reveal that the HASTM is facile to implement and reduce the computational work to a tangible level while still maintaining a higher calibre of precision.

2.3.1 Example

Consider the following problem of one–dimensional time–fractional fourth–order PDE [128]:

\[ D_t^\alpha U(x, t) + \left( \frac{1}{x} + \frac{x^4}{120} \right) \frac{\partial^4 U(x, t)}{\partial x^4} = 0, \quad \frac{1}{2} < x < 1; \quad t > 0, \quad 1 < \alpha \leq 2, \]

(2.3.1)

subject to the initial and boundary conditions

\[
\begin{align*}
U(x, 0) &= 0, \quad \frac{\partial U(x, t)}{\partial t} = 1 + \frac{x^5}{120}; \\
U\left(\frac{1}{2}, t\right) &= \left(1 + \frac{0.5^5}{120}\right) \sin(t, \alpha); \\
\frac{\partial^2 U}{\partial x^2}\left(\frac{1}{2}, t\right) &= \frac{1}{6} \frac{1}{2^\alpha} \sin(t, \alpha); \\
U(1, t) &= \frac{121}{120} \sin(t, \alpha); \\
\frac{\partial^2 U}{\partial x^2}(1, t) &= \frac{1}{6} \sin(t, \alpha);
\end{align*}
\]

(2.3.2)

where the function \( \sin(t, \alpha) \) is defined as

\[
\sin(t, \alpha) = \sum_{i=0}^{\infty} \frac{(-1)^i t^{i\alpha+1}}{\Gamma(i\alpha + 2)}. \quad (2.3.3)
\]
The initial condition is initiated as

$$U_0 (x, t) = U (0, t) + tU_t (0, t) = \left( 1 + \frac{x^5}{120} \right) t.$$ \hfill (2.3.4)

Operating the Sumudu transform of both sides of Eq. (2.3.4) and after using the differentiation property of Sumudu transform for fractional derivative, we get

$$S [D_t^\alpha U (x, t)] + S \left[ \left( \frac{1}{x} + \frac{x^4}{120} \right) \frac{\partial^4 U (x, t)}{\partial x^4} \right] = 0,$$ \hfill (2.3.5)

or

$$S [U (x, t)] + u^\alpha S \left[ \left( \frac{1}{x} + \frac{x^4}{120} \right) \frac{\partial^4 U (x, t)}{\partial x^4} \right] = 0.$$ \hfill (2.3.6)

The nonlinear operator is

$$N [\varphi (x, t; q)] = S [\varphi (x, t; q)] + u^\alpha S \left[ \left( \frac{1}{x} + \frac{x^4}{120} \right) \frac{\partial^4 \varphi (x, t; q)}{\partial x^4} \right]$$ \hfill (2.3.7)

and thus,

$$R_m \left( \vec{U}_{m-1}, x, t \right) = S [U_{m-1} (x, t)] + u^\alpha S \left[ \left( \frac{1}{x} + \frac{x^4}{120} \right) \frac{\partial^4 U_{m-1} (x, t)}{\partial x^4} \right].$$ \hfill (2.3.8)

The $m^{th}$- order deformation equation is given by

$$S [U_m (x, t) - \chi_m U_{m-1} (x, t)] = \hbar H (x, t) R_m \left( \vec{U}_{m-1} \right).$$

Applying the inverse Sumudu transform,

$$U_m (x, t) = \chi_m U_{m-1} (x, t) + S^{-1} \left[ \hbar H (x, t) R_m \left( \vec{U}_{m-1} \right) \right].$$ \hfill (2.3.9)
On solving above equation for \( m = 1, 2, \ldots \),

\[
U_1(x,t) = h \left(1 + \frac{x^5}{120}\right) \frac{t^{\alpha+1}}{\Gamma(\alpha + 2)},
\]

(2.3.10)

\[
U_2(x,t) = h \left(1 + \frac{x^5}{120}\right) \frac{t^{\alpha+1}}{\Gamma(\alpha + 2)} + h^2 \left(1 + \frac{x^5}{120}\right) \left[\frac{t^{\alpha+1}}{\Gamma(\alpha + 2)} + \frac{t^{2\alpha+1}}{\Gamma(2\alpha + 2)}\right],
\]

(2.3.11)

\[
U_3(x,t) = h \left(1 + \frac{x^5}{120}\right) \frac{t^{\alpha+1}}{\Gamma(\alpha + 2)} + 2h^2 \left(1 + \frac{x^5}{120}\right) \left[\frac{t^{\alpha+1}}{\Gamma(\alpha + 2)} + \frac{t^{2\alpha+1}}{\Gamma(2\alpha + 2)}\right] + 2h^3 \left[\frac{t^{2\alpha+1}}{\Gamma(2\alpha + 2)} + \frac{t^{3\alpha+1}}{\Gamma(3\alpha + 2)}\right],
\]

(2.3.12)

and so on. In the same manner the rest of the components of the series \( m \geq 4 \) can be obtained.

Finally, the solution of (2.3.1) is given as

\[
U(x,t) = U_0(x,t) + \sum_{m=1}^{\infty} U_m(x,t).
\]

(2.3.13)

As pointed out by Liao [162], the accuracy and convergence of the HAM series solution depends on the careful selection of the auxiliary parameter \( h \). Here we choose \( h = -1 \), then

\[
U(x,t) = \left(t - \frac{t^{\alpha+1}}{\Gamma(\alpha + 2)} + \frac{t^{2\alpha+1}}{\Gamma(2\alpha + 2)} - \frac{t^{3\alpha+1}}{\Gamma(3\alpha + 2)} + \ldots + (-1)^n \frac{t^{n\alpha+1}}{\Gamma(n\alpha + 2)} - \ldots\right).
\]

(2.3.14)

For \( h = -1 \), the above expression is exactly the same as those given by the Decomposition method by Khan [128].
Figure 2.1: Plot of $U(x, t)$ w.r.t. control parameter $h$ at $\alpha = 1.5, 1.75, 2.0$

Figure 2.2: Plot of $U(x, t)$ w.r.t. $x$ and $t$ at $\alpha = 1.5$. 
Figure 2.3: Plot of $U(x, t)$ w.r.t. $x$ and $t$ at $\alpha = 1.75$.

Figure 2.4: Plot of $U(x, t)$ w.r.t. $x$ and $t$ at $\alpha = 2.0$. 
Figure 2.5: Plot of exact solution of $U(x,t)$ w.r.t. $x$ and $t$.

Fig. 2.1 Shows that the curve between approximate solution $U(x,t)$ and convergence control parameter $\hbar$ for the different values of fractional order $\alpha$ viz. $\alpha = 1.5, 1.75, 2.0$. The convergence lies between the range $-1 \leq \hbar < 0$ at $t = 0.5$ and $x = 1$ for example 2.3.1.

Fig. 2.2 shows that the three dimensional plot between $U(x,t)$ for independent variables $x$ and $t$ at $\alpha = 1.5$. Similarly, Fig. 2.3 and Fig. 2.4 shows the corresponding slight changes for different fractional Brownian motions of $\alpha = 1.75, 2.0$ respectively.

Fig. 2.5 is plotted for the exact solution of $U(x,t)$ which is equal to the Fig. 2.4. However, most of the results given by the Adomian decomposition method, Laplace decomposition method, homotopy perturbation method and homotopy perturbation transform method converge to the corresponding numerical solutions in small region. But, different from these methods, the new homotopy analysis Sumudu transform method provides us with a simple way to adjust and control the convergence region of solution series by choosing a proper value for the auxiliary parameter $\hbar$. So, the valid region for $\hbar$, where the series converge is
the horizontal segment of each \( h \) curve. When we choose \( \alpha = 2 \), then clearly, one can conclude that the obtained solution \( \sum_{m=0}^{\infty} U_m(x,t) \) converges to the exact solution \( U(x,t) = \sin t \left( 1 + \frac{x^2}{120} \right) \) which is obtained by Wazwaz [249, 250].

### 2.3.2 Example

Consider the following problems of one-dimensional time-fractional fourth-order PDE [128]

\[
D_t^\alpha U(x,t) + \left( \frac{x}{\sin x} - 1 \right) \frac{\partial^4 U(x,t)}{\partial x^4} = 0; \quad 0 < x < 1, \quad t > 0, \quad 1 < \alpha \leq 2, \quad (2.3.15)
\]

subject to the initial and boundary conditions

\[
\begin{aligned}
U(x,0) &= x - \sin x; \\
\frac{\partial U}{\partial t}(x,0) &= -x + \sin x; \\
U(0,t) &= 0; \\
\frac{\partial^2 U}{\partial x^2}(0,t) &= 0; \\
U(1,t) &= \text{Exp}(t, \alpha) (1 - \sin 1); \\
\frac{\partial^2 U}{\partial x^2}(1,t) &= \text{Exp}(t, \alpha) \sin 1;
\end{aligned}
\]

(2.3.16)

where the function \( \text{Exp}(t, \alpha) = (-1)^i \frac{i^{\alpha/2}}{\Gamma(\frac{\alpha}{2}+1)} \).

The initial condition is \( U_0(x,t) = U(0,t) + tU_t(0,t) = (1 - t) \left( x - \sin x \right) \).

Operating the Sumudu transform on both sides of (2.3.15) and after using the differentiation property of Sumudu transform for fractional derivative, we get

\[
\mathbb{S}[D_t^\alpha U(x,t)] + \mathbb{S} \left[ \left( \frac{x}{\sin x} - 1 \right) \frac{\partial^4 U(x,t)}{\partial x^4} \right] = 0. \quad (2.3.17)
\]
Applying the initial and boundary conditions of (2.3.15) from (2.3.16) to (2.3.17), we obtain

\[ S[U(x, t)] - (1 - \chi_m)(1 - u)(x - \sin x) + u^\alpha S \left[ \left( \frac{x}{\sin x} - 1 \right) \frac{\partial^4 U(x, t)}{\partial x^4} \right] = 0. \]

The nonlinear operator is

\[ N[\varphi(x, t; q)] = S[\varphi(x, t; q)] - (1 - u)(x - \sin x) + u^\alpha S \left[ \left( \frac{x}{\sin x} - 1 \right) \frac{\partial^4 \varphi(x, t; q)}{\partial x^4} \right] \]  

and thus

\[ R_m[\overrightarrow{U}_{m-1}, x, t] = S[U_{m-1}(x, t)] - (1 - u)(x - \sin x) + u^\alpha S \left[ \left( \frac{x}{\sin x} - 1 \right) \frac{\partial^4 U_{m-1}(x, t)}{\partial x^4} \right]. \]

The \( m^{th} \)-order deformation equation is

\[ S[U_m(x, t) - \chi_m U_{m-1}(x, t)] = \hbar H(x, t) R_m \left( \overrightarrow{U}_{m-1}(x, t) \right). \]

Applying the inverse Sumudu transform,

\[ U_m(x, t) = \chi_m U_{m-1}(x, t) + S^{-1} \left[ \hbar H(x, t) R_m \left( \overrightarrow{U}_{m-1}(x, t) \right) \right] \]  

(2.3.20)

On solving above equation for \( m = 1, 2, ..., \)

\[ U_1(x, t) = \hbar (x - \sin x) \left[ \frac{t^{\alpha+1}}{\Gamma(\alpha+2)} - \frac{t^\alpha}{\Gamma(\alpha+1)} \right], \]  

(2.3.21)
\[ U_2 (x, t) = h (x - \sin x) \left[ \left( \frac{t^{\alpha+1}}{\Gamma (\alpha + 2)} - \frac{t^\alpha}{\Gamma (\alpha + 1)} \right) + h \left( \frac{t^{\alpha+1}}{\Gamma (\alpha + 2)} - \frac{t^\alpha}{\Gamma (\alpha + 1)} \right) \right], \]

\[ U_3 (x, t) = h (x - \sin x) \left[ \left( \frac{t^{\alpha+1}}{\Gamma (\alpha + 2)} - \frac{t^\alpha}{\Gamma (\alpha + 1)} \right) + h \left( \frac{t^{\alpha+1}}{\Gamma (\alpha + 2)} - \frac{t^\alpha}{\Gamma (\alpha + 1)} \right) \right] \]

and so on, in the same manner, the rest of the components of the series \( m \geq 4 \) can be obtained.

Finally, the solution of Eq. (2.3.15) is given as

\[ U(x, t) = U_0(x, t) + \sum_{m=1}^{\infty} U_m(x, t). \] (2.3.24)

As pointed out by Liao [162], the accuracy and convergence of the HAM series solution depend on the careful selection of the auxiliary parameter \( h \). If one can choose \( h = -1 \), then

\[ U(x, t) = \left( 1 - t + \frac{t^\alpha}{\Gamma (\alpha + 1)} - \frac{t^{\alpha+1}}{\Gamma (\alpha + 2)} + \frac{t^{2\alpha}}{\Gamma (2\alpha + 1)} - \frac{t^{2\alpha+1}}{\Gamma (2\alpha + 2)} + \frac{t^{3\alpha}}{\Gamma (3\alpha + 1)} - \frac{t^{3\alpha+1}}{\Gamma (3\alpha + 2)} + \ldots \right) (x - \sin x). \] (2.3.25)
For $\hbar = -1$, the above expressions are exactly the same as those given by the Decomposition method by Khan [128].

By choosing $\alpha = 2$, the researcher clearly concludes that the obtained solution $\sum_{m=0}^{\infty} U_m (x, t)$ converges to the exact solution $U (x, t) = e^{-t} \sin t$ obtained by Wazwaz [249, 250], and Biazar and Ghavini [34].

Figure 2.6: Plot of $U (x, t)$ w.r.t. $\hbar$ at $t = 0.1$ and $x = 1$.

Figure 2.7: Plot of $U (x, t)$ w.r.t. $x$ and $t$ at $\alpha = 1.5$. 

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Figure 2.8: Plot of $U(x, t)$ w.r.t. $x$ and $t$ at $\alpha = 1.75$.

Figure 2.9: Plot of $U(x, t)$ w.r.t. $x$ and $t$ at $\alpha = 2.0$. 
In the subsequent manner the plot of example 2.3.2, \( h \) curve for \( U(x, t) \) lies between \(-1.3 \leq h < 0\) in Fig. 2.6. and Figures 2.7, 2.8, 2.9 are shown, the plot of approximate solution corresponding to two independent variables \( x \) and \( t \) verses \( \alpha = 1.5, 1.75, 2.0 \). Fig. 2.10 is plotted for exact solution which shows the same plot as Fig. 2.9.

2.3.3 Example

Consider the following problem of two-dimensional time-fractional fourth-order PDE [128]

\[
D_t^\alpha U(x, y, t) + 2 \left( \frac{1}{x^2} + \frac{x^4}{6!} \right) \frac{\partial^4 U(x, y, t)}{\partial x^4} + 2 \left( \frac{1}{y^2} + \frac{y^4}{6!} \right) \frac{\partial^4 U(x, y, t)}{\partial x^4} = 0,
\]

\[
\frac{1}{2} < x, y < 1, \quad t > 0, \quad 1 < \alpha \leq 2,
\]

subject to the initial and boundary conditions:
\[
\begin{align*}
U(x, y, 0) &= 0; \\
\frac{\partial U}{\partial t}(x, y, 0) &= 2 + \frac{x^4}{6!} + \frac{y^4}{6!}; \\
U\left(\frac{1}{2}, y, t\right) &= \left(2 + \frac{x^4}{6!} + \frac{y^4}{6!}\right) \sin \left(t, \alpha\right); \\
U(1, y, t) &= \left(2 + \frac{x^4}{6!} + \frac{y^4}{6!}\right) \sin \left(t, \alpha\right); \\
\frac{\partial^2 U}{\partial x^2}\left(\frac{1}{2}, y, t\right) &= \frac{4}{\sqrt{\pi}} \sin \left(t, \alpha\right); \\
U(1, y, t) &= \frac{1}{24} \sin \left(t, \alpha\right).
\end{align*}
\] (2.3.27)

The initial condition is started with \(U_0(x, y, t) = U(x, y, 0) + tU_t(x, y, 0) = t \left(2 + \frac{x^4}{6!} + \frac{y^4}{6!}\right)\).

Operating the Sumudu transform on both sides in (2.3.26) and after using the differentiation property of Sumudu transform for fractional derivative,

\[
\mathcal{S}\left[D_t^\alpha U(x, y, t)\right] + 2\mathcal{S}\left[\left(\frac{1}{x^2} + \frac{x^4}{6!}\right) \frac{\partial^4 U(x, y, t)}{\partial x^4}\right] + 2\mathcal{S}\left[\left(\frac{1}{y^2} + \frac{y^4}{6!}\right) \frac{\partial^4 U(x, y, t)}{\partial y^4}\right] = 0. 
\] (2.3.28)

Applying the initial and boundary conditions of equation (2.3.26) from equation (2.3.27)

We get

\[
\mathcal{S}\left[U(x, y, t)\right] + 2u^\alpha \mathcal{S}\left[\left(\frac{1}{x^2} + \frac{x^4}{6!}\right) \frac{\partial^4 U(x, y, t)}{\partial x^4}\right] + 2u^\alpha \mathcal{S}\left[\left(\frac{1}{y^2} + \frac{y^4}{6!}\right) \frac{\partial^4 U(x, y, t)}{\partial y^4}\right] = 0. 
\] (2.3.29)

The nonlinear operator is

\[
N[\phi(x, y, t; q)] = \mathcal{S}[\phi(x, y, t; q)] + 2u^\alpha \mathcal{S}\left[\left(\frac{1}{x^2} + \frac{x^4}{6!}\right) \frac{\partial^4 \phi(x, y, t; q)}{\partial x^4}\right] \\
+ 2u^\alpha \mathcal{S}\left[\left(\frac{1}{y^2} + \frac{y^4}{6!}\right) \frac{\partial^4 \phi(x, y, t; q)}{\partial y^4}\right],
\] (2.3.30)
and thus

\[
R_m \left[ \hat{U}_{m-1}(x, y, t) \right] = S \left[ U_{m-1}(x, y, t) \right] + 2u^\alpha S \left[ \left( \frac{1}{x^2} + \frac{x^4}{6!} \right) \frac{\partial^4 U_{m-1}(x, y, t)}{\partial x^4} \right]
+ 2u^\alpha S \left[ \left( \frac{1}{y^2} + \frac{y^4}{6!} \right) \frac{\partial^4 U_{m-1}(x, y, t)}{\partial y^4} \right].
\]

(2.3.31)

The \(m\)\(^{th}\) - order deformation equation is

\[
S \left[ U_m(x, y, t) - \chi_m U_{m-1}(x, y, t) \right] = \hbar H(x, t) R_m \left( \hat{U}_{m-1}(x, y, t) \right).
\]

Applying the inverse Sumudu transform,

\[
U_m(x, t) = \chi_m U_{m-1}(x, t) + S^{-1} \left[ \hbar H(x, t) R_m \left( \hat{U}_{m-1}(x, y, t) \right) \right].
\]

(2.3.32)

On solving above equation for \(m = 1, 2, \ldots\)

\[
U_1(x, y, t) = \hbar \left( 2 + \frac{x^6}{6!} + \frac{y^6}{6!} \right) \frac{t^{\alpha+1}}{\Gamma(\alpha + 2)},
\]

(2.3.33)

\[
U_2(x, y, t) = \hbar \left( 2 + \frac{x^6}{6!} + \frac{y^6}{6!} \right) \frac{t^{\alpha+1}}{\Gamma(\alpha + 2)} + \hbar^2 \left( 2 + \frac{x^6}{6!} + \frac{y^6}{6!} \right) \frac{t^{2\alpha+1}}{\Gamma(2\alpha + 2)}
+ \hbar^3 \left( 2 + \frac{x^6}{6!} + \frac{y^6}{6!} \right) \frac{t^{\alpha+1}}{\Gamma(\alpha + 2)},
\]

(2.3.34)
are obtained and so on. In the same manner the rest of the components of the series $m \geq 4$ can be obtained.

Finally, the solution of Eq. (2.3.26) is given as

$$U (x, t) = U_0 (x, t) + \sum_{m=1}^{\infty} U_m (x, t).$$  

(2.3.36)

As pointed out by Liao [162], the accuracy and convergence of the HAM series solution depend on the careful selection of the auxiliary parameter $\hbar$, here, we choose $\hbar = -1$, then

$$U (x, y, t) = \left( t - \frac{\hbar^{\alpha+1}}{\Gamma (\alpha+2)} + \frac{\hbar^{2\alpha+1}}{\Gamma (2\alpha+2)} - \frac{\hbar^{3\alpha+1}}{\Gamma (3\alpha+1)} + \ldots \right) \left( 2 + \frac{x^6}{6!} + \frac{y^6}{6!} \right).$$  

(2.3.37)

For $h = -1$, the above expressions are exactly the same as those given by the Decomposition method by Khan [128].
Figure 2.11: Plot of \( U(x,t) \) w.r.t. \( h \) at \( t = 0.1 \) and \( x = 1 \).

Figure 2.12: Plot of \( U(x,t) \) w.r.t \( x \) and \( t \) at \( \alpha = 1.5 \).
Figure 2.13: Plot of $U(x,t)$ w.r.t. $x$ and $t$ at $\alpha = 1.75$.

Figure 2.14: Plot of $U(x,t)$ w.r.t. $x$ and $t$ at $\alpha = 2.0$. 
Figs. 2.11, 2.12, 2.13, 2.14, 2.15 show the evaluation results of the approximate analytical solution for the Example 2.3.3. These figures also show the behaviour of the approximate solution obtained by the proposed method for different fractional Brownian motions $\alpha = 1.5, 1.75, 2.0$ and the convergence region for convergence control parameter $\hbar$ and approximate solutions.

Solutions at integral value at $\alpha = 2.0$ in all above mentioned plots are shown the same as obtained by Wazwaz [249, 250] and Biazar and Gazvini [34] and by Khan et al. [128].

### 2.4 Conclusion

The incipient modification of HASTM is potent implement to probe the solution of sundry linear and nonlinear quandaries arising in science and engineering. The
main aim of this chapter is to provide the approximate solution and additionally analytic approximation utilizing the proposed method for fourth order boundary value quandaries. The analytical results have been given in terms of a potency series with facilley computed terms. The method surmounts the arduousness in other methods because it is efficient. Three examples were investigated to demonstrate the facileness and multifariousness of the incipient approach. The illustrative examples show that the method is facile to utilize and is an efficacious implement to solve fractional partial differential equations numerically.