CHAPTER 3

DAR(1)/D/s queue with Discrete Skew Laplace as marginal distribution

3.1 Introduction

Discrete time queue with DAR(1) input having only positive customers are studied by many researchers. Hwang et al. (2002) obtained the waiting time distribution of the discrete time single server queue with DAR(1) input. Again Hwang and Sohraby (2003) obtained the closed form expression for the stationary probability generating function of the system size of the discrete time single server queue with DAR(1) input. Choi and Kim (2004) analyzed a multiserver queue fed by DAR(1) input. Tail behavior of the queue size and waiting time in a queue with discrete autoregressive arrivals is described by Kim et al.

Some results included in this chapter form part of the paper : Bindu and Jose in the proceedings of the International Workshop on Cyber Security(IWCS 2k11),(2012a) and Proceedings of the National seminar on Discrete Mathematics and Computational Statistics, (2011).

(2006). A queueing system with discrete autoregressive arrivals is analyzed by Kim et al. (2007). Regularly varying tails in a queue with discrete autoregressive arrivals of order p is analyzed by Kim et al. (2007). Mean queue size in a queue with discrete autoregressive arrivals of order p is obtained by Kim et al. (2008). Analytic approximations of queues with lightly- and heavily-correlated autoregressive service times is discussed in Dieter et al. (2011). Jose and Bindu (2011) analyzed the DAR(1)/D/s queue with Quasi negative Binomial -II distribution as marginal. Second-order performance analysis of discrete-time queues fed by DAR(2) sources with a focus on the marginal effect of the additional traffic parameter is analyzed by Daniel and Lee (2012).

3.2 G-queues

Queueing systems with negative arrivals are used as a control mechanism in many telecommunication and computer networks. The introduction of the concept of negative customers is by Gelenbe (1991). So the queues with positive and negative customers is termed as G-queues. Glynn et al. (1991) analyzed queues with negative customers. Networks with positive and negative arrivals, called G- networks is obtained by Artalejo (2000). Many continuous-time queueing models with negative arrivals have been discussed during the last years, but the analysis of discrete-time queueing models has received considerable attention in the scientific literature over the past years in view of its applicability in the study of many computer and communication systems in which time is slotted, for instance ATM and B-ISDN

Shin (2003) described a queue with positive and negative arrivals governed by a Markov chain. The work about negative customers in discrete-time can be found in Atencia (2004) and Moreno (2005) where the authors considered the single-server discrete-time queue with negative arrivals and various killing disciplines caused by the negative customers. Jinting et al. (2009) considered a discrete-time retrial queue with negative customers and unreliable server. A discrete time queueing system with negative customers and single working vacation is analyzed by Songfang and Chen in (2011).

Research on queueing systems with negative arrivals has been greatly motivated by some practical applications such as computer networks, neural networks, manufacturing systems and communication networks etc. In neural networks a positive customer is interpreted as excitation while a negative one as inhibition. In a computer network or a database, negative customers can represent viruses or commands to delete some transaction. In a manufacturing system, negative customers can represent orders of demand.

Input traffic consist of two types of arrivals; positive customers, negative customers whose arrival processes are correlated. At a negative arrival epoch the system is affected if only if customers are present; in which case the customer is removed from the system. Intuitively, the introduction of negative arrivals makes the system less congested than if they were not present. Positive customers are ordinary ones who, upon arrival, join the queue with the intention of being served. Only positive customers can form a queue and negative customers just reduce system congestion. Negative customer is a signal to delete a positive customer in the system if any presents, and disappeared immediately. discrete skew Laplace distribution can accommodate both positive and negative values. So the VBR coded Teleconference traffic is modeled as discrete autoregressive process of order 1 [DAR(1)] with discrete skew Laplace distribution [DSL(p, q)] with parameters p and q as marginal distribution.

In a multiserver queue any customer finding all servers busy upon arrival may leave the service area and re-apply for service after some random time. The one step transition probability matrix of this Markov process is of M/G/1 type as in Neuts (1989). Construct a Markov renewal process at embedded epochs from the original Markov process. Then compute the stationary distribution of the constructed Markov renewal process by matrix analytic methods. From this stationary distribution calculate the stationary distribution of the system size of the original Markov process by the theory of Markov regenerative processes.

A multiserver queue with s servers$(s > 0)$ having constant service rate in which the input is ATM multiplexer with VBR coded teleconference traffic with both positive and negative arrivals is analyzed in this chapter. Here the input traffic of the queue is considered as DAR(1) with discrete skew Laplace distribution as the marginal distribution. The closed form expression for the stationary distributions of the system size and the waiting time of an arbitrary packet are obtained by using matrix analytic methods and the Markov regenerative theory. The quantitative effect of the stationary distribution of system size and waiting time on the autocorrelation function as well as the parameters of the input traffic is illustrated numerically. The model is applied to a real data with positive and negative (virus) arrivals in an ATM network and is established that the model well suits this data.

3.3 Discrete Skew Laplace DSL (p, q) distribution

Inusah and Kozubowski (2006) introduced the discrete skew Laplace random variable. A skew Laplace random variable has the same distribution as the difference of two independent exponential random variables whereas the discrete skew Laplace (DSL) variable has a similar representation as the difference of two independent geometric random variables. The simplicity of this model and its connection with geometric distribution and the skew Laplace distribution lead to immediate application of DSL distribution.

This distribution is symmetric unimodal, with explicit forms of the densities, distribution functions, characteristic functions and moments. This distribution is infinitely divisible, geometrically infinitely divisible and stable with respect to geometric compounding. DSL distribution is useful in answering questions on whether the positive and negative arrivals have the same geometric distribution. Seethalakshmi and Jose (2008) studied autoregressive processes with discrete skew Laplace distribution and various properties are explored. DSL distribution is very useful for analyzing VBR coded teleconference traffic with positive and negative arrivals. A random variable X has discrete skew Laplace distribution with parameter p and $q \in (0, 1)$ denoted by DSL(p, q) if

$$f(k/p, q) = P(X = k) = \frac{(1-p)(1-q)}{1-pq} \begin{cases} p^k & k = 0, 1, \ldots \\ q^{|k|} & k = 0, -1, -2, \ldots \end{cases} \qquad (3.3.1)$$

The cumulative distribution function of X is given by

$$F(x/p,q) = P(X \leq x) = \begin{cases} \dfrac{(1-p)q^{[x]}}{1-pq}, & \text{if } x < 0 \\ 1 - \dfrac{(1-q)p^{[x]+1}}{1-pq}, & \text{if } x \geq 0 \end{cases}$$

where$[.]$ is the greatest integer function.
The characteristic function of X is given by

$$\Phi(t) = E(e^{itx}) = \frac{(1-p)(1-q)}{(1-pe^{it})(1-qe^{-it})}, t \in R$$

The probability generating function is given as

$$P(z) = \frac{(1-p)(1-q)}{(1-pz)(1-\frac{q}{z})}$$

The moments are

$$E[X] = \frac{p}{1-p} - \frac{q}{1-q}$$

$$V[X] = \frac{1}{(1-p^2)(1-q^2)} \left(\frac{q(1-p)^3(1+q) + p(1-q)^3(1+p)}{1-pq} - (p-q)^2 \right)$$

3.3.1 Maximum likelihood estimation

Let $X_1, X_2, \ldots, X_n$ be i.i.d variables from the DSL(p, q) distribution with density (3.3.1). Let and X^+ and X^- are the positive and the negative parts of X, respectively.

$$\text{and } \bar{X_n}^+ = \frac{1}{n}\sum_{i=1}^{n} \bar{X_i}^+, \quad \bar{X_n}^- = \frac{1}{n}\sum_{i=1}^{n} \bar{X_i}^-$$

Let c_1 and c_2 be defined as

$$c_1 = \bar{X_n}^+ - \bar{X_n}^- = \bar{X_n} = \text{ sample mean} \tag{3.3.2}$$

$$\text{and } c_2 = \bar{X_n}^+ + \bar{X_n}^- = \mid \bar{X} \mid_n = \text{sample first absolute moment} \tag{3.3.3}$$

Then the MLE's of p and q are unique values given by equations (3.3.4), (3.3.5)

$$q = \frac{(c_2 - c_1)(1 + c_1)}{1 + (c_2 - c_1)c_1 + \sqrt{1 + (c_2 - c_1)(c_2 + c_1)}}, \quad p = \frac{q + c_1(1 - q)}{1 + c_1(1 - q)}, \quad c_1 \geq 0 \tag{3.3.4}$$

and

$$p = \frac{(c_2 + c_1)(1 - c_1)}{1 - (c_2 + c_1)c_1 + \sqrt{1 + (c_2 + c_1)(c_2 - c_1)}}, \quad p = \frac{p - c_1(1 - p)}{1 - c_1(1 - p)}, \quad c_1 \leq 0 \tag{3.3.5}$$

which gives the estimates as

$$\hat{q}_n = \frac{2\bar{X_n}^-(1 + \bar{X}_n)}{1 + 2\bar{X_n}^-\bar{X}_n + \sqrt{1 + 4\bar{X_n}^-\bar{X_n}^+}}, \quad \hat{p}_n = \frac{\hat{q}_n + \bar{X}_n(1 - \hat{q}_n)}{1 + \bar{X}_n(1 - \hat{q}_n)}, \bar{X}_n \geq 0$$

$$\hat{p}_n = \frac{2\bar{X_n}^+(1 - \bar{X}_n)}{1 - 2\bar{X_n}^-\bar{X}_n + \sqrt{1 + 4\bar{X_n}^-\bar{X_n}^+}}, \quad \hat{q}_n = \frac{\hat{p}_n - \bar{X}_n(1 - \hat{p}_n)}{1 - \bar{X}_n(1 - \hat{p}_n)}, \bar{X}_n \leq 0$$

3.4 Input traffic as DAR(1) with DSL (p, q) as marginal distribution

The input ATM multiplexer with VBR coded Teleconference traffic is assumed to be DAR(1) with discrete skew Laplace DSL (p, q) distribution as marginal. Consider a sequence $\{Y(t) : t = 0, 1, \ldots\}$ of i.i.d random variables. $Y(t)$ assume positive and negative values and

$$b_x = P[Y(t) = x], x = 0, \pm 1, \pm 2, \ldots$$

When the input process has discrete skew Laplace DSL (p, q) distribution as marginal, then $b(x)$ has the pmf as in (3.3.1).

From McKenzie (2003), the first-order autoregressive equation can be in the form.

$$X(t) = (1 - Z(t))X(t - 1) + Z(t)Y(t), t = 1, 2, \ldots \tag{3.4.1}$$

where $\{Z(t) : t = 1, 2, \ldots\}$ are i.i.d. binary r.v.s with $P[Z(t) = 0] = \beta(0 \leq \beta < 1)$

and $P[Z(t) = 1] = 1 - \beta$. $\{Z(t) : t = 1, 2, \ldots\}$ is assumed to be independent of $\{Y(t) : t = 0, 1, \ldots\}$.

If $X(0)$ is also sampled from DSL (p, q) then (3.4.1) generates a stationary process $X(t)$ whose marginal distribution is DSL (p, q). The model defines the current observation to be a mixture of two independent r.v.s: it is either the last observation, with probability β, or another independent sample from the same distribution. Thus the input ATM multiplexer with VBR coded Teleconference traffic is assumed to be DAR(1) with discrete skew Laplace distribution as marginal. DAR(1) is determined by the parameter β and the distribution $\{b_m : m = 0, \pm 1, \pm 2, \ldots\}$ of $Y(t)$, where $b_m = P[Y(t) = m], m = 0, \pm 1, \pm 2, \ldots$.DAR(1) is modeled as

$$\begin{aligned} X(0) &= Y(0) \\ X(t) &= \begin{cases} X(t-1) & \text{with probability} \quad \beta \\ Y(t) & \text{with probability} \quad 1-\beta \end{cases} \end{aligned}$$

The properties of DAR(1) are as follows
i. $\{X(t) : t = 0, 1, \ldots\}$ is stationary
ii. The probability distribution of $X(t)$ is the same as the distribution of $Y(t)$

$$P[X(t) = m] = b_m, m = 0, \pm 1, \pm 2, \ldots$$

iii.The autocorrelation function $\gamma(t)$ for $X(t)$ at lag t is $\beta^t, t = 0, 1, 2, ...$ where the parameter β related to the decay rate of the autocorrelation function .

The simulated sample path of DAR(1) with discrete skew Laplace as marginal distribution for different values of β is shown in figure (3.1).

The model (3.4.1) extends to higher orders in an obvious way. The p^{th} order model [DAR(p)] as in Kim et al. (2008) is given by

$$X(t) = (1 - Z(t))X(t - A(t)) + Z(t)Y(t), t = 1, 2, \ldots$$

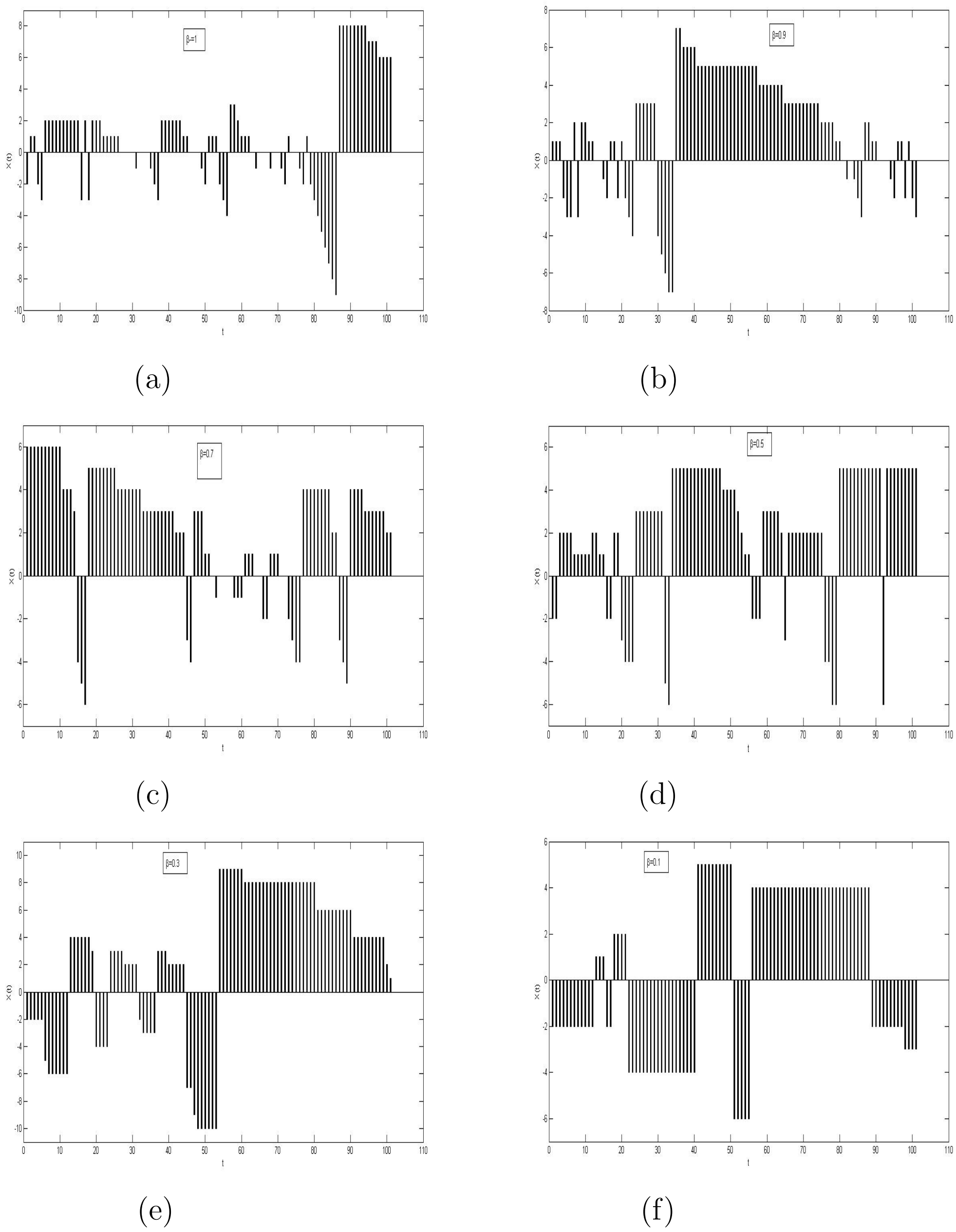


Figure 3.1: Simulated sample path of the DAR(1) process with discrete skew Laplace as marginal distribution for $p = 0.665, q = 0.335, n = 100$ (a) $\beta = 1$ (b) $\beta = 0.9$(c) $\beta = 0.7$ (d) $\beta = 0.5$(e) $\beta = 0.3$ (f) $\beta = 0.1$

$A(t)$ are i.i.d. r.v.s defined on the set $(1, 2, \ldots p)$ so that $P(A(t) = k) = \phi_k$. Thus, the current value is either one of the last p observed values, chosen stochastically, or an independent choice $Y(t)$. The autocorrelation function satisfies the usual form of Yule-Walker equations for an AR(p), given by

$$r_X(k) = \beta \sum_{i=1}^{p} \phi_i r_X(k-i)$$

3.5 Analysis of DAR(1)/D/s queue with DSL(p, q) as marginal

In this integer valued time queue, the time is divided into slots of equal size and one slot is needed to serve a packet by a server. Assume that packet arrivals occur at the beginning of slots and departures occur at the end of the slots. Let $N(t)$ be the number of packets in the system say system size, immediately before arrivals at the beginning of the tth slot. A negative arrival does not receive service and has the effect of removing a customer from the queue. If it arrives when the queue is empty, it has no effect on the system and it is lost. For a fixed number k, when a negative arrival of size $l(< k)$ arrives at the system, it removes min $(l; N(t))$ customers in the queue. However, when a negative arrival of size $l(\geq k)$ arrives at the system, it removes min $(k, N(t))$ customers in the queue. Thus only positive arrivals can form a queue and negative arrivals just reduce system congestion. So $N(t)$ can assume only positive integer values.

Here $\{X(t) : t = 0, 1, \ldots\}$ represents packet arrivals so that $X(t)$ is the number of packets arriving at the beginning of the tth slot. $X(t)$ can assume both positive as well as negative integer values.

Then $\{(N(t), X(t)) : t = 0, 1, \ldots\}$ is a two dimensional Markov process of M/G/1 queue type. The state space is

$$\bigcup_{n\geq 0} l(n) = \bigcup_{n,i\geq 0} \{(n,i)\} = E\{0,1,2,\ldots\} \times \{0, \pm 1, \pm 2, \ldots\}$$

The number of phases is infinity. The computation of stationary distribution of the

Markov Process $\{(N(t), X(t)) : t = 0, 1, \ldots\}$is not easy to work out. To overcome this difficulty first compute the stationary distribution of a new process at the embedded epochs $\{t_\tau, \tau = 0, 1, \ldots\}$ $0 < t_0 < t_1 < t_2 < t_3 \ldots$ with the help of matrix analytical method and using the theory Markov regenerative processes as follows.

$$t_\tau = \begin{cases} 0, & \tau = 0 \\ inf\{t > t_{\tau-1} : Z(t) = 1 \text{ or } -k \leq X(t) \leq s-1\}, & \tau = 1, 2, \ldots \end{cases}$$

Let $N_\tau = N(t_\tau), \tau = 0, 1, \ldots$. Then we have

$$\begin{aligned} J_0 &= s \\ J_\tau &= \begin{cases} X(t_\tau), & \text{if } Z(t_\tau) = 0 \quad \tau = 1, 2, \ldots \\ \\ s, & \text{if } Z(t_\tau) = 1, \quad \tau = 1, 2, \ldots \end{cases} \end{aligned}$$

The packet arrivals at and after t_τ are independent of the information prior to t_τ given J_τ. From this, it is observed that $\{(N_\tau, J_\tau) : \tau = 0, 1, 2, \ldots\}$ is the new Markov renewal process with state space $E = \{0, 1, \ldots\} \times \{-k, \ldots, 0, \ldots, s\}$ The probability transition matrix of the Markov renewal process is computed as follows

1. For $n = 0, 1, \ldots$ and $i = -k, \ldots, 0, \ldots, s-1$

$$(n, i) \rightarrow \begin{cases} (\max\{n-s+i, 0\}, i) & \text{with probability} \quad \beta \\ \\ (\max\{n-s+i, 0\}, s) & \text{with probability} \quad 1-\beta \end{cases}$$

2. For $n = 0, 1, \ldots$

$$(n, s) \rightarrow \begin{cases} (\max\{n-s+i, 0\}, i) & \text{with probability } b_i\beta \qquad -k \leq i \leq s-1, \\ (n-s+i, s) & \text{with probability } b_i(1-\beta) \quad s-n+1 \leq i \leq s-1, \\ (0, s) & \text{with probability } \sum_{i=-k}^{min\{s-n, s-1\}} b_i(1-\beta) + g_0\delta_{n0} \\ (n+l, s) & \text{with probability } g_l \qquad l \geq 0, , n+l > 0 \end{cases}$$

$$\text{where } \delta_{n0} = \begin{cases} 1 & \text{if } n = 0 \\ 0 & \text{if } n \geq 1 \end{cases}$$

$$g_0 = b_s$$

$$g_l = \sum_{i|l} b_{i+s}(1-\beta)\beta^{\frac{l}{i}} - 1, l = 1, 2, \ldots$$

Each level n represents the set of different states such as $((n, -k), \ldots (n, 0), \ldots (n, s-1), (n, s))$ where $n = 0, 1, \ldots$ Then the probability transition matrix P is given below

$$P = \begin{array}{c} \\ 0 \\ 1 \\ \vdots \\ s-1 \\ s \\ s+1 \\ s+2 \\ \vdots \\ 2s-1 \\ 2s \\ 2s+1 \\ \vdots \\ \\ \\ \end{array} \overset{\begin{array}{cccccccc} 0 & 1 & 2 & .... & s-1 & s & s+1 & ..... \end{array}}{\left(\begin{array}{cccccccc} B_s & A_{s+1} & A_{s+2} & \ldots & A_{2s-1} & A_{2s} & A_{2s+1} & \ldots \\ B_{s-1} & A_s & A_{s+1} & \ldots & A_{2s-2} & A_{2s-1} & A_{2s} & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ldots \\ B_1 & A_2 & A_3 & \ldots & A_s & A_{s+1} & A_{s+2} & \ldots \\ B_0 & A_1 & A_2 & \ldots & A_{s-1} & A_s & A_{s+1} & \ldots \\ B_{-1} & A_0 & A_1 & \ldots & A_{s-2} & A_{s-1} & A_s & \ldots \\ B_{-2} & A_{-1} & A_0 & \ldots & A_{s-3} & A_{s-2} & A_{s-1} & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ldots \\ B_{(-k+1)} & A_{(-k+2)} & A_{(-k+3)} & \ldots & A_0 & A_1 & A_2 & \ldots \\ A_{-k} & A_{(-k+1)} & A_{(-k+2)} & \ldots & A_{-1} & A_0 & A_1 & \ldots \\ 0 & A_{-k} & A_{(-k+1)} & \ldots & A_{-2} & A_{-1} & A_{-3} & \ldots \\ 0 & 0 & A_{-k} & \ldots & A_{-3} & A_{-2} & A_{-1} & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \\ 0 & 0 & 0 & \ldots & 0 & 0 & A_{-k} & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \end{array} \right)}$$

Where

$$A_i = \begin{array}{c} \\ -k \\ \vdots \\ i \\ \vdots \\ s \end{array} \begin{array}{c} \begin{array}{ccccc} -k & ... & i & ... & s \end{array} \\ \begin{pmatrix} 0 & .... & 0 & ... & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & ... & \beta & \vdots & 1-\beta \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \vdots & b_i\beta & \vdots & b_i(1-\beta) \end{pmatrix} \end{array} -k \leq i \leq s-1$$

$$A_i = \begin{array}{c} \\ -k \\ \vdots \\ s \end{array} \begin{array}{c} \begin{array}{ccc} -k & ... & s \end{array} \\ \begin{pmatrix} 0 & ... & 0 \\ \vdots & \vdots & \vdots \\ 0 & ... & g_{i-s} \end{pmatrix} \end{array} i \geq s, B_i = \sum_{j=-k}^{i} A_j \qquad -k+1 \leq i \leq s$$

3.5.1 Stability condition

In order to derive the stationary distribution, the stability condition $\rho = \frac{\lambda}{s\mu} < 1.$ should be satisfied. Here the service rate is constant then the offered load should be $\frac{\lambda}{s} < 1$ which means

$$\lambda = E[X(t)] = \sum_{x=-\infty}^{\infty} x b_x = \left[\frac{p}{1-p} - \frac{q}{1-q}\right] < s$$

is the stability condition to be satisfied in the DAR(1)/D/s queue with discrete skew Laplace distribution as the marginal distribution. To get a fixed mean the parameters of the distribution $\hat{p}, \hat{q}$ should be derived for a given s, the number of servers

3.5.2 Stationary distribution of the Markov renewal process

Consider the Markov renewal process $\{N_\tau, J_\tau, \tau = 0, 1, \ldots\}$, with limiting probabilities

$$\pi_{ni} = lim_{\tau\rightarrow\infty} P\{N_\tau = n, J_\tau = i\}, n \geq 0, -k \leq i \leq s$$

We apply matrix analytic method to find the limiting probabilities. The transition probability matrix P has infinite order, so that it would have to be truncated before implementing matrix analytic method. Assume that there exists some index N such that $A_N = 0$ for all $n > N$. That is assume that the Markov chain does not jump more than N steps at a time so that the matrix is of finite order as in Latouche et al. (1991). For an illustration consider the case when $s = 5$ and $N = 14$. Then the transition probability matrix P can be obtained as

$$
\begin{pmatrix}
B_5 & A_6 & A_7 & A_8 & A_9 & | & A_{10} & A_{11} & A_{12} & A_{13} & A_{14} & | & A_{15} & A_{16} & A_{17} & A_{18} & A_{19} \\
B_4 & A5 & A_6 & A_7 & A_8 & | & A_9 & A_{10} & A_{11} & A_{12} & A_{13} & | & A_{14} & A_{15} & A_{16} & A_{17} & A_{18} \\
B_3 & A_4 & A_5 & A_6 & A_7 & | & A_8 & A_9 & A_{10} & A_{11} & A_{12} & | & A_{13} & A_{14} & A_{15} & A_{16} & A_{17} \\
B_2 & A_3 & A_4 & A5 & A_6 & | & A_7 & A_8 & A_9 & A_{10} & A_{11} & | & A_{12} & A_{13} & A_{14} & A_{15} & A_{16} \\
B_1 & A_2 & A_3 & A_4 & A_5 & | & A_6 & A_7 & A_8 & A_9 & A_{10} & | & A_{11} & A_{12} & A_{13} & A_{14} & A_{15} \\
-- & -- & -- & -- & -- & -- & -- & -- & -- & -- & -- & -- & -- & -- & -- & -- & -- \\
B_0 & A_1 & A_2 & A_3 & A_4 & | & A_5 & A_6 & A_7 & A_8 & A_9 & | & A_{10} & A_{11} & A_{12} & A_{13} & A_{14} \\
B_{-1} & A_0 & A_1 & A_2 & A_3 & | & A_4 & A_5 & A_6 & A_7 & A_8 & | & A_9 & A_{10} & A_{11} & A_{12} & A_{13} \\
B_{-2} & A_{-1} & A_0 & A_1 & A_2 & | & A_3 & A_4 & A_5 & A_6 & A_7 & | & A_8 & A_9 & A_{10} & A_{11} & A_{12} \\
B_{-3} & A_{-2} & A_{-1} & A_0 & A_1 & | & A_2 & A_3 & A_4 & A_5 & A_6 & | & A_7 & A_8 & A_9 & A_{10} & A_{11} \\
B_{-4} & A_{-3} & A_{-2} & A_{-1} & A_0 & | & A_1 & A_2 & A_3 & A_4 & A_5 & | & A_6 & A_7 & A_8 & A_9 & A_{10} \\
-- & -- & -- & -- & -- & -- & -- & -- & -- & -- & -- & -- & -- & -- & -- & -- & -- \\
A_{-5} & A_{-4} & A_{-3} & A_{-2} & A_{-1} & | & A_0 & A_1 & A_2 & A_3 & A_4 & | & A_5 & A_6 & A_7 & A_8 & A_9 \\
0 & A_{-5} & A_{-4} & A_{-3} & A_{-2} & | & A_{-} & A_0 & A_1 & A_2 & A_3 & | & A_4 & A_5 & A_6 & A_7 & A_8 \\
0 & 0 & A_{-5} & A_{-4} & A_{-3} & | & A_{-2} & A_{-1} & A_0 & A_1 & A_2 & | & A_3 & A_4 & A_5 & A_6 & A_7 \\
0 & 0 & 0 & A_{-5} & A_{-4} & | & A_{-3} & A_{-2} & A_{-1} & A_0 & A_1 & | & A_2 & A_3 & A_4 & A_5 & A_6 \\
0 & 0 & 0 & 0 & A_{-5} & | & A_{-4} & A_{-3} & A_{-2} & A_{-1} & A_0 & | & A_1 & A_2 & A_3 & A_4 & A_5 \\
-- & -- & -- & -- & -- & -- & -- & -- & -- & -- & -- & -- & -- & -- & -- & -- & -- \\
0 & 0 & 0 & 0 & 0 & | & A_{-5} & A_{-4} & A_{-3} & A_{-2} & A_{-1} & | & A_0 & A_1 & A_2 & A_3 & A_4 \\
0 & 0 & 0 & 0 & 0 & | & 0 & A_{-5} & A_{-4} & A_{-3} & A_{-2} & | & A_{-1} & A_0 & A_1 & A_2 & A_3 \\
0 & 0 & 0 & 0 & 0 & | & 0 & 0 & A_{-5} & A_{-4} & A_{-3} & | & A_{-2} & A_{-1} & A_0 & A_1 & A_2 \\
0 & 0 & 0 & 0 & 0 & | & 0 & 0 & 0 & A_{-5} & A_{-4} & | & A_{-3} & A_{-2} & A_{-1} & A_0 & A_1 \\
0 & 0 & 0 & 0 & 0 & | & 0 & 0 & 0 & 0 & A_{-5} & | & A_{-4} & A_{-3} & A_{-2} & A_{-1} & A_0 \\
-- & -- & -- & -- & -- & -- & -- & -- & -- & -- & -- & -- & -- & -- & -- & -- & -- \\
0 & 0 & 0 & 0 & 0 & | & 0 & 0 & 0 & 0 & 0 & | & A_{-5} & A_{-4} & A_{-3} & A_{-2} & A_{-1} \\
0 & 0 & 0 & 0 & 0 & | & 0 & 0 & 0 & 0 & 0 & | & 0 & A_{-5} & A_{-4} & A_{-3} & A_{-2} \\
0 & 0 & 0 & 0 & 0 & | & 0 & 0 & 0 & 0 & 0 & | & 0 & 0 & A_{-5} & A_{-4} & A_{-3} \\
0 & 0 & 0 & 0 & 0 & | & 0 & 0 & 0 & 0 & 0 & | & 0 & 0 & 0 & A_{-5} & A_{-4} \\
0 & 0 & 0 & 0 & 0 & | & 0 & 0 & 0 & 0 & 0 & | & 0 & 0 & 0 & 0 & A_{-5}
\end{pmatrix}
$$

By partitioning the transition probability matrix into $(s \times s)$ matrices we get

$$P = \begin{pmatrix} \hat{B}_0 & \hat{B}_1 & \hat{B}_2 \\ \hat{C}_0 & \hat{C}_1 & \hat{C}_2 \\ \hat{A}_0 & \hat{A}_1 & \hat{A}_2 \\ 0 & \hat{A}_0 & \hat{A}_1 \\ 0 & 0 & \hat{A}_0 \end{pmatrix} \text{ or equivalently } P = \begin{pmatrix} \hat{B}_0 & \hat{A}_3 & \hat{A}_4 \\ \hat{C}_0 & \hat{A}_2 & \hat{A}_3 \\ \hat{A}_0 & \hat{A}_1 & \hat{A}_2 \\ 0 & \hat{A}_0 & \hat{A}_1 \\ 0 & 0 & \hat{A}_0 \end{pmatrix}$$

In general we can symbolize the transition matrix P as

$$P = \begin{pmatrix} \hat{B}_0 & \hat{B}_1 & \hat{B}_2 & .... & \hat{B}_{n^*-2} \\ \hat{C}_0 & \hat{C}_1 & \hat{C}_2 & .... & \hat{C}_{n^*-2} \\ \hat{A}_0 & \hat{A}_1 & \hat{A}_2 & .... & \hat{A}_{n^*-2} \\ 0 & \hat{A}_0 & \hat{A}_1 & \vdots & \hat{A}_{n^*-3} \\ 0 & 0 & \hat{A}_0 & \vdots & \hat{A}_{n^*-4} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & ... & \hat{A}_0 \end{pmatrix}, n^* = \frac{N+s+1}{s} - 1$$

or equivalently

$$P = \begin{pmatrix} \hat{B}_0 & \hat{A}_3 & \hat{A}_4 & .... & \hat{A}_{n^*} \\ \hat{C}_0 & \hat{A}_2 & \hat{A}_3 & .... & \hat{A}_{n^*-1} \\ \hat{A}_0 & \hat{A}_1 & \hat{A}_2 & .... & \hat{A}_{n^*-2} \\ 0 & \hat{A}_0 & \hat{A}_1 & \vdots & \hat{A}_{n^*-3} \\ 0 & 0 & \hat{A}_0 & \vdots & \hat{A}_{n^*-4} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & ... & \hat{A}_0 \end{pmatrix}, n^* = \frac{N+s+1}{s} - 1$$

A matrix of this structure is said to be of M/G/1 type, which underlines the similarity to the

Embedded Markov chain of the M/G/1 queue. With respect to the levels , the Markov chain is called skip free to the left, since in one transition the level can be reduced only one. The elements of P can be written as

$$\hat{B}_0 = \begin{pmatrix} B_s & A_{s+1} & ... & A_{2s-1} \\ B_{s-1} & A_s & ... & A_{2s-2} \\ \vdots & \vdots & \vdots & \vdots \\ B_1 & A_2 & ... & A_s \end{pmatrix}, \hat{C}_0 = \begin{pmatrix} B_0 & A_1 & ... & A_{s-1} \\ B_{-1} & A_0 & ... & A_{s-2} \\ \vdots & \vdots & \vdots & \vdots \\ B_{-k+1} & A_{-k+2} & ... & A_0 \end{pmatrix}$$

$$\hat{A}_n = \begin{pmatrix} A_{s(n-1} & A_{s(n-1)+1} & \cdots & A_{sn-1} \\ A_{s(n-1)-1} & A_{s(n-1)} & \cdots & A_{sn-2} \\ \vdots & \vdots & \vdots & \vdots \\ A_{s(n-2)+1} & A_{s(n-2)+2} & \cdots & A_{s(n-1)} \end{pmatrix}, n = 0, 1, 2...n^*$$

$$A_l = O \text{ for } l < -k$$

$$\hat{B}_n = \hat{A}_{n+2}, n = 1, 2, 3, ...n^*, \hat{C}_n = \hat{A}_{n+1}, n = 1, 2, 3, ...n^*.$$

Step 1: Find the minimal nonnegative solution G of the matrix equation

$$G = \sum_{n=0}^{\infty} \hat{A}_n G^n$$

G can be given by the following iteration as in Breuer (2005).
Start with

$$\begin{aligned} G_0 &= 0 \\ G_1 &= \hat{A}_0 \\ G_m &= \textstyle\sum_{n=1}^{m-1} \hat{A}_n G_{m-1}^n, m = 0, 1, \ldots \\ G &= \textstyle\sum_{m=1}^{\infty} G_m \end{aligned}$$

G is a stochastic matrix, so stop the iteration procedure when $\|\mathbf{1} - G \cdot \mathbf{1}\| < \varepsilon$ reaches where $\varepsilon = 0.0001$. The upper limit of m and $n^* = m - 1$ are obtained from this iteration. From this n^* the truncated index N at which G become stochastic is found out.

Step 2: Find

$$H = \sum_{n=0}^{\infty} \hat{B}_n G^n \text{ and } D = \sum_{n=0}^{\infty} \hat{C}_n G^n$$

Find a positive row vector h satisfying

$$hH = h$$

and positive row vector d satisfying

$$dD = d$$

Step 3: Then take

$$\begin{aligned} x_0 &= h \\ x_1 &= d \\ x_n &= \Big(x_0 (\textstyle\sum_{i=0}^{\infty} \hat{B}_{n-1+i} G^i) + x_1 (\sum_{i=0}^{\infty} \hat{C}_{n-1+i} G^i) + \sum_{l=2}^{n-1} x_l \sum_{i=0}^{\infty} \hat{A}_{n-l+i+1} G^i \Big) \times \\ &\quad \Big(I - \textstyle\sum_{i=0}^{\infty} \hat{A}_{i+1} G^i \Big)^{-1}, n = 2, 3, \ldots, n^* \end{aligned}$$

Step 4: Finally

$$((\pi_{ns,-k}, \ldots, \pi_{ns,0}, \ldots, \pi_{ns,s}), \ldots, (\pi_{(n+1)s-1,-k}, \ldots \pi_{(n+1)s-1,0}, \ldots, \pi_{(n+1)s-1,s})) = Cx_n,$$

$$n = 0, 1, \ldots, n^*$$

Where $C = \left[\sum_{n=0}^{n^*} x_n e \right]^{-1}$ and e is a column vector whose components are all ones.

3.6 Stationary distribution of $\{N(t), X(t), t = 0, 1, \ldots\}$

Let $\{((N_\tau, J_\tau), t_\tau), \tau = 0, 1, \ldots\}$ is a Markov renewal sequence in the queue and $\{((N(t + t_\tau), X(t + t_\tau)) : t = 0, 1, \ldots\} / \{(N(\upsilon), X(\upsilon)), 0 \leq \upsilon < t_\tau, N_\tau, J_\tau) = (n, i)\}$ is stochastically equivalent to $\{N(t), X(t) : t = 0, 1, \ldots\}$ / $\{N_0, J_0 = (n, i).\}$. Hence $\{N(t), X(t) : t = 0, 1, \ldots\}$ is a discrete time Markov regenerative process with the Markov renewal sequence $\{((N_\tau, J_\tau), t_\tau) : \tau = 0, 1, \ldots\}$.

$p_{nj} = lim_{t\to\infty} P\{(N(t), X(t)) = (n, j)\} n, j = 0, 1, \ldots$ of $\{(N(t), X(t)) : t = 0, 1, \ldots\}$ are the limiting probabilities which are derived by the following equation

$$p_{nj} = \frac{\sum_{l=0}^{\infty} \sum_{i=-k}^{s} \pi_{li} E\left[\sum_{t=t_\tau}^{t_{\tau+1}-1} 1_{(N(t),X(t))=(n,j)} \mid (N_\tau, J_\tau) = (l, i)\right]}{\sum_{l=0}^{\infty} \sum_{i=-k}^{s} \pi_{li} E\left[t_{\tau+1} - t_\tau \mid (N_\tau, J_\tau) = (l, i)\right]} \tag{3.6.1}$$

The numerator of equation (3.6.1) is

$$\begin{cases} \pi_{nj} + \pi_{ns} b_j, & -k \le j \le s-1 \\ \frac{\pi_{ns} b_s}{1-\beta}, & j = s \\ \sum_{i=0}^{\left|\frac{n}{j-s}\right|} \pi_{n-i(j-s),s} b_j \beta^i, & j \ge s+1 \end{cases}$$

The denominator of equation (3.6.1) is

$$\sum_{l=0}^{\infty} \sum_{i=-k}^{s-1} \pi_{li} + \sum_{l=0}^{\infty} \pi_{ls} \left(\sum_{r=-k}^{s-1} b_r + \sum_{r=s}^{\infty} b_r \frac{1}{1-\beta} \right) \tag{3.6.2}$$

Where $\pi = \left(\sum_{l=0}^{\infty} \pi_{l(-k)}, \ldots \sum_{l=0}^{\infty} \pi_{ls}\right)$ is the stationary probability vector of the Markov process$\{J_\tau : \tau = 0, 1, \ldots\}$ whose transition probability matrix is

$$(P(J_{\tau+1} = j \mid J_\tau = i))_{-k \le i,j \le s} = \begin{array}{c} \\ -k \\ \vdots \\ 0 \\ \vdots \\ s-1 \\ s \end{array} \begin{array}{c} \begin{array}{cccccc} -k & \ldots & 0 & \ldots & s-1 & s \end{array} \\ \begin{pmatrix} \beta & \ldots & 0 & \ldots & 0 & 1-\beta \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \ldots & \beta & \ldots & 0 & 1-\beta \\ & & & & & \\ 0 & \ldots. & 0 & \ldots. & \beta & 1-\beta \\ \beta b_{-k} & \ldots & \beta b_0 & \ldots & \beta b_{s-1} & 1 - \beta \sum_{r=-k}^{s-1} b_r \end{pmatrix} \end{array} \tag{3.6.3}$$

The infinitesimal generator matrix of (3.6.3) is

$$Q = \begin{array}{c} \\ -k \\ \vdots \\ 0 \\ \vdots \\ s-1 \\ s \end{array} \begin{array}{c} \begin{array}{cccccc} -k & \dots & 0 & \dots & s-1 & s \end{array} \\ \begin{pmatrix} -(1-\beta) & \dots & 0 & \dots & 0 & (1-\beta) \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \dots & -(1-\beta) & \dots & 0 & (1-\beta) \\ & & & & & \\ 0 & \dots. & 0 & \dots. & -(1-\beta) & (1-\beta) \\ \beta b_{-k} & \dots & \beta b_0 & \dots & \beta b_{s-1} & -\beta \sum_{r=-k}^{s-1} b_r \end{pmatrix} \end{array}$$

The balance equations are $\pi\mathcal{Q} = 0$ and $\pi e = 1$ where e is a column vector whose components are all ones. The stationary distribution of $\{J_\tau : \tau = 0, 1, \ldots\}$ is obtained by solving the balance equations.

$$\sum_{l=0}^{\infty} \pi_{li} = \begin{cases} \frac{\beta b_i}{1-\beta\sum_{r=s}^{\infty} b_r}, & -k \leq j \leq s-1 \\ \frac{1-\beta}{1-\beta\sum_{r=s}^{\infty} b_r}, & i = s \end{cases} \tag{3.6.4}$$

By substituting (3.6.4) into (3.6.2) the denominator of the right hand side of(3.6.1) can be obtained as

$$\left(1 - \beta \sum_{r=s}^{\infty} b_r\right)^{-1}$$

Theorem 3.6.1. *The stationary distribution or the limiting probabilities, namely* $p_{nj} = lim_{t\to\infty} P\{(N(t), X(t)) = (n, j)\}, n, j = 0, 1, \ldots$ *of* $\{(N(t), X(t)) : t = 0, 1, \ldots\}$ *are given by*

$$p_{nj} = \begin{cases} \mu^{-1}(\pi_{nj} + \pi_{ns} b_j), & -k \leq j \leq s-1 \\ \mu^{-1} \frac{\pi_{ns} b_s}{1-\beta}, & j = s \\ \mu^{-1} \sum_{i=0}^{\lfloor \frac{n}{j-s} \rfloor} \pi_{n-i(j-s),s} b_j \beta^i, & j \geq s+1 \end{cases}$$

$$where \left(1 - \beta \sum_{r=s}^{\infty} b_r\right)^{-1} = \mu$$

3.7 Stationary distribution of waiting time of a packet

Let W be the waiting time of an arbitrary packet at steady state. Then for $\omega = 0, 1, \ldots$

$$P(W = \omega) = \frac{\text{Mean number of arrivals in a slot at steady state whose waiting time is } \omega}{\text{Mean number of arrivals in a slot}} \tag{3.7.1}$$

Suppose that there are n packets immediately before arrivals at the beginning of the t^{th} slot and that the number of packet arrivals is j at the beginning of the t^{th} slot, i.e., $N(t) = n$ and $X(t) = j$. Then the number of packets whose waiting time is ω among the ones who arrive at the beginning of the t^{th} slot is

$$\begin{cases} min\{s(\omega+1) - n, j\}, & s\omega < n < s(\omega+1) \\ min\{n = j - s\omega, s\}, & n \leq s\omega \leq n = j \\ 0 & \text{otherwise} \end{cases}$$

The mean number of arrivals in a slot at steady state whose waiting time ω is

$$\sum_{n=0}^{s\omega} \sum_{j=s\omega-n+1}^{\infty} p_{nj} \text{ min } \{n = j - s\omega, s\} + \sum_{n=s\omega+1}^{s(\omega+1)-1} \sum_{j=1}^{\infty} p_{nj} \text{ min } \{s(\omega+1) - n, j\}$$

The mean number of arrivals in a slot is λ. Then from (3.7.1)

Theorem 3.7.1. *The distribution of the waiting time W of an arbitrary packet is given by $P(W = \omega)$*

$$= \frac{1}{\lambda} \left(\sum_{n=0}^{s\omega} \sum_{j=s\omega-n+1}^{\infty} p_{nj}\ min\ \{n = j - s\omega, s\} + \sum_{n=s\omega+1}^{s(\omega+1)-1} \sum_{j=1}^{\infty} p_{nj}\ min\ \{s(\omega+1) - n, j\} \right)$$

$$\omega = 0, 1, \ldots$$

3.8 Empirical Analysis

Assume the number of servers to be 2 and to get a fixed mean $\lambda = 1.8, p$ and q are derived . Here the offered load = $\frac{\lambda}{s} = 0.9 < 1$. Hence the system satisfied the stability condition. Assume different values for the autocorrelation coefficient β of the DAR(1) process.

Table (3.1),(3.2) and (3.3) display the probability distribution of stationary system size for different values of β ,λ, q and $s = 2$. Figure (3.2) displays the Complementary distribution function of the stationary system size when $q = 0.1$ and $\beta = 0.2, 0.4, 0.6$, and 0.8 and the Complementary distribution function of the stationary system size when β=0.1 and $q = 0.1, 0.3, 0.5$ and 0.7 respectively.

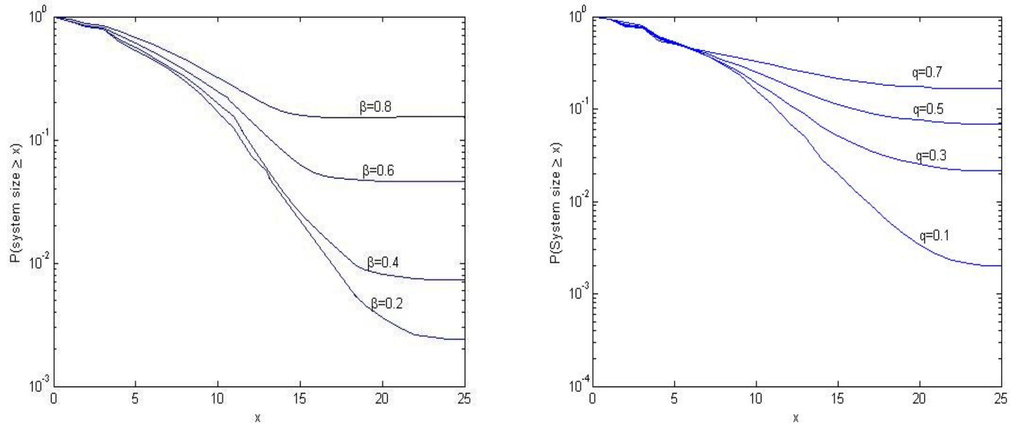
The parameter β gives the information on the strength of correlation of the input process. Stationary system size is larger for the large β . The stationary system size increases as the parameter $q(p$ and q change in the same direction) of the input process increases . Thus the stationary system size for the case that the stationary distribution of the input process has a small variance $(q = 0.7$, variance=0.8176) are stochastically larger than those for the case that the stationary distribution of the input process has a large variance $(q = 0.1$, variance=1.0390). Figure (3.2) supports this intuitive facts.

Table (3.4) and (3.5) display the probability distribution of waiting time of an arbitrary packet for different values β ,λ, q and $s = 2$. Figure (3.3) displays the Complementary distribution function of the waiting time of an arbitrary packet,when $q = 0.1$ and $\beta = 0.2, 0.4, 0.6$ and 0.9 and the Complementary distribution function of the stationary system size when $\beta = 0.1 q = 0.1, 0.3, 0.5$ and 0.7 respectively.

Stationary waiting time of an arbitrary packet, is larger for large β. Also stationary waiting time of an arbitrary packet, is stochastically increases as the parameter q of the input process increases. Figure (3.3) supports this intuitive fact.

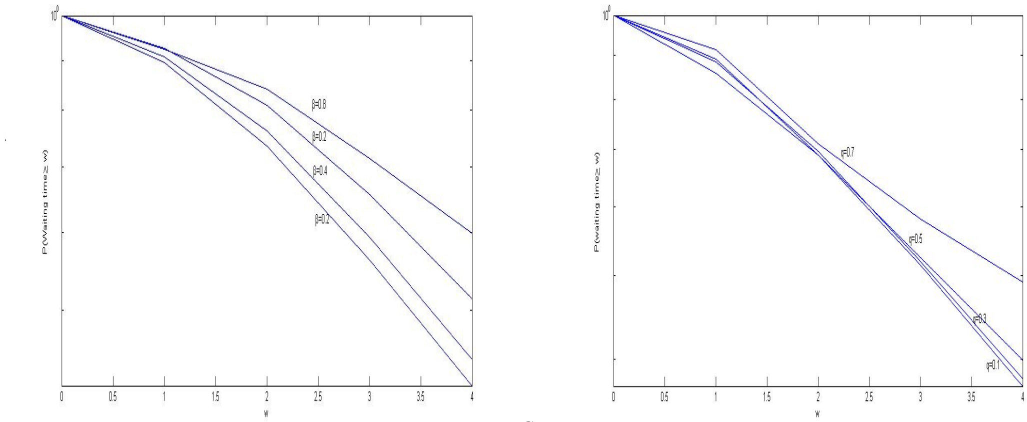
3.9 Real data analysis

Apply the model to a pilot scheme data from Box and Jenkins (2008) as input traffic



(a) (b)

Figure 3.2: Complementary distribution function of the stationary system size in DAR(1)/D/s with DSL(p, q) as marginal when (a)$\lambda = 1.8, q = 0.1$ and $s = 2$ (b) $\lambda = 1.8$, $\beta = 0.1$ and $s = 2$



s

(a) (b)

Figure 3.3: Complementary distribution function of the waiting time of an arbitrary packet in DAR(1)/D/s with DSL(p, q) as marginal for (a) different β, $\lambda = 1.8, q = 0.1$ and $s = 2$ (b) different q, $\lambda = 1.8$, β=0.1 and $s = 2$

							j						
n	⋯	-2	-1	0	1	2	3	4	5	6	7	8	⋯
0	⋯	0.0006	0.0064	0.0639	0.0480	0.0289	0.0176	0.0119	0.0080	0.0054	0.0037	0.0025	⋯
1	⋯	0.0001	0.0007	0.0060	0.0476	0.0030	0.0036	0.0013	0.0008	0.0006	0.0004	0.0003	⋯
2	⋯	0.0001	0.0007	0.0018	0.1037	0.0008	0.0009	0.0024	0.0002	0.0002	0.0001	0.0001	⋯
3	⋯	0.0004	0.0045	0.0480	0.0396	0.0219	0.0134	0.0092	0.0069	0.0041	0.0028	0.0019	⋯
4	⋯	0.0003	0.0026	0.0258	0.0197	0.0118	0.0085	0.0050	0.0034	0.0028	0.0015	0.0010	⋯
5	⋯	0.0002	0.0024	0.0244	0.0168	0.0112	0.0076	0.0055	0.0031	0.0022	0.0018	0.0010	⋯
6	⋯	0.0002	0.0023	0.0232	0.0158	0.0106	0.0072	0.0049	0.0036	0.0020	0.0014	0.0012	⋯
7	⋯	0.0002	0.0022	0.0227	0.0154	0.0104	0.0070	0.0131	0.0032	0.0024	0.0000	0.0009	⋯
8	⋯	0.0002	0.0022	0.0223	0.0151	0.0102	0.0069	0.0047	0.0032	0.0022	0.0016	0.0009	⋯
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋱

Table 3.1: p_{nj} in DAR(1)/D/s with DSL(p, q) as marginal for $\lambda = 1.8$, $\beta = 0.1$, $q = 0.1$ and $s = 2$

							j						
n	⋯	-2	-1	0	1	2	3	4	5	6	7	8	⋯
0	⋯	0.0101	0.0143	0.0347	0.0181	0.0159	0.0127	0.0113	0.0100	0.0089	0.0079	0.0070	⋯
1	⋯	0.0007	0.0010	0.1305	0.0011	0.0011	0.0021	0.0008	0.0007	0.0006	0.0005	0.0005	⋯
2	⋯	0.0021	0.0028	0.0024	0.1622	0.0019	0.0017	0.0019	0.0012	0.0010	0.0009	0.0008	⋯
3	⋯	0.0072	0.0104	0.0162	0.0292	0.0128	0.0104	0.0091	0.0090	0.0071	0.0063	0.0056	⋯
4	⋯	0.0022	0.0031	0.0046	0.0066	0.0036	0.0039	0.0028	0.0023	0.0029	0.0018	0.0016	⋯
5	⋯	0.0017	0.0024	0.0034	0.0034	0.0027	0.0026	0.0028	0.0018	0.0016	0.0021	0.0012	⋯
6	⋯	0.0015	0.0022	0.0031	0.0028	0.0024	0.0022	0.0020	0.0024	0.0015	0.0013	0.0018	⋯
7	⋯	0.0014	0.0021	0.0030	0.0027	0.0023	0.0021	0.0071	0.0017	0.0020	0.0001	0.0011	⋯
8	⋯	0.0014	0.0020	0.0029	0.0026	0.0023	0.0020	0.0018	0.0016	0.0015	0.0017	0.0011	⋯
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋱

Table 3.2: p_{nj} in DAR(1)/D/s with DSL(p, q) as marginal for $\lambda = 1.8$, $\beta = 0.1$, $q = 0.7$ and $s = 2$

							j						
n	⋯	-2	-1	0	1	2	3	4	5	6	7	8	⋯
0	⋯	0.0006	0.0063	0.0828	0.0424	0.0279	0.0113	0.0076	0.0052	0.0035	0.0024	0.0016	⋯
1	⋯	0.0000	0.0004	0.0328	0.0017	0.0019	0.0053	0.0005	0.0004	0.0002	0.0002	0.0001	⋯
2	⋯	0.0003	0.0023	0.0028	0.0863	0.0013	0.0026	0.0036	0.0002	0.0002	0.0001	0.0001	⋯
3	⋯	0.0003	0.0031	0.0401	0.0508	0.0183	0.0085	0.0052	0.0055	0.0023	0.0016	0.0011	⋯
4	⋯	0.0002	0.0026	0.0286	0.0319	0.0131	0.0087	0.0050	0.0026	0.0030	0.0011	0.0008	⋯
5	⋯	0.0002	0.0022	0.0265	0.0235	0.0121	0.0084	0.0054	0.0023	0.0016	0.0020	0.0007	⋯
6	⋯	0.0002	0.0021	0.0228	0.0186	0.0104	0.0076	0.0048	0.0041	0.0014	0.0009	0.0012	⋯
7	⋯	0.0002	0.0018	0.0206	0.0158	0.0094	0.0069	0.0428	0.0028	0.0021	0.0000	0.0006	⋯
8	⋯	0.0002	0.0017	0.0189	0.0140	0.0087	0.0065	0.0043	0.0025	0.0023	0.0014	0.0005	⋯
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋱

Table 3.3: p_{nj} in DAR(1)/D/s with DSL(p, q) as marginal for $\lambda = 1.8$, $\beta = 0.4$, $q = 0.1$ and $s = 2$

	β			
ω	0.2	0.4	0.6	0.8
0	0.1041	0.0935	0.0740	0.0766
1	0.1603	0.1435	0.1164	0.0827
2	0.1731	0.1687	0.1537	0.1265
3	0.1439	0.1491	0.1422	0.1149
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

Table 3.4: $P(W = \omega)$ in DAR(1)/D/s with DSL(p, q) as marginal for different values of β and $q = 0.1$, $\lambda = 1.8$ and $s = 2$

	q			
ω	0.1	0.3	0.5	0.7
0	0.1425	0.1165	0.1099	0.0881
1	0.1683	0.1872	0.2013	0.2014
2	0.1749	0.1765	0.1646	0.1288
3	0.1416	0.1400	0.1244	0.0895
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

Table 3.5: $P(W = \omega)$ in DAR(1)/D/s with DSL(p, q) as marginal for different values of $q, \beta = 0.1$ and $s = 2$

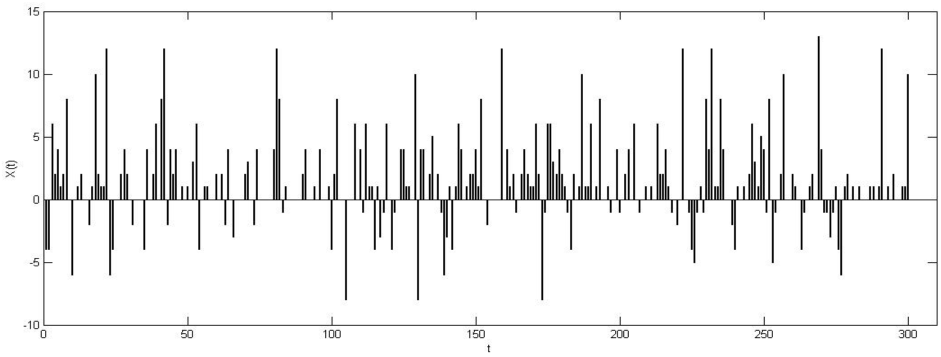


Figure 3.4: Real data with positive and negative arrivals as DAR(1).

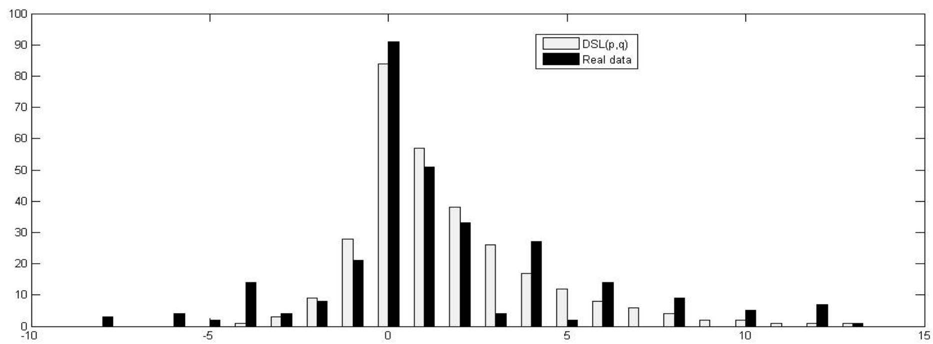


Figure 3.5: The Probability histogram of real data and the DAR(1) with marginal as discrete skew Laplace distribution with $\hat{p} = 0.675$ and $\hat{q} = 0.335$

consist of both positive as well as negative arrivals. Let $X(t)$ be the number of arrivals waiting for the service in time t which is displayed in the figure (3.4).

We assume that the number of arrivals is DAR(1) with marginal distribution as discrete skew Laplace distribution with parameters p and q both $\in (0, 1]$. Thus the data set can be fitted to the DAR(1) with Discrete Skew Laplace as marginal distribution. To test whether there is a significant difference between an observed distribution and the DAR(1) with marginal as discrete skew Laplace distribution, use Kolmogorov-Smirnov [K.S.] test for H_0: DAR(1) with marginal as discrete skew Laplace distribution with parameter $\hat{p} = 0.675$

w	p(w)
0	0.2073
1	0.2052
2	0.0596
3	0.0363
⋮	⋮

Table 3.6: $P(W = \omega)$ in real data when $\beta = 0.3$, $\lambda = 1.29$ and $s = 2$

						j					
n	⋯	-2	-1	0	1	2	3	4	5	6	⋯
0	⋯	0.0322	0.0217	0.0146	0.0778	0.0066	0.0031	0.0021	0.0014	0.0010	⋯
1	⋯	0.0004	0.0003	0.0001	0.1586	0.0001	0.0010	0.0000	0.0000	0.0000	⋯
2	⋯	0.0190	0.0115	0.0041	0.1713	0.0019	0.0012	0.0007	0.0004	0.0003	⋯
3	⋯	0.0176	0.0127	0.0110	0.0566	0.0050	0.0027	0.0016	0.0015	0.0007	⋯
4	⋯	0.0064	0.0047	0.0039	0.0188	0.0018	0.0017	0.0009	0.0004	0.0006	⋯
5	⋯	0.0029	0.0021	0.0017	0.0065	0.0008	0.0009	0.0007	0.0003	0.0001	⋯
6	⋯	0.0014	0.0010	0.0008	0.0023	0.0004	0.0004	0.0004	0.0005	0.0001	⋯
7	⋯	0.0008	0.0006	0.0004	0.0009	0.0002	0.0002	0.0187	0.0002	0.0002	⋯
8	⋯	0.0005	0.0003	0.0003	0.0004	0.0001	0.0002	0.0002	0.0001	0.0002	⋯
9	⋯	0.0003	0.0002	0.0002	0.0002	0.0001	0.0001	0.0063	0.0002	0.0000	⋯
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋱

Table 3.7: p_{nj} in real data when $\beta = 0.3$, $\lambda = 1.29$ and $s = 2$.

and $\hat{q} = 0.335$ is a good fit for the given data. Here the calculated value of the K.S. test statistic is 0.0767 and the critical value corresponding to the significance level 0.01 is 0.0941 showing that the assumption that the data follow DAR(1) with marginal as discrete skew Laplace distribution is valid as in figure (3.5).

The stationary distribution of system size and waiting time of an arbitrary customer for the DAR/D/s queue with discrete skew Laplace distribution as marginal is obtained by matrix analytic method. Here the mean = $\lambda = 1.29$. To satisfy the stability condition the number of servers is assumed as $s = 2$, so that $\lambda/s = 0.645 < 1$ Also the value of autocorrelation function β is assumed as 0.3, $\hat{p} = 0.675$ and $\hat{q} = 0.335$. Table (3.6) and (3.7) display the stationary distribution of waiting time of an arbitrary customer and system size.

3.10 Conclusions

A multiserver queue with s servers having constant service rate in which the input is ATM multiplexer with VBR coded teleconference traffic with both positive and negative arrivals is analyzed. The input traffic is considered as the DAR(1) with discrete skew Laplace distribution as the marginal distribution. Based on the matrix analytic methods and the Markov regenerative theory, the stationary distributions of the system size and the waiting time of an arbitrary packet are obtained. The larger the parameter β , the slower the decay of the autocorrelation of the input process. So the stationary system size and waiting time for the case of large β are stochastically larger than those for the case of small β. Also the stationary system size and waiting time is stochastically increases as the parameter q(or p) of the input process increases. Thus the stationary system size and waiting time of an arbitrary packet for the case when the stationary distribution of the input process has a small variance are stochastically larger than those for the case when the stationary distribution of the input process has a large variance.

3.11 Appendix

Program in MATLAB to simulate DAR(1) process with discrete skew Lapalce as marginal distribution

```
% program to simulate DAR(1) process with discrete skew Lapalce  as marginal  distribution
close all
clc
p = 0.675;
q = 0.335;
i=1;
beta=0.1;
k=1;s=[];
while(k<=5100)
u1=rand;
if i~=0
if u1>=1/2;
j=-(i-1);
else
j=+(i+1);
end
```

```
else
i=0;
if u1<1/2;
j=0;
else
j=1;
end
end
d=((1-p)*(1-q))/(1-p*q);
if i>=0
f1=d*p^i;
else
f1=d*q^(abs(i));
end
if  j>=0
f2=d*p^j;
else
f2=d*q^(abs(j));
end
r=f1/f2;
if r>=1;
sample=j;
else
u2=rand;
if u2<r;
sample=j;
else
sample =i;
end
end;
i=sample;
ax=[];e=[];
if k>=5000
i=sample;
s1=-i;
s=[s;s1];
e1=binornd(1,beta);
e=[e;e1];
ax1=(1-e1)*ax0+e1*s1
ax=[ax;ax1];
ax0=ax1;
else
i=sample;
```

```
s0=i;
s=[s;s0];
ax0=s0;
ax=[ax;ax0];
end
k=k+1;
end
```

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