CHAPTER - 0
INTRODUCTION

Chaos theory originated with the work of Edward Lorenz, a meteorologist in 1960 when he was working on the problem of weather prediction with a set of twelve equations; which express the relationships between temperature and pressure, pressure and wind speed; to model the weather in his computer set up at Massachusetts Institute of Technology. A Scientist considers himself lucky if he can get measurements with accuracy to three decimal places. The fourth and fifth may be difficult to measure using reasonable methods can't have a huge effect on the outcome of the experiment. But Lorenz proved that this idea is wrong, i.e. approximation may lead to disorder. The consequence of Lorenz experiment is known as butterfly effect. Flapping of a butterfly's wings today may produce after some period a storm. i.e. a small change in the initial conditions may cause a big change in the output. This phenomenon is also known as sensitive dependence on initial conditions. The phenomenon which changes order into disorder is known as Chaos. In 1964, A.N. Sharkovsky[67] defined a partial order on the set of positive integers and proved that for a continuous map on a closed interval there is a point of period m if there is a point of period n where m precedes n in that order[Result 1:1:14]. Existence of such periodic points is a necessary condition for a system to be Chaotic. But a precise Mathematical definition of Chaos is given by Tien-Yien Li and James. A Yorke [71] in 1975. They defined Chaotic functions [Def:1:1:5] and proved that if f is a continuous map on a closed interval such that f has a point of period 3 then f is Chaotic[Result:1:1:13]. Since there are other definitions of Chaos, let us call this definition Li Yorke (LY) Chaos. In 1977, Melvyn.B. Nathanson[56, 57] proved that the condition that there is a point of period 3 is not necessary for Li-Yorke Chaos. He proved that if f has a point of period which is divisible by 3, 5 or 7 then f is Li-Yorke Chaotic. In 1979, Frederick J. Fuglister[40], proved that if f has a point of period p where p is not a power of 2, then f is Li-Yorke Chaotic. He also showed an example of a non Chaotic function with points of period p, for each p which is a power of 2. In 1980, Joseph Auslander and James A. Yorke[48] defined Chaotic functions in Metric Spaces [Definition:1:1:9]. In that definition
existence of a point with dense orbit is needed. This paper can be considered as first generalization of Chaos into metric spaces. Defining topological transitivity of a map, they proved that for a Compact Metric Space topological transitivity is equivalent to existence of dense orbit \[\text{Result:1:1:11}\]. In [31] J.Doyne Farmer, Edward Ott and James A. Yorke have illustrated some applications of Li-Yorke Chaos. Marcy Barge and Joe Martin[54, 55], stated some conditions for dense periodic orbits, dense set of periodic points on a closed interval. Topological entropy defined by R.T. Adler, A.G. Konhecm and M.H. Mc Andrew[2] played an important role in the complexity of the space later[Definition:1:1:10]. J. Smital[69] gives a relationship between topological entropy and Li-Yorke Chaos. In 1986, K. Jankova and J. Smital[46] stated some equivalent conditions for Li-Yorke Chaos. In 1989, R.L. Devaney[30] defined Chaotic functions in general metric spaces, which became popular[Definition:1:1:7]. There are three conditions namely transitivity, dense set of periodic points and sensitive dependence for a function to be Chaotic. But in 1994, J. Banks, J. Brooks, G. Cairns, G. Davis and P. Stacey[14] proved a surprising result that the first two conditions of Devaney’s definition imply third one. Note that third one is well known as butterfly effect. In Devaney’s definition the first two conditions are topological and the third one depends on the metric. By the Result of J. Banks et al. it has been proved that property of being Chaotic is purely topological. It is unfortunate that the property from which a theory originated is found to be unnecessary later. In 1994, Michel Vellekoop and Raoul Berglund[59] proved that if we restrict Devaney’s definition in to intervals on \( \mathbb{R} \), a function is chaotic if it is transitive. In 1995, G.L. Forti, L. Paganmi and J. Smital [39] extended Li-Yorke chaos in to \( \mathbb{R}^2 \) and obtained some equivalent condition. Chaotic nature of triangular maps has also been studied in that paper. To get a visual idea of chaotic nature see [27]. Jan. M. Arts and Fons. G. M. Daalderooop, in 1998[1] studied chaotic homeomorphisms on manifolds. L. Alseda, S. Kolyada, J. Llibre and L. V. Snoha in 1999[3] obtained some connection between transitivity, density of the set of periodic points and topological entropy. In 1999 we have extended, Devaney’s definition of chaos in to topological spaces.
Theory of fractals has its origin with the work of Benoît Mandelbrot in 1977[52]. Roughly saying, fractals are “irregular” objects. By ‘regular’ we means objects which have nice geometric properties. According to a Mandelbrot, a set is called a fractal if its Hausdorff dimension is strictly greater than topological (small inductive) dimension. But the calculation of Hausdorff dimension is a tedious task. So a question arises- How can we characterise fractals? In 1981, John. E. Hutchinson [47] introduced self similar sets [Def: 1:2:2]. If every part of a set looks like the whole then it is called a self similar set. Some fractals are self similar; but not all. Some extra ordinary sets like Cantor set, Von- Koch curve, Sierpenski gasket which had their birth in early 20th century were found to be self similar fractals later. In the same way some Julia sets, due to the work of P.Fatou and G. Julia in 1920’s are self similar. Self similar sets, holomorphic dynamics and dimension properties are thrust area in the study of fractals. In 1982, F.M. Dekking [28] introduced recurrent sets as a generalization of self similar sets. Christoph Bandt [6-13] published a series of papers on self similar sets (1, 2, 3, 4, 5, 6, 7, 8) from 1989 to 1992. In 1987, W.J. Gilbert introduced partial self similar sets as a generalization of self similar sets in some other direction. A set is partial self similar if it consists of copies of different parts of it. In 1995, Kenneth Falconer [34] introduced subself similar sets as a generalization of self similar sets [Def: 1: 2: 3]. A set is subself similar if it is contained in a union of similar copies of it self. In 1995, R. Daniel Mauldin [24] extended theory of finite iterated function systems into infinite iterated function systems by introducing conformal iterated function systems. James Keesling[44] in 1999, studied the properties of boundaries of self similar sets. We can measure the irregularity of fractals by assigning it to a real number. Usually, Hausdorff dimension is used for this purpose. Similarity dimension measures the irregularity. Hausdorff dimension was introduced by Felix Hausdorff [41] in 1919. But its importance as a measure of irregularity was realized only after the introduction of fractals. In 1982, Claude Tricot[74] introduced packing dimension which may also be a non-integer. In 1984, Curtis Mcmullen [22] calculated the Hausdorff dimension of general Sierpinski carpets. Tim Bedford[ 72 ] in 1986 calculated Hausdorff dimension of some recurrent sets. The work of James Keesling [45] on Hausdorff dimension is from topological point of view. R.D. Mauldin and


We are not claming that the above references are complete. The literature is so vast that we can’t quote all the references. But the above quoted references are those which we used in our work and to prepare the thesis.

In this thesis, in chapter 1, we quote some definitions and results in the chaos theory (section 1:1) and theory of fractals (section 1:2) which are already in the literature. In chapter 2, we introduce chaos in topological spaces, we study some properties of chaos spaces which also include hyper spaces. Topological entropy is a measure to determine the complexity of the space. We study different properties of topological entropy in chaos spaces. By defining a measure on chaos spaces, we compare different chaos spaces. In chapter 3 we study some properties of self similar sets and partial self similar sets. We can associate a directed graph to each partial selfsimilar set. Dimension properties of partial self similar sets are studied using this graph. We introduce superself similar sets as a generalization of self similar sets. We
also prove that chaotic self similar self are dense in the hyper space. In chapter 4, we define Julia sets and Mandelbrot set on general topological spaces. We define generalized Julia sets and Mandelbrot sets on chaos spaces. In chapter 5, we study some relationships between different kinds of dimension and fractals. By defining regular sets through packing dimension in the same way as regular sets defined by K. Falconer through Hausdorff dimension, we examine different properties of regular sets.