Chapter 7

Mean Labeling
Chapter 7. Mean Labeling

7.1 Introduction

In this chapter we investigate three new results on the concept of mean labeling of graphs.

7.2 Mean Labeling

Definition 7.2.1. A function $f$ is called a mean labeling of a graph $G$ if $f : V(G) \rightarrow \{0, 1, 2, \ldots, q\}$ is injective and the induced function $f^* : E(G) \rightarrow \{1, 2, \ldots, q\}$ defined as

$$f^*(e = uv) = \frac{f(u) + f(v)}{2}; \text{ if } f(u) + f(v) \text{ is even}$$

$$= \frac{f(u) + f(v) + 1}{2}; \text{ if } f(u) + f(v) \text{ is odd}$$

is bijective. A graph which admits mean labeling is called a mean graph.

In the Figure 7.1 cycle $C_5$ and its mean labeling is shown.

![Figure 7.1](image)

7.3 Some Existing Results on Mean Labeling

Somasundaram and Ponraj[37, 46–49] have introduced the notion of mean labeling and they proved that

- $P_n$ is a mean graph for any $n \in N$. 
• $C_n$ is a mean graph for any $n \in N$.

• $K_{2,n}$ is a mean graph for any $n \in N$.

• $C_m \cup P_n$ is a mean graph for any $m, n \in N$.

• $P_m \times P_n$ is a mean graph for any $m, n \in N$.

• $P_m \times C_n$ is a mean graph for any $m, n \in N$.

• $K_n$ is a mean graph if and only if $n < 3$.

• $K_{1,n}$ is a mean graph if and only if $n < 3$.

• Bistars $B_{m,n}$ is a mean graph if and only if $m < n + 2$.

• The subdivision graph of the star $K_{1,n}$ is a mean graph if and only if $n < 4$.

• The wheel $W_n$ is not a mean graph for $n > 3$.

Vaidya and Lekha[57] have proved that

• The graph obtained by duplicating an arbitrary vertex of cycle $C_n$ admits mean labeling.

• The graph obtained by duplicating an arbitrary edge in cycle $C_n$ is a mean graph.

• The joint sum of two copies of cycle $C_n$ admits mean labeling.

• Fusion of two vertices $v_i$ and $v_j$ (where $d(v_i, v_j) \geq 3$) in cycle $C_n$ produces a mean graph.
7.4 Some New Mean Graphs

Theorem 7.4.1. The graph obtained by the path union of \( k \) copies of cycle \( C_n \) is a mean graph.

Proof. Let \( G \) be the path union of cycle \( C_n \) and \( G_1, G_2, \ldots, G_k \) be \( k \) copies of the cycle \( C_n \). Let us denote the successive vertices of the graph \( G_i \) by \( u_{i1}, u_{i2}, \ldots, u_{in} \). Let \( e_i = u_{i1}u_{(i+1)1} \) be the edge joining \( G_i \) and \( G_{i+1} \) for \( i = 1, 2, \ldots, k-1 \). We note that \( |V(G)| = nk \) and \( |E(G)| = nk + k - 1 \).

To define mean labeling \( f : V(G) \rightarrow \{0, 1, 2, \ldots, nk+(k-1)\} \) we consider following cases.

Case 1. If \( n \equiv 0(\text{mod}2) \)

\[
\begin{align*}
  f(u_{ij}) &= in + (i-1); \text{if } j = 1 \\
  f(u_{ij}) &= 2(n - j) + 3 + (\frac{i}{2} - 1)(2n + 2); 2 \leq j \leq \frac{n}{2} + 1 \\
  &= 2j + (\frac{i}{2} - 1)(2n + 2); \frac{n}{2} + 1 < j \leq n \\
\end{align*}
\]

\[
\begin{align*}
  f(u_{ij}) &= (i-1)(n+1); \text{if } j = 1 \\
  f(u_{ij}) &= (2j - 1) + ((\frac{i+1}{2}) - 1)(2n + 2); 2 \leq j \leq \frac{n}{2} \\
  &= 2(n - j + 1) + (\frac{i+1}{2} - 1)(2n + 2); \frac{n}{2} < j \leq n \\
\end{align*}
\]

Case 2. If \( n \equiv 1(\text{mod}2) \)

\[
\begin{align*}
  f(u_{ij}) &= in + (i-1); \text{if } j = 1 \\
  f(u_{ij}) &= 2(n - j) + 3 + (\frac{i}{2} - 1)(2n + 2); 2 \leq j \leq \frac{n-1}{2} \\
  &= 2j + (\frac{i}{2} - 1)(2n + 2); \frac{n-1}{2} < j \leq n \\
\end{align*}
\]
\[ f(u_{ij}) = (i - 1)(n + 1); \text{if } j = 1 \]
\[ f(u_{ij}) = (2j - 1) + (\frac{i+1}{2} - 1)(2n + 2) \]
where \( 2 \leq j \leq \frac{n+1}{2} \)
\[ = 2(n - j + 1) + (\frac{i+1}{2} - 1)(2n + 2) \]
where \( \frac{n+1}{2} < j \leq n \)

The labeling pattern defined above covers all the possibilities and in each case the graph \( G \) under consideration admits mean labeling. That is, the graph obtained by the path union of \( k \) copies of cycle \( C_n \) is a mean graph. \( \square \)

**Illustration 7.4.2.** Consider a graph \( G \) obtained by path union of three copies of cycle \( C_7 \). It is the case related to \( n \equiv 1(\text{mod}2) \). The mean labeling is as shown in Figure 7.2.

**Theorem 7.4.3.** The graph obtained by joining two copies of cycle \( C_n \) by a path \( P_k \) is a mean graph.

**Proof.** Let \( G \) be the graph obtained by joining two copies of cycle \( C_n \) by a path \( P_k \). Let \( u_1, u_2, \ldots, u_n \) be the vertices of first copy of cycle \( C_n \) and \( v_1, v_2, \ldots, v_n \) be the vertices of second copy of cycle \( C_n \). Let \( w_1, w_2, \ldots, w_k \) be the vertices of path \( P_k \) with \( u_1 = w_1 \) and \( v_1 = w_k \). We note that \( |V(G)| = 2n + k - 2 \) and \( |E(G)| = 2n + k - 1 \).

To define mean labeling \( f : V(G) \to \{1, 2, \ldots, 2n, 2n + 1, \ldots, 2n + k - 1\} \) we consider following cases.
Case 1. If \( n \equiv 0 \pmod{2} \)

Subcase I \( k \equiv 0 \pmod{2} \)

\[ f(u_1) = 0; \]
\[ f(u_i) = 2i - 1; 1 < i \leq \frac{n}{2} \]
\[ = 2(n - i + 1); \frac{n}{2} < i \leq n \]
\[ f(v_1) = 2n + 1; \]
\[ f(v_i) = 2(n - i) + 3; 1 < i \leq \frac{n}{2} + 1 \]
\[ = 2i; \frac{n}{2} + 1 < i \leq n \]
\[ f(w_j) = 2(n + j - 1); 1 < j \leq \frac{k}{2} \]
\[ = 2(n + k - j) + 1; \frac{k}{2} < j < k \]

Subcase II \( k \equiv 1 \pmod{2} \)

\[ f(u_1) = 0; \]
\[ f(u_i) = 2i - 1; 1 < i \leq \frac{n}{2} \]
\[ = 2(n - i + 1); \frac{n}{2} < i \leq n \]
\[ f(v_1) = 2n + 1; \]
\[ f(v_i) = 2(n - i) + 3; 1 < i \leq \frac{n}{2} + 1 \]
\[ = 2i; \frac{n}{2} + 1 < i \leq n \]
\[ f(w_j) = 2(n + j - 1); 1 < j \leq \frac{k+1}{2} \]
\[ = 2(n + k - j) + 1; \frac{k+1}{2} < j < k \]

Case 2. If \( n \equiv 1 \pmod{2} \)

Subcase I \( k \equiv 0 \pmod{2} \)

\[ f(u_1) = 0; \]
\[ f(u_i) = 2i - 1; 1 < i \leq \frac{n+1}{2} \]
\[ = 2(n - i + 1); \frac{n+1}{2} < i \leq n \]
\[ f(v_1) = 2n + 1; \]
\[ f(v_i) = 2(n - i) + 3; 1 < i \leq \frac{n+1}{2} \]
\[ = 2(n - i + 2); \frac{n+1}{2} < i \leq \frac{n+3}{2} \]
\[ = 2i; \frac{n+3}{2} < i \leq n \]
\[ f(w_j) = 2(n + j - 1); 1 < j \leq \frac{k}{2} \]
\[ = 2(n + k - j) + 1; \frac{k}{2} < j < k \]
Subcase II \( k \equiv 1 \pmod{2} \)

\[
f(u_i) = 2i - 1; \quad 1 < i \leq \frac{n+1}{2}
\]

\[
= 2(n - i + 1); \quad \frac{n+1}{2} < i \leq n
\]

\[
f(v_1) = 2n + 1;
\]

\[
f(v_i) = 2(n - i) + 3; \quad 1 < i \leq \frac{n-1}{2}
\]

\[
= 2(n - i + 2); \quad \frac{n-1}{2} < i \leq \frac{n+3}{2}
\]

\[
= 2i; \quad \frac{n+3}{2} < i \leq n
\]

\[
f(w_j) = 2(n + j - 1); \quad 1 < j \leq \frac{k+1}{2}
\]

\[
= 2(n + k - j) + 1; \quad \frac{k+1}{2} < j < k
\]

The labeling pattern defined above covers all the possibilities and in each case the graph \( G \) under consideration admits mean labeling. That is, the graph obtained by joining two copies of cycle \( C_n \) by a path \( P_k \) is a mean graph. \( \square \)

Illustration 7.4.4. Consider a graph \( G \) obtained by joining two copies of cycle \( C_{10} \) by a path \( P_3 \). It is the case related to \( n \equiv 0 \pmod{2} \) and \( k \equiv 1 \pmod{2} \). The corresponding mean labeling is shown in Figure 7.3.

Theorem 7.4.5. The graph obtained by arbitrary supersubdivision of any path \( P_n \) of length \( n - 1 \) is a mean graph.

Proof. Let \( P_n \) be the path of length \( n - 1 \) with vertices \( v_1, v_2, v_3, \ldots, v_n \). Let \( e_i \) denote the edge \( v_i v_{i+1} \) of the path \( P_n \) for \( 1 \leq i \leq n - 1 \).
Let $G$ be a graph obtained by arbitrary supersubdivision of path $P_n$. That is, for $1 \leq i \leq n - 1$ each edge $e_i$ of the path $P_n$ is replaced by a complete bipartite graph $K_{2,mi}$ where $m_i$ is any positive integer. Let $u_{ij}$ be the vertices which are used for arbitrary supersubdivision where $1 \leq i \leq n - 1$, $1 \leq j \leq m_i$. We note that $|V(G)| = m_1 + m_2 + \ldots + m_{n-1} + n$ and $|E(G)| = 2(m_1 + m_2 + \ldots + m_{n-1})$.

Let us define mean labeling $f : V(G) \to \{0, 1, 2, \ldots, 2(m_1 + m_2 + \ldots + m_{n-1})\}$ as follows.

- $f(v_1) = 0$
- $f(v_i) = 2(m_1 + m_2 + \ldots + m_{i-1}) - 1$; $1 < i \leq n$
- $f(u_{ij}) = 2j$; $i = 1$
- $f(u_{ij}) = f(v_i) + 2j + 1$; $1 < i < n$, $1 \leq j \leq m_i$

The labeling pattern defined above covers all the possibilities and in each one the graph $G$ under consideration admits mean labeling. That is, the graph obtained by arbitrary supersubdivision of any path $P_n$ of length $n - 1$ is a mean graph. □

Illustration 7.4.6. In the Figure 7.4, the arbitrary supersubdivision of $P_4$ and its mean labeling is shown, where $m_1 = 2$, $m_2 = 3$ and $m_3 = 4$. 

![Figure 7.4](image)
7.5 Conclusion and Scope

Shee and Ho[45] have discussed cordial labeling for the path union of various graphs. Vaidya et al.[55, 56, 58] have also discussed cordial labeling in the context of path union of various graphs while we discuss here mean labeling in the context of path union of cycle. Sethuraman and Selvaraju[44] have discussed graceful labeling of paths in the context of arbitrary supersubdivision while we prove that arbitrary supersubdivision of path $P_n$ admits mean labeling. We conclude this chapter as well as thesis with the following open problems.

1. Is it possible that every connected graph has at least one arbitrary supersubdivision which is mean graph?

2. Are there any other graphs whose arbitrary supersubdivision are mean graphs?

3. Is it possible to derive similar results in the context of various labeling techniques?

The content of this chapter give rise to following research paper.


The reprint of above research paper is provided in annexure.