Chapter 4

Product Cordial Labeling
4.1 Introduction

The previous chapter was devoted to the cordial labeling of graph while this chapter is aimed to discuss a labeling having cordial theme. Here the induced edge labels are product of vertex labels unlike the absolute difference in cordial labeling and the labeling is called product cordial. We contribute twelve new results in the context of product cordial labeling.

4.2 Product Cordial Labeling

We adhere all the notations which are introduced in the concept of cordial labeling except the induced edge labeling function. Here $f^* : E(G) \rightarrow \{0, 1\}$ defined as $f^*(e) = f(u)f(v)$.

**Definition 4.2.1.** A binary vertex labeling of graph $G$ is called a product cordial labeling if $|v_f(0) - v_f(1)| \leq 1$ and $|e_f(0) - e_f(1)| \leq 1$. A graph with a product cordial labeling is called a product cordial graph.

In the Figure 4.1 the star $K_{1,4}$ and its product cordial labeling is shown.

![Figure 4.1](image-url)
4.3 Some Existing Results on Product Cordial Labeling

The notion of product cordial labeling was introduced by Sundaram et al.\cite{50} and they proved that

- Every tree is product cordial.
- Every unicyclic graph of odd order is product cordial.
- Every triangular snake is product cordial.
- Dragons and helms are product cordial.
- $P_m \cup P_n$ is product cordial for all $m$ and $n$.
- $C_m \cup P_n$ is product cordial for all $m$ and $n$.
- $P_m \cup K_{1,n}$ is product cordial for all $m$ and $n$.
- $W_n \cup C_n$ is product cordial for all $m$ and $n$.
- $W_n \cup F_n$ is product cordial for all $m$ and $n$.
- $K_{1,m} \cup K_{1,n}$ is product cordial for all $m$ and $n$.
- $W_m \cup K_{1,n}$ is product cordial for all $m$ and $n$.
- The total graph of path $P_n$ is product cordial.
- The total graph of cycle $C_n$ is product cordial if and only if $n < 4$.
- The one point union of $t$ copies of $C_n$ is product cordial provided $t$ is even or both $t$ and $n$ are even.
- $K_2 + mK_1$ is product cordial if and only if $m$ is odd.
- $K_{m,n} \cup P_s$ is product cordial if $s > mn$.
- $C_{n+2} \cup K_{1,n}$ is product cordial if $n$ is odd.
• $K_n \cup K_{n-1,r}$ is product cordial if $n$ is even.

• $P_n^2$ is product cordial if and only if $n$ is odd.

• $K_{m,n}$ ($m, n > 2$) is not product cordial.

• $P_m \times P_n$ ($m, n > 2$) is not product cordial.

• Wheels $W_n$ are not product cordial.

• A graph $G$ with $p$ vertices and $q$ edges with $p \geq 4$ is product cordial then $q < \frac{p^2 - 1}{4}$.

Vaidya and Dani [53] proved that

• Graph $< S_n^{(1)} : S_n^{(2)} >$ is product cordial.

• Graph $< S_n^{(1)} : S_n^{(2)}, \ldots, S_n^{(k)} >$ is product cordial except for odd $k$ and even $n$.

• Graph $< K_{1,n}^{(1)} : K_{1,n}^{(2)} >$ is product cordial.

• Graph $< K_{1,n}^{(1)} : K_{1,n}^{(2)}, \ldots, K_{1,n}^{(k)} >$ is product cordial.

4.4 Product Cordial Labeling of Some Cycle Related Graphs

Theorem 4.4.1. The path union of $k$ copies of cycle $C_n$ is a product cordial graph except for odd $k$ and even $n$.

Proof. Let $G_1, G_2, \ldots, G_k$ be $k$ copies of the cycle $C_n$ and $G$ be the path union of cycle $C_n$. Let us denote the successive vertices of the $i^{th}$ copy graph $G_i$ by $u_{i1}, u_{i2}, \ldots, u_{in}$. Let $e_i = u_{i1}u_{i+1}$ be the edge joining $G_i$ and $G_{i+1}$ for $i = 1, 2, \ldots, k - 1$. We note that $|V(G)| = nk$ and $|E(G)| = nk + k - 1$.

To define binary vertex labeling $f : V(G) \rightarrow \{0, 1\}$ we consider following cases.
Case 1. $n \equiv 0 (mod 2)$

Subcase I $k \equiv 0 (mod 2)$

$$f(u_{ij}) = 0, 1 \leq j \leq n \quad 1 \leq i \leq \frac{k}{2}$$

$$f(u_{ij}) = 1, 1 \leq j \leq n \quad \frac{k}{2} < i \leq k$$

In view of the above defined labeling pattern $v_f(0) = v_f(1) = \frac{nk}{2}$ and $e_f(0) = e_f(1) + 1 = \frac{nk + k}{2}$.

Thus the graph $G$ satisfies the condition for product cordial graph. That is, $G$ admits product cordial labeling when both $n$ and $k$ are even.

Subcase II $k \equiv 1 (mod 2)$

In this case we observe that the graph $G$ under consideration is having three types of vertices

(i) 2 vertices of degree three,

(ii) $k - 2$ vertices of degree four,

(iii) $nk - k$ vertices of degree two.

If the graph under consideration admits product cordial labeling then it must have $v_f(0) = v_f(1) = \frac{nk}{2}$ as $n$ is even and $k$ is odd. It is obvious that the label of either of the end vertices is 0 then the induced edged labels are 0. Any pattern assigning vertex labels satisfying vertex condition will induce edge labels for $nk + k - 1$ number of edges in such a way that $|e_f(0) - e_f(1)| \geq 2$, that is edge condition for product cordial graph is violated.

Thus the graph $G$ under consideration is not a product cordial graph when $n$ is even and $k$ is odd.
Case 2. $n \equiv 1(\text{mod}2)$

**Subcase I** $k \equiv 0(\text{mod}2)$

\[
\begin{align*}
  f(u_{ij}) &= 0; \quad 1 \leq j \leq n \quad \left\{ \begin{array}{l}
    1 \leq i \leq \frac{k}{2}
  
\end{array} \right. \\
  f(u_{ij}) &= 1; \quad 1 \leq j \leq n \quad \left\{ \begin{array}{l}
    \frac{k}{2} < i \leq k
  
\end{array} \right.
\end{align*}
\]

In view of the above defined labeling pattern $v_f(0) = v_f(1) = \frac{nk}{2}$ and $e_f(0) = e_f(1) + 1 = \frac{nk+k}{2}$.

Thus the graph $G$ satisfies the condition for product cordial graph. That is, $G$ admits product cordial labeling when $n$ is odd and $k$ is even.

**Subcase II** $k \equiv 1(\text{mod}2)$

\[
\begin{align*}
  f(u_{ij}) &= 0; \quad 1 \leq j \leq n \quad \left\{ \begin{array}{l}
    1 \leq i \leq \frac{k+1}{2}
  
\end{array} \right. \\
  f(u_{ij}) &= 1; \quad 1 \leq j \leq \frac{n+1}{2} \quad \left\{ \begin{array}{l}
    i = \frac{k+1}{2} \\
    0, \quad \frac{n+1}{2} < j \leq n
  
\end{array} \right.
\end{align*}
\]

\[
\begin{align*}
  f(u_{ij}) &= 1; \quad 1 \leq j \leq n \quad \left\{ \begin{array}{l}
    \frac{k+1}{2} < i \leq k
  
\end{array} \right.
\end{align*}
\]

In view of the above defined labeling pattern $v_f(0) + 1 = v_f(1) = \frac{nk+1}{2}$, and $e_f(0) = e_f(1) + 1 = \frac{nk+k}{2}$.

Thus the graph $G$ satisfies the condition for product cordial graph. That is, $G$ admits product cordial labeling when $n$ and $k$ both are odd. \qed
**Illustration 4.4.2.** Consider a graph $G$ obtained by a path union of four copies of cycle $C_6$. It is the case when both $n$ and $k$ are even. The product cordial labeling is shown in Figure 4.2.

![Figure 4.2](image)

**Theorem 4.4.3.** The graph obtained by joining two copies of cycle $C_n$ by path $P_k$ admits product cordial labeling.

*Proof.* Let $G$ be the graph obtained by joining two copies of cycle $C_n$ by path $P_k$. Let $u_1, u_2, \ldots, u_n$ be the vertices of first copy of cycle $C_n$ and $v_1, v_2, \ldots, v_n$ be the vertices of second copy of cycle $C_n$. Let $w_1, w_2, \ldots, w_k$ be the vertices of path $P_k$ with $u_1 = w_1$ and $v_1 = w_k$. We note that $|V(G)| = 2n + k - 2$ and $|E(G)| = 2n + k - 1$.

To define binary vertex labeling $f : V(G) \to \{0, 1\}$ we consider following cases.

**Case 1.** $k \equiv 0 \, (mod\, 2)$

$f(u_i) = 0; 1 \leq i \leq n$

$f(v_i) = 1; 1 \leq i \leq n$

$f(w_j) = 0; 1 < j \leq \frac{k}{2}$

$= 1; \frac{k}{2} < j < k$

**Case 2.** $k \equiv 1 \, (mod\, 2)$

$f(u_i) = 0; 1 \leq i \leq n$

$f(v_i) = 1; 1 \leq i \leq n$

$f(w_j) = 0; 1 < j \leq \frac{k-1}{2}$

$= 1; \frac{k-1}{2} < j < k$
In view of above defined labeling pattern the graph $G$ under consideration admits product cordial labeling and in each case it satisfies the condition for product cordiality as shown in Table 4.1. That is, the graph obtained by joining two copies of cycle $C_n$ by path $P_k$ admits product cordial labeling.

(In the following table $n = 2a + b, k = 2c + d$ where $a, c \in N$.)

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</tr>
</tbody>
</table>

Table 4.1

**Illustration 4.4.4.** Consider a graph $G$ obtained by joining two copies of cycle $C_8$ by a path $P_3$. It is the case related to $n \equiv 0 (mod 2), k \equiv 1 (mod 2)$. The labeling is shown in Figure 4.3.

![Figure 4.3](image)

**Theorem 4.4.5.** The path union of $k$ copies $D_2(C_n)$ is a product cordial graph except for odd $k$.

**Proof.** Let $D_2(C_n)$ be the shadow graph of the cycle $C_n$. Let $G$ be the path union of $k$ copies of $G_1, G_2, \ldots, G_k$ of shadow graph of $D_2(C_n)$. Let $G'_1, G'_2, \ldots, G'_k$ be the first $k$ copies of cycle $C_n$ and $G''_1, G''_2, \ldots, G''_k$ be the second $k$ copies of cycle $C_n$ in $D_2(C_n)$. Let us denote the successive vertices of $G'_j$ by $u'_1, u'_2, \ldots, u'_m$ and $G''_j$ by $u''_1, u''_2, \ldots, u''_m$. 
Let \( e_i = u'_{i} u''_{i+1} \) be the edge joining \( G_i \) and \( G_{i+1} \) for \( i = 1, 2, \ldots, k - 1 \). We note that \(|V(G)| = 2nk\) and \(|E(G)| = 4nk + k - 1\).

To define binary vertex labeling \( f : V(G) \to \{0, 1\} \) we consider following cases.

**Case 1.** \( k \equiv 0 \text{(mod2)} \)

\[
\begin{align*}
  f(u'_{ij}) &= 0 & 1 \leq i \leq \frac{k}{2} \\
  f(u''_{ij}) &= 0 & 1 \leq j \leq n \\
  f(u'_{ij}) &= 1 & \frac{k}{2} < i \leq n \\
  f(u''_{ij}) &= 1 & 1 \leq j \leq n
\end{align*}
\]

In view of the above defined labeling pattern \( v_f(0) = v_f(1) = nk \) and \( e_f(0) = e_f(1) + 1 = 2nk + \frac{k}{2} \).

Thus the graph \( G \) satisfies the condition for product cordial graph. That is, \( G \) admits product cordial labeling when \( k \) is even.

**Case 2.** \( k \equiv 1 \text{(mod2)} \)

In this case \(|V(G)| = 2nk\) is even. Therefore, in order to satisfy the vertex condition for product cordiality and to minimize the edge labels with label 0, we label the vertices of first \( \frac{k-1}{2} \) copies of \( D_2(C_n) \) by 0 and last \( \frac{k-1}{2} \) copies of \( D_2(C_n) \) by 1. Now for the \( \left( \frac{k+1}{2} \right)^{th} \) copy of \( D_2(C_n) \), we label \( n \) vertices of degree four by 0 and remaining \( n \) vertices by 1 then \( |e_f(0) - e_f(1)| > 2 \). It is easy to verify that any other pattern to assign vertex labels satisfying the vertex condition will increase the difference between \( e_f(0) \) and \( e_f(1) \).

Thus the graph \( G \) under consideration is not a product cordial graph when \( k \) is odd. \( \square \)

**Illustration 4.4.6.** Consider a graph \( G \) obtained by the path union of two copies of shadow graph \( D_2(C_4) \) of cycle \( C_4 \). It is the case related to \( k \) is even. The labeling is shown in Figure 4.4.
Figure 4.4

Theorem 4.4.7. The graph $G$ obtained by joining two copies of shadow graph $D_2(C_n)$ by a path of arbitrary length is a product cordial graph.

Proof. Let $D_2(C_n)$ be the shadow graph of cycle $C_n$. Let $G$ be the graph obtained by joining two copies of $D_2(C_n)$ by a path $P_k$ of arbitrary length $k - 1$. Let $u'_1, u'_2, \ldots, u'_n$ be the vertices of $C'_n$ and $u''_1, u''_2, \ldots, u''_n$ be the vertices of $C''_n$ in first copy of $D_2(C_n)$ in $G$. Let $v'_1, v'_2, \ldots, v'_n$ be the vertices of $C'_n$ and $v''_1, v''_2, \ldots, v''_n$ be the vertices of $C''_n$ in second copy of $D_2(C_n)$ in $G$. Let $w_1, w_2, \ldots, w_k$ be the vertices of path $P_k$ with $u''_1 = w_1$ and $v''_1 = w_k$. We note that $|V(G)| = 4n + k - 2$ and $|E(G)| = 8n + k - 1$.

To define binary vertex labeling $f : V(G) \to \{0, 1\}$ we consider following cases.

Case 1. $k \equiv 0(\text{mod}2)$

\[
\begin{align*}
  f(u'_i) &= 0 \quad 1 \leq i \leq n \\
  f(u''_i) &= 0 \\
  f(v'_i) &= 1 \quad 1 \leq i \leq n \\
  f(v''_i) &= 1
\end{align*}
\]
Case 2. \( k \equiv 1 \mod 2 \)

\[
\begin{align*}
  f(w_j) &= 0 \quad 1 < j \leq \frac{k}{2} \\
  f(w_j) &= 1 \quad \frac{k}{2} < j < k
\end{align*}
\]

The labeling pattern defined above exhausts all the possibilities. In each case the graph \( G \) under consideration satisfies the conditions for product cordiality as shown in Table 4.2. That is, the graph \( G \) obtained by joining two copies of shadow graph \( D_2(C_n) \) by a path of arbitrary length is a product cordial graph.  

(In the following table \( n = 2a + b, k = 2c + d \) where \( a, c \in \mathbb{N} \).)

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</tr>
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\[\text{Table 4.2}\]
Illustration 4.4.8. Consider a graph $G$ obtained by joining two copies of shadow graph $D_2(C_5)$ by a path $P_3$. It is the case related with $k \equiv 1 \pmod{2}$. The labeling pattern is shown in Figure 4.5.

\[\text{Figure 4.5}\]

Theorem 4.4.9. The path union $k$ copies of $C_n(C_n)$ is a product cordial graph except for odd $k$.

Proof. Let $G$ be the path union of $k$ copies of $C_n(C_n)$. Let $G_1, G_2, ..., G_k$ be $k$ copies of cycle $C_n$ and $G'_1, G'_2, ..., G'_k$ be $k$ copies of cycle $C_n$ which are obtained by joining each newly inserted vertices of adjacent edges by an edge. Next denote the successive vertices of $G_i$ by $u_{i1}, u_{i2}, ..., u_{in}$ and corresponding vertices of $G'_i$ by $u'_{i1}, u'_{i2}, ..., u'_{in}$. Let $e_i = u_{i1}u_{(i+1)1}$ be the edge joining $i^{th}$ copy and $(i+1)^{th}$ copy of $C_n(C_n)$ for $i = 1, 2, \ldots, k-1$. Here we note that $|V(G)| = 2nk$ and $|E(G)| = 3nk + k - 1$.

To define binary vertex labeling $f : V(G) \to \{0, 1\}$ we consider following cases.

Case 1. $k \equiv 0 \pmod{2}$

\[
\begin{align*}
  f(u'_{ij}) &= 0; 1 \leq j \leq n \\
  f(u''_{ij}) &= 0; 1 \leq j \leq n \\
  f(u'_{ij}) &= 1; 1 \leq j \leq n \\
  f(u''_{ij}) &= 1; 1 \leq j \leq n
\end{align*}
\]

\[1 \leq i \leq \frac{k}{2} \]

\[\frac{k}{2} < i \leq k \]
In view of above defined labeling pattern \( v_f(0) = v_f(1) = nk \) and \( e_f(0) = e_f(1) + 1 = \frac{3nk+k}{2} \).

**Case 2.** \( k \equiv 1(\text{mod}2) \)

In this case \( |V(G)| = 2nk \) is even. Therefore in order to satisfy the vertex condition for product cordiality and to minimize the edge labels with label 0, we label vertices of first \( \frac{k-1}{2} \) copies of \( C_n(C_n) \) by 0 and last \( \frac{k-1}{2} \) copies of \( C_n(C_n) \) by 1. Now for the \( \left( \frac{k+1}{2} \right) \) th copy of \( C_n(C_n) \) we label \( n \) adjacent vertices by 0 and remaining \( n \) vertices by 1 then \( |e_f(0) - e_f(1)| > 2 \). It is easy to verify that any other pattern to assign vertex labels satisfying the vertex condition will increase the difference between \( e_f(0) \) and \( e_f(1) \). Thus the graph under consideration is not a product cordial graph when \( k \) is odd.

**Illustration 4.4.10.** In the Figure 4.6 the product cordial labeling for the path union of two copies of \( C_6(C_6) \) is demonstrated.(The hollow vertices are newly inserted vertices for barycentric subdivision of \( C_6 \).)

![Figure 4.6](image)

**Theorem 4.4.11.** The graph obtained by joining two copies of \( C_n(C_n) \) by a path of arbitrary length is a product cordial graph.

**Proof.** Let \( G \) be the graph obtained by joining two copies of \( C_n(C_n) \) by a path of arbitrary length \( k - 1 \). Let \( u_1, u_2, \ldots, u_n \) be the vertices of cycle \( C_n \) and \( u'_1, u'_2, \ldots, u'_n \) be the corresponding vertices of the cycle which is obtained by joining newly inserted vertices...
of adjacent edges in cycle $C_n$. Next denote the corresponding vertices in second copy of $C_n(C_n)$ by $v_1, v_2, \ldots, v_n$ and $v'_1, v'_2, \ldots, v'_n$ respectively. Let $w_1, w_2, \ldots, w_k$ be the vertices of path $P_k$ with $u_1 = w_1$ and $v_1 = w_k$. Here we note that $|V(G)| = 4n + k - 2$ and $|E(G)| = 6n + k - 1$.

To define binary vertex labeling $f : V(G) \rightarrow \{0, 1\}$ following cases are in our consideration.

**Case 1.** If $k \equiv 0 (\text{mod} 2)$

\[
\begin{align*}
    f(u_i) &= 0 & 1 \leq i \leq n \\
f(u'_i) &= 0 & 1 \leq i \leq n \\
f(v_i) &= 1 & 1 \leq i \leq n \\
f(v'_i) &= 1 & 1 \leq i \leq n \\
f(w_j) &= 0 & 1 < i \leq \frac{k}{2} \\
f(w_j) &= 1 & \frac{k}{2} < j \leq k
\end{align*}
\]

**Case 2.** If $k \equiv 1 (\text{mod} 2)$

\[
\begin{align*}
    f(u_i) &= 0 & 1 \leq i \leq n \\
f(u'_i) &= 0 & 1 \leq i \leq n \\
f(v_i) &= 1 & 1 \leq i \leq n \\
f(v'_i) &= 1 & 1 \leq i \leq n
\end{align*}
\]
The labeling pattern defined above includes all possible arrangement of vertices. In each case the graph $G$ under consideration satisfies the conditions for product cordiality as shown in Table 4.3. That is, the graph obtained by joining two copies of $C_n(C_n)$ by a path of arbitrary length is a product cordial graph.

(In the following table $n = 2a + b, k = 2c + d$ where $a, c \in N$.)

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**Table 4.3**

**Illustration 4.4.12.** Consider a graph $G$ obtained by joining two copies of $C_4(C_4)$ by a path $P_3$. The labeling pattern is shown in Figure 4.7. (The hollow vertices are newly inserted vertices for barycentric subdivision of $C_4$.)

![Figure 4.7](image-url)
4.5 Product Cordial Labeling of Some Wheel and Shell Related Graphs

**Theorem 4.5.1.** The path union of \( k \) copies of wheel \( W_n \) is a product cordial graph except for odd \( k \).

**Proof.** Let \( G_1, G_2, \ldots, G_k \) be \( k \) copies of the wheel \( W_n \) and \( G \) be the path union of wheel \( W_n \). Let us denote the successive rim vertices of the \( i^{th} \) copy of \( G_i \) by \( u_{1i}, u_{2i}, \ldots, u_{ni} \) and the apex vertex by \( u_{0i} \). Let \( e_i = u_{1i}u_{(i+1)i} \) be the edge joining \( G_i \) and \( G_{i+1} \) for \( i = 1, 2, \ldots, k-1 \). We note that \( |V(G)| = (n+1)k \) and \( |E(G)| = 2nk + k - 1 \).

To define binary vertex labeling \( f : V(G) \rightarrow \{0, 1\} \) we consider following cases.

**Case 1.** \( k \equiv 0 \pmod{2} \)

\[
f(u_{ij}) = 0 ; 0 \leq j \leq n, 1 \leq i \leq \frac{k}{2}
\]

\[
f(u_{ij}) = 1 ; 0 \leq j \leq n, \frac{k}{2} < i \leq k
\]

In view of the above defined labeling pattern \( v_f(0) = v_f(1) = \frac{(n+1)k}{2} \) and \( e_f(0) = e_f(1) + 1 = nk + \frac{k}{2} \).

Thus the graph \( G \) satisfies the condition for product cordial graph. That is, \( G \) admits product cordial labeling for even \( k \).

**Case 2.** \( k \equiv 1 \pmod{2} \)

**Subcase I** \( n \equiv 0 \pmod{2} \)

In this case \( |V(G)| = (n+1)k \) is odd. Therefore, in order to satisfy the vertex condition for product cordiality and to minimize the number of edges with label 0 we label vertices of first \( \frac{n-1}{2} \) copies of \( W_n \) by 0 and last \( \frac{n-1}{2} \) copies of \( W_n \) by 1. Now for the \( (\frac{k+1}{2})^{th} \) copy of \( W_n \) we label \( \frac{n}{2} \) vertices of degree three by 0 and remaining \( \frac{n+2}{2} \) vertices by 1. Then \( v_f(0) + 1 = v_f(1) = \frac{(n+1)k}{2} \) but \( |e_f(0) - e_f(1)| \geq 2 \). It is easy to verify that all other pattern to assign vertex labels satisfying the vertex condition will increase the difference between \( e_f(0) \) and \( e_f(1) \).

**Subcase II** \( n \equiv 1 \pmod{2} \)

In this case \( |V(G)| = (n+1)k \) is even. Therefore, in order to satisfy the vertex condition
for product cordiality to minimize the number of edges with label 0 we label vertices of first \( \frac{k-1}{2} \) copies of \( W_n \) by 0 and last \( \frac{k-1}{2} \) copies of \( W_n \) by 1. Now for the \(( \frac{k+1}{2} )\)th copy of \( W_n \) we label \( \frac{n+1}{2} \) vertices of degree three by 0 and remaining \( \frac{n+1}{2} \) vertices by 1. Then \( v_j(0) = v_j(1) = \frac{(n + 1)k}{2} \) but \( |e_f(0) - e_f(1)| > 2 \). It is easy to verify that all other pattern to assign vertex labels satisfying the vertex condition will increase the difference between \( e_f(0) \) and \( e_f(1) \).

Thus we conclude that the graph \( G \) under consideration is not a product cordial graph for odd \( k \).

Illustration 4.5.2. Consider the graph \( G \) obtained by path union of four copies of wheel \( W_5 \). Here \( n = 5 \) and \( k = 4 \). It is the case related to odd \( n \) and even \( k \). The product cordial labeling is shown in Figure 4.8.

![Figure 4.8](image)

Theorem 4.5.3. The graph obtained by joining two copies of wheel \( W_n \) by a path \( P_k \) admits product cordial labeling.

Proof. Let \( G \) be the graph obtained by joining two copies of wheel \( W_n \) by a path \( P_k \). Let \( u_1, u_2, \ldots, u_n \) be the vertices of first copy of wheel \( W_n \) and \( v_1, v_2, \ldots, v_n \) be the vertices of second copy of wheel \( W_n \). Let \( u_0 \) and \( v_0 \) be the apex vertices of first copy and second copy of \( W_n \) respectively. Let \( w_1, w_2, \ldots, w_k \) be the vertices of path \( P_k \) with \( u_1 = w_1 \) and \( v_1 = w_k \). We note that \( |V(G)| = 2n + k \) and \( |E(G)| = 4n + k - 1 \).

To define binary vertex labeling \( f : V(G) \rightarrow \{0, 1\} \) we consider following cases.

Case 1. \( k \equiv 0(\text{mod}2) \)

\[
f(u_i) = 0; 0 \leq i \leq n
\]

\[
f(v_i) = 1; 0 \leq i \leq n
\]
$f(w_j) = 0; 1 < j \leq \frac{k}{2}$
$= 1; \frac{k}{2} < j < k$

**Case 2.** If $k \equiv 1 (mod\ 2)$

$f(u_i) = 0; 0 \leq i \leq n$
$f(v_i) = 1; 0 \leq i \leq n$
$f(w_j) = 0; 1 < j \leq \frac{k-1}{2}$
$= 1; \frac{k-1}{2} < j < k$

The labeling pattern defined above includes all possible arrangement of vertices. In each case the graph $G$ under consideration satisfies the conditions for product cordiality as shown in Table 4.4. That is, the graph obtained by joining two copies of wheel $W_n$ by a path $P_k$ admits product cordial labeling.

(In the following table $n = 2a + b$, $k = 2c + d$ where $a, c \in N$.)

<table>
<thead>
<tr>
<th>b</th>
<th>d</th>
<th>Vertex conditions</th>
<th>Edge conditions</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>$v_f(0) = v_f(1) = \frac{2n+k}{2}$</td>
<td>$e_f(0) = e_f(1) + 1 = \frac{4n+k}{2}$</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>$v_f(0) + 1 = v_f(1) = \frac{2n+k+1}{2}$</td>
<td>$e_f(0) = e_f(1) = \frac{4n+k-1}{2}$</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>$v_f(0) = v_f(1) = \frac{2n+k}{2}$</td>
<td>$e_f(0) = e_f(1) + 1 = \frac{4n+k}{2}$</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>$v_f(0) + 1 = v_f(1) = \frac{2n+k+1}{2}$</td>
<td>$e_f(0) = e_f(1) = \frac{4n+k-1}{2}$</td>
</tr>
</tbody>
</table>

**Table 4.4**

**Illustration 4.5.4.** Consider a graph $G$ obtained by joining two copies of wheel $W_7$ by a path $P_4$. The product cordial labeling is shown in Figure 4.9.

![Figure 4.9](image-url)
Theorem 4.5.5. The path union of \( k \) copies of shell \( S_n \) is a product cordial graph except for odd \( k \) and even \( n \).

**Proof.** Let \( G_1, G_2, \ldots, G_k \) be \( k \) copies of shell \( S_n \) and \( G \) be the path union of \( S_n \). Let us denote the successive vertices of \( i^{th} \) copy \( G_i \) by \( u_{i1}, u_{i2}, \ldots, u_{in} \). Let \( u_{i1} \) be the apex vertex of \( G_i \). Without loss of generality we start the label assignment to vertices of \( G_i \) in clockwise direction. Let \( e_i = u_{i1}u_{(i+1)1} \) be the edge joining \( G_i \) and \( G_{i+1} \) for \( i = 1, 2, \ldots, k - 1 \). We note that \(|V(G)| = nk\) and \(|E(G)| = 2k(n - 1) - 1\).

To define binary vertex labeling \( f : V(G) \to \{0, 1\} \) we consider following cases.

**Case 1.** \( n \equiv 0(\text{mod}2) \)

**Subcase I** \( k \equiv 0(\text{mod}2) \)

\[
f(u_{ij}) = 0; \quad 1 \leq j \leq n \quad \text{for} \quad 1 \leq i \leq \frac{k}{2}
\]

\[
f(u_{ij}) = 1; \quad 1 \leq j \leq n \quad \text{for} \quad \frac{k}{2} < i \leq k
\]

In view of the above defined labeling pattern \( v_f(0) = v_f(1) = \frac{nk}{2} \) and \( e_f(0) = e_f(1) + 1 = k(n - 1) \).

Thus the graph \( G \) satisfies the condition for product cordial graph. That is, \( G \) admits product cordial labeling when \( n \) and \( k \) are even.

**Subcase II** \( k \equiv 1(\text{mod}2) \)

In this case we observe that the graph \( G \) under consideration is having four types of vertices

- 2\( k \) vertices of degree two,
- \((n - 3)k\) vertices of degree three,
- 2 vertices of degree \( n \),
- \( k - 2 \) vertices of degree \( n + 1 \).
If the graph under consideration admits product cordial labeling then it must have \( v_f(0) = v_f(1) = \frac{n+1}{2} \) as \( n \) is even and \( k \) is odd. It is obvious that label of the either end vertices is 0 then the induced edge label is zero. Any pattern of assigning vertex labels satisfying vertex condition will induce edge labels for \( 2k(n-1) - 1 \) edges in such a way that \( |e_f(0) - e_f(1)| > 2 \).

Thus the graph \( G \) under consideration will violate the edge condition for product cordiality. That is, \( G \) is not a product cordial graph when \( k \) is odd and \( n \) is even.

**Case 2.** \( n \equiv 1(\text{mod}2) \)

**Subcase I** \( k \equiv 0(\text{mod}2) \)

\[
\begin{align*}
   f(u_{ij}) &= 0; \ 1 \leq j \leq n \quad 1 \leq i \leq \frac{k}{2} \\
   f(u_{ij}) &= 1; \ 1 \leq j \leq n \quad \frac{k}{2} < i \leq k
\end{align*}
\]

In view of the above defined labeling pattern \( v_f(0) = v_f(1) = \frac{n+1}{2} \) and \( e_f(0) = e_f(1) + 1 = k(n-1) \).

Thus the graph \( G \) satisfies the condition for product cordial graph. That is, \( G \) admits product cordial labeling when \( n \) is odd and \( k \) is even.

**Subcase II** \( k \equiv 1(\text{mod}2) \)

\[
\begin{align*}
   f(u_{ij}) &= 0; \ 1 \leq j \leq n \quad 1 \leq i \leq \frac{k-1}{2} \\
   f(u_{ij}) &= 1; \ j = 1 \\
   f(u_{ij}) &= 0; \ j = 2 \\
   f(u_{ij}) &= 1; \ 3 < j \leq \frac{n+3}{2} \quad i = \frac{k+1}{2} \\
   f(u_{ij}) &= 0; \ \frac{n+3}{2} < j \leq n - 1 \\
   f(u_{ij}) &= 0; \ j = n
\end{align*}
\]
In view of the above defined labeling pattern \( v_f(0) + 1 = v_f(1) = \frac{nk+1}{2} \) and \( e_f(0) = e_f(1) + 1 = k(n-1) \).

Thus we conclude that the graph \( G \) admits product cordial labeling except for the case when \( n \) is even and \( k \) is odd. \( \square \)

**Illustration 4.5.6.** In the Figure 4.10 the product cordial labeling for the path union of three copies of shell \( S_7 \) is demonstrated.

![Figure 4.10](image).

**Theorem 4.5.7.** The graph \( G \) obtained by joining two copies of shell \( S_n \) by a path of arbitrary length is a product cordial graph.

**Proof.** Let \( G \) be the graph obtained by joining two copies of shell \( S_n \) by a path \( P_k \). Let \( u_1, u_2, \ldots, u_n \) be the vertices of first copy of shell \( S_n \) and \( v_1, v_2, \ldots, v_n \) be the vertices of second copy of shell \( S_n \). Let \( w_1, w_2, \ldots, w_k \) be the vertices of path \( P_k \) with \( u_1 = w_1 \) and \( v_1 = w_k \). We note that \( |V(G)| = 2n + k - 2 \) and \( |E(G)| = 4n + k - 9 \).

To define binary vertex labeling \( f : V(G) \to \{0, 1\} \) we consider following cases.

**Case 1.** \( k \equiv 0 (mod 2) \)

\[
f(u_i) = 0; 1 \leq i \leq n \\
f(v_i) = 1; 1 \leq i \leq n \\
f(w_j) = 0; 1 < j \leq \frac{k}{2} \\
= 1; \frac{k}{2} < j < k
\]
**Case 2.** \( k \equiv 1 \pmod{2} \\

\begin{align*}
f(u_i) &= 0; 1 \leq i \leq n \\
f(v_i) &= 1; 1 \leq i \leq n \\
f(w_j) &= 0; 1 < j \leq \frac{k-1}{2} \\
&= 1; \frac{k-1}{2} < j < k
\end{align*}

The labeling pattern defined above includes all possible arrangement of vertices. In each case the graph \( G \) under consideration satisfies the conditions for product cordiality as shown in Table 4.5. That is, the graph \( G \) obtained by joining two copies of shell \( S_n \) by a path of arbitrary length is a product cordial graph.

\( \Box \)

(In the following table \( n = 2a + b, k = 2c + d \) where \( a, c \in \mathbb{N} \).)

<table>
<thead>
<tr>
<th>b</th>
<th>d</th>
<th>Vertex conditions</th>
<th>Edge conditions</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>( v_f(0) = v_f(1) = \frac{2n+k-2}{2} )</td>
<td>( e_f(0) = e_f(1) + 1 = \frac{4n-6+k}{2} )</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>( v_f(0) = v_f(1) = \frac{2n+k-3}{2} )</td>
<td>( e_f(0) = e_f(1) = \frac{4n-7+k}{2} )</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>( v_f(0) = v_f(1) = \frac{2n+k-2}{2} )</td>
<td>( e_f(0) = e_f(1) + 1 = \frac{4n-6+k}{2} )</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>( v_f(0) + 1 = v_f(1) = \frac{2n+k-1}{2} )</td>
<td>( e_f(0) = e_f(1) = \frac{4n-7+k}{2} )</td>
</tr>
</tbody>
</table>

**Table 4.5**

**Illustration 4.5.8.** Consider a graph \( G \) obtained by joining two copies of shell \( S_8 \) by a path \( P_3 \). The labeling pattern is shown in Figure 4.11.
4.6 Product Cordial Labeling of Some Graphs Related to Petersen Graph

**Theorem 4.6.1.** The path union of $k$ copies of Petersen graph is a product cordial graph except for odd $k$.

**Proof.** Let $G$ be the path union of $k$ copies $G_1, G_2, \ldots, G_k$ of Petersen graph. Let us denote the external vertices of $G_i$ by $u_{i1}, u_{i2}, u_{i3}, u_{i4}, u_{i5}$ and internal vertices by $u_{i6}, u_{i7}, u_{i8}, u_{i9}, u_{i10}$. Let $e_i = u_{i1}u_{i(i+1)1}$ be the edge joining $G_i$ and $G_{i+1}$ for $i = 1, 2, \ldots, k - 1$. We note that $|V(G)| = 10k$ and $|E(G)| = 16k - 1$.

To define binary vertex labeling $f : V(G) \to \{0, 1\}$ we consider following cases.

**Case 1.** $k \equiv 0 (\text{mod} 2)$

- $f(u_{ij}) = 0; 1 \leq i \leq \frac{k}{2}, 1 \leq j \leq 10$
- $f(u_{ij}) = 1; \frac{k}{2} < i \leq k, 1 \leq j \leq 10$

In view of above defined labeling pattern $v_f(0) = v_f(1) = 5k$ and $e_f(0) = e_f(1) + 1 = 8k$.

**Case 2.** $k \equiv 1 (\text{mod} 2)$

In this case $|V(G)| = 10k$ is even. Therefore, in order to satisfy the vertex condition for product cordiality and to minimize the edge labels with label 0, we label the vertices of first $\frac{k-1}{2}$ copies of $G$ by 0 and last $\frac{k-1}{2}$ copies of $G$ by 1. Now for the $(\frac{k+1}{2})th$ copy of $G$, we label 5 vertices of degree three by 0 and remaining 5 vertices by 1 then $|e_f(0) - e_f(1)| > 2$.

It is easy to verify that any other pattern to assign vertex labels satisfying the vertex condition will increase the difference between $e_f(0)$ and $e_f(1)$.

Thus we conclude that the graph $G$ under consideration is not a product cordial graph for odd $k$. \hfill \Box

**Illustration 4.6.2.** In the Figure 4.12 the product cordial labeling for the path union of two copies of Petersen graph is demonstrated.
Theorem 4.6.3. The graph $G$ obtained by joining two copies of Petersen graph by a path of arbitrary length is a product cordial graph.

Proof. Let $G_1$ and $G_2$ be two copies of a Petersen graph. Let $G$ be the graph obtained by joining $G_1$ and $G_2$ by a path of arbitrary length $k - 1$. Let $u_1, u_2, u_3, u_4, u_5$ be the external vertices of $G_1$ and $u_6, u_7, u_8, u_9, u_{10}$ be the internal vertices of $G_1$. Let $v_1, v_2, v_3, v_4, v_5$ be the external vertices of $G_2$ and $v_6, v_7, v_8, v_9, v_{10}$ be the internal vertices of $G_2$. Let $w_1, w_2, \ldots, w_k$ be the vertices of path $P_k$ with $u_1 = w_1$ and $v_1 = w_k$. We note that $|V(G)| = 20 + k - 2$ and $|E(G)| = 30 + k - 1$.

To define binary vertex labeling $f : V(G) \rightarrow \{0, 1\}$ we consider following cases.

**Case 1.** $k \equiv 0(mod 2)$

$f(u_i) = 0; 1 \leq i \leq 10$

$f(v_i) = 1; 1 \leq i \leq 10$

$f(w_j) = 0; 1 < j \leq \frac{k}{2}$

$= 1; \frac{k}{2} < j < k$

**Case 2.** $k \equiv 1(mod 2)$

$f(u_i) = 0; 1 \leq i \leq 10$

$f(v_i) = 1; 1 \leq i \leq 10$

$f(w_j) = 0; 1 < j \leq \frac{k-1}{2}$

$= 1; \frac{k-1}{2} < j < k$
The labeling pattern defined above exhausts all the possibilities. In each case the graph $G$ under consideration satisfies the conditions for product cordiality as shown in Table 4.6. That is, the graph $G$ obtained by joining two copies of Petersen graph by a path of arbitrary length is a product cordial graph.

(In the following table $k = 2c + d$ where $c \in N$.)

<table>
<thead>
<tr>
<th>$d$</th>
<th>Vertex conditions</th>
<th>Edge conditions</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>$v_f(0) = v_f(1) = 10 + \frac{k}{2}$</td>
<td>$e_f(0) = e_f(1) + 1 = 15 + \frac{k}{2}$</td>
</tr>
<tr>
<td>1</td>
<td>$v_f(0) + 1 = v_f(1) = 10 + \frac{k+1}{2}$</td>
<td>$e_f(0) = e_f(1) = 15 + \frac{k+1}{2}$</td>
</tr>
</tbody>
</table>

Table 4.6

**Illustration 4.6.4.** Consider a graph $G$ obtained by joining two copies of Petersen graph by a path $P_6$. The labeling pattern is shown in Figure 4.13.
4.7 Conclusion and Scope

The product cordial labeling is discussed in detail and some existing results are reported. We have investigated twelve new results.

Similar investigations can be carried out in the context of different labeling techniques and to investigate necessary and sufficient condition for the existence of product cordial labeling is an open area of research. The content of this chapter give rise to following research paper.

  (www.eashwarpublications.com)

The reprint of this research paper is given in annexure.

The next chapter is focused on strongly multiplicative labeling of graphs.