CHAPTER 4

STATIC ANALYSIS AND OPTIMIZATION

4.1 Static Analysis of Axisymmetric Shells

The finite element formulation of a family of linear, quadratic and cubic, variable thickness C(0) elements for shells of revolution based on Mindlin-Reissner shell theory is described. Only axisymmetric behavior is considered.

4.1.1 General Perspective:

Owing to their distinct geometrical and structural form, shells of revolution with axisymmetric loading, boundary conditions and material disposition can be idealized as effectively one dimensional problems and can be solved efficiently using computer based numerical procedures. Among the various numerical methods, the FE method has been the most prominent in the analysis of shells of revolution.

4.1.2 Basic Formulation:

Consider the axisymmetric MR shell shown in Fig. 4.1.1 The displacement components \( u_\ell \) and \( w_\ell \), expressed in terms of axes that are tangential and normal to the shell, may be written in terms of global displacements \( u \) and \( w \) as

\[
\begin{align*}
  u_\ell &= ucosa + wsina \\
  w_\ell &= -usina + wcosa
\end{align*}
\]
where $\alpha$ is as shown in the Fig. 4.1.1

![Fig. 4.1.1 Axisymmetric Shell](image)

The radius of curvature $R$ may be obtained from the expression

$$\frac{\text{d}\alpha}{\text{d}\ell} = -\frac{1}{R}$$

(4.3)

The total potential energy for a typical axisymmetric MR shell is given as

$$\Pi(u_{\ell}, w_{\ell}, \theta) = \pi \int \left[ \left( \varepsilon_m^T D_m \varepsilon_m + \left( \varepsilon_b^T D_b \varepsilon_b + \left( \varepsilon_s^T D_s \varepsilon_s \right) \right) \right] \text{d}\ell - 2\pi \int w_{\ell} q r \text{d}\ell - (M\ddot{\theta} + N\ddot{u}_\ell + Q\ddot{w}_{\ell})$$

(4.4)

The loading in the above equation consists of a distributed ring pressure loading $q$, couples $M$, axial forces $N$ or lateral forces $Q$
applied along a circumferential circle at \( \ell = \ell \). \( u_\ell, w_\ell \) and \( \bar{\theta} \) are the corresponding displacements and rotation value at \( \ell = \bar{\ell} \).

The membrane strains are given by

\[
\varepsilon_m = \begin{bmatrix} \varepsilon_{\ell} & \varepsilon_\psi \end{bmatrix}^T
\]

(4.5)
in which the radial strain may be expressed in terms of the local displacement as

\[
\varepsilon_{\ell} = \frac{du_\ell}{d\ell} + \frac{w_\ell}{R}
\]

(4.6.1)
or in terms of the global displacements as

\[
\varepsilon_{\ell} = \frac{du}{d\ell} \cos\alpha + \frac{dw}{d\ell} \sin\alpha
\]

(4.6.2)

and the hoop or circumferential strain as

\[
\varepsilon_\psi = \frac{\{u_\ell \cos\alpha - w_\ell \sin\alpha\}}{r}
\]

(4.7.1)
or

\[
\varepsilon_\psi = \frac{u}{r}
\]

(4.7.2)

The bending strains or curvatures are given by

\[
\varepsilon_b = \begin{bmatrix} X_r & X_\psi \end{bmatrix}^T
\]

(4.8)

where the radial curvature

\[
X_r = -\frac{d\theta}{d\ell}
\]

(4.9)
and the hoop or circumferential curvature

\[
X_\psi = -\frac{\theta \cos\alpha}{r}
\]

(4.10)

The radial transverse shear strain is given by

\[
\varepsilon_s = \frac{dw_\ell}{d\ell} - \theta \frac{u_\ell}{R}
\]

(4.11.1)
or 

\[ \epsilon_s = -\theta - \frac{du}{d\ell} \sin \alpha + \frac{dw}{d\ell} \cos \alpha \]  

(4.11.2)

For an isotropic material of Elastic modulus \( E \), Poisson’s ratio \( \nu \) and Thickness \( t \), the matrix of membrane rigidities may be written as

\[ \mathbf{D}_m = \frac{Et}{(1-\nu^2)} \begin{pmatrix} 1 & \nu \\ \nu & 1 \end{pmatrix} \]  

(4.12)

The matrix of flexural rigidities has the form

\[ \mathbf{D}_f = \frac{Et^3}{12(1-\nu^2)} \begin{pmatrix} 1 & \nu \\ \nu & 1 \end{pmatrix} \]  

(4.13)

and the shear rigidity is given as

\[ D_s = \kappa \frac{Et}{2(1+\nu)} \]  

(4.14)

where \( \kappa \) is the shear modification factor and is taken as \( 5/6 \) for an isotropic material.

4.1.3 Illustrative Example

4.1.3.1 Problem Definition:

The example problem is taken from Reference^60. The shape of a spherical shell subjected to a concentrated ring load (u.d.l.) of intensity 255 kN/m on the upper free edge is optimized. The material properties used are as given below.

- Young’s Modulus = \( 2.1 \times 10^5 \) N / mm²
- Poisson’s ratio = 0.3
- Radius of the spherical shell = 10 m
- Radius of apex hole = 2.5 m.
- Thickness of the shell = 25 mm.

The geometry of the spherical shell along with the boundary conditions & loading is shown in Fig. 4.1.2(a). The deflected shape of
the shell is shown in Fig. 4.1.2(b). The von-Mises stress intensities of the shell were shown in Fig 4.1.2(c).

In the finite element method (FEM), the static behaviour of a structure is represented by $K \mathbf{u} = \mathbf{P}$ \hspace{1cm} (4.15)
where, $K$ is the global stiffness matrix, $\mathbf{u}$ is the global nodal displacement vector and $\mathbf{P}$ is the nodal load vector.

The strain energy of the structure is defined as $SE = \frac{1}{2} \mathbf{P}^T \mathbf{u}$ \hspace{1cm} (4.16)
This is commonly used as the inverse measure of the overall stiffness of the structure. It is obvious that maximizing the overall stiffness is equivalent to minimizing the strain energy or vice versa.

The stress leveling index is defined as $S = \int \left( \sigma - \sigma_{avg} \right)^2 dA$ \hspace{1cm} (4.17)
where $\sigma$ is the stress in the element and $\sigma_{avg}$ is the average stress of all the elements and $A$ is the area of the entire structure. After stress leveling, a more uniform stress distribution is usually obtained throughout the structure and this is important in situations where the effective stress initially varies considerably along the cross section of the structure. It enables the use of structural material more efficiently.

4.1.3.2 Validation of Results:

The spherical shell analyzed by Rao and Hinton\textsuperscript{60} is investigated in this research work for the validation. The volume of the shell obtained by them is 15 m\textsuperscript{3}. The volume of the shell obtained in this thesis is 15.19 m\textsuperscript{3}. Comparing the results with the benchmark problem, it is observed that an excellent agreement is found, thus validating the present concept.
(a) Loading diagram

(b) Deflected shape

(c) von Mises stresses

Fig. 4.1.2 Spherical shell – Initial conditions
4.2 Static Analysis of Prismatic Shells with Rectangular Planform

The linear elastic analysis of prismatic folded plate and shell structures supported on diaphragms at two opposite edges with the other two edges arbitrarily restrained is presented. The theoretical formulation is presented for right prismatic structures and then extended to prismatic folded plate and shell structures that are of curved planform.

4.2.1 General Perspective:

Prismatic structures such as folded plates and shells of constant transverse cross-section with diaphragm ends are quite common. These structures are of rectangular planform and can have complex cross-sections. The present work focuses on a general formulation for the analysis of variable thickness, rectangular planform prismatic structures that may have components which have curved cross-sections.

4.2.2 Basic Formulation:

Consider the MR curved shell strip shown in Fig. 4.2.1. Displacement components $u_i$, $v_i$ and $w_i$ are associated with movements in the $\ell$, $y$ and $n$ directions respectively.

The displacement components $u_\ell$ and $w_\ell$ may be written in terms of global displacements $u$ and $w$ in the $x$ and $z$ directions as

$$u_\ell = uc_\alpha + w\sin \alpha$$  \hspace{1cm} (4.1)

$$w_\ell = -us_\alpha + w\cos \alpha$$ \hspace{1cm} (4.2)
where $\alpha$ is the angle between the $x$ and $\ell$ axis.

The radius of curvature $R$ may be obtained from the expression

$$\frac{d\alpha}{d\ell} = -\frac{1}{R}$$

(4.3)

Fig. 4.2.1 Primatic Shells with Rectangular Planform

The total potential energy for a typical curved MR strip of length $b$ shown in Fig. 4.2.1 is given in terms of the global displacements $u, v, w$ and the rotations $\phi$ and $\psi$ of the midsurface normal in the $\ell n$ and $yn$ planes respectively by the expression

$$I(u,v,w,\phi,\psi) = \frac{1}{2} \int_0^b \left( [\varepsilon_m] D_m \varepsilon_m + [\varepsilon_b] D_b \varepsilon_b + [\varepsilon_s] D_s \varepsilon_s + kw^2 \right) dl dy$$

$$- \left[ u^T g dl dy - \left[ u^T g dy \right]_0^b \right]$$

(4.18)
where the membrane strains $\varepsilon_m$ are given by $\varepsilon_m = [\varepsilon_x, \varepsilon_y, \gamma_{xy}]^T$ (4.19)

in which the strain in the $\ell$ direction may be expressed in terms of the

local displacement as $\varepsilon_x = \frac{\partial u_x}{\partial \ell} + \frac{w_y}{R}$ (4.6.1)

or in terms of the global displacement as $\varepsilon_x = \frac{\partial u}{\partial \ell} \cos \alpha + \frac{\partial w}{\partial \ell} \sin \alpha$ (4.6.2)

The longitudinal strain is expressed as $\varepsilon_y = \frac{\partial v}{\partial y}$ (4.20)

and the shear strain is expressed as $\gamma_{xy} = \frac{\partial u_y}{\partial y} + \frac{\partial v}{\partial \ell}$ (4.21.1)

or $\gamma_{xy} = \frac{\partial u}{\partial y} \cos \alpha + \frac{\partial w}{\partial y} \sin \alpha + \frac{\partial v}{\partial \ell}$ (4.21.2)

The bending strains or curvatures are given by $\varepsilon_b = [k_x, k_y, k_{xy}]^T$ (4.22)

where the curvature in the $\ell$ direction is $k_{\ell} = -\frac{\partial \phi}{\partial \ell}$ (4.23)

The longitudinal curvature is given as $k_y = -\frac{\partial \psi}{\partial y}$ (4.24)

and the twisting curvature $k_{xy} = -\left(\frac{\partial \phi}{\partial y} + \frac{\partial \psi}{\partial \ell}\right) + \frac{\partial u_y}{\partial y} \frac{1}{R}$ (4.25.1)

or $k_{xy} = -\left(\frac{\partial \phi}{\partial y} + \frac{\partial \psi}{\partial \ell}\right) - \left(\frac{\partial u}{\partial y} \cos \alpha + \frac{\partial w}{\partial y} \sin \alpha\right) \frac{d\alpha}{d\ell}$ (4.25.2)

The transverse shear strains are given by $\varepsilon_{\ell n} = [\gamma_{\ell n}, \gamma_{n\ell}]^T$ (4.26)

in which the transverse shear strain in the $\ell n$ plane is

$$\gamma_{\ell n} = \frac{\partial w}{\partial \ell} - \phi - \frac{u_y}{R}$$ (4.27.1)

or

$$\gamma_{\ell n} = -\frac{\partial u}{\partial \ell} \sin \alpha + \frac{\partial w}{\partial \ell} \cos \alpha - \phi$$ (4.27.2)

The longitudinal transverse shear strain in the $yn$ plane is
\[ \gamma_{mn} = \frac{\partial w_i}{\partial y} - \psi \]  \hspace{1cm} (4.28.1)

or

\[ \gamma_{mn} = -\frac{\partial u}{\partial y} \sin \alpha + \frac{\partial w}{\partial y} \cos \alpha - \psi \]  \hspace{1cm} (4.28.2)

For an isotropic material of elastic modulus \( E \), Poisson’s ratio \( \nu \) and thickness \( t \), the matrix of membrane rigidities has the form

\[
D_m = \frac{Et}{(1-\nu^2)} \begin{pmatrix}
1 & \nu & 0 \\
\nu & 1 & 0 \\
0 & 0 & (1-\nu)/2
\end{pmatrix}
\]  \hspace{1cm} (4.29)

The matrix of flexural rigidities may be expressed as

\[
D_b = \frac{Et^3}{12(1-\nu^2)} \begin{pmatrix}
1 & \nu & 0 \\
\nu & 1 & 0 \\
0 & 0 & (1-\nu)/2
\end{pmatrix}
\]  \hspace{1cm} (4.30)

and the matrix of shear rigidities is given as

\[
D_s = \frac{kEt}{2(1+\nu)} \begin{pmatrix}
1 & 0 \\
0 & 1
\end{pmatrix}
\]  \hspace{1cm} (4.31)

where \( k \) is the shear modification factor and is taken as 5/6 for an isotropic material.

The displacement components \( u \) are listed as \( u = [u, v, w, \phi, \psi]^T \)

and the corresponding distributed loadings \( g \) may be written as

\[ g = [g_u, g_v, g_w, g_\phi, g_\psi]^T \]

The distributed line loadings are \( \overline{g} = [\overline{F}_u, \overline{F}_v, \overline{F}_w, \overline{F}_\phi, \overline{F}_\psi]^T \) in which the line forces are \( \overline{F}_u, \overline{F}_v \), and \( \overline{F}_w \). The distributed line couples are \( \overline{F}_\phi \) and \( \overline{F}_\psi \). These loadings are applied at \( \ell = \mathbf{\ell} \), where the corresponding displacements are \( \overline{u} = [\overline{u}, \overline{v}, \overline{w}, \overline{\phi}, \overline{\psi}] \).
4.2.3 Illustrative Example

4.2.3.1 Problem Definition:

The example problem is taken from Reference\textsuperscript{67}. The shape of a cylindrical shell subjected to a central point load of 11.34 kN is optimized. The material properties used are as follows.

\begin{itemize}
  \item Young’s Modulus = $2.109 \times 10^7$ kN/m\textsuperscript{2}
  \item Poisson’s ratio = 0.15
  \item Radius of the shell = 9.144 m
  \item Span of the shell = 15.24 m.
  \item Thickness of the shell = 0.0762 m.
\end{itemize}

Due to the symmetry, only a quarter of the shell is analyzed. The geometry of the cylindrical shell along with the boundary conditions and loading is shown in Fig. 4.2.2(a). The deflected shape of the shell is shown in Fig. 4.2.2(b). The von Mises stress contour of the shell is shown in Fig 4.2.2(c).

4.2.3.2 Validation of Results:

The cylindrical shell analyzed by Lee\textsuperscript{67} is investigated in this research work for the validation. The initial volume and strain energy of the shell obtained by Lee\textsuperscript{67} are 3.705 m\textsuperscript{3} and 0.0392 kN.m respectively. The initial volume and strain energy of the shell obtained in the present study are 3.704 m\textsuperscript{3} and 0.0393 kN.m respectively. Comparing the results with the benchmark problem, it is observed that an excellent agreement is found, thus validating the present concept.
Fig. 4.2.2 Cylindrical shell – Initial conditions

(a) Loading diagram (one – quarter of the shell)

(b) Deflected shape

(c) von Mises stresses (x 0.0488 kN/m$^2$)
4.3 Static Analysis of Prismatic Shells with Curved Planform

Prismatic structures such as bridge decks, slabs some shells are often of curved planform and can have complex cross sections. The general formulation for the analysis of variable thickness prismatic structures that have curved cross section and are curved in plan is presented as below.

4.3.1 General Perspective:

Prismatic structures such as folded plates and shells of constant transverse cross-section with diaphragm ends are quite common. These structures are of curved planform and can have complex cross-sections. The present work focuses on a general formulation for the analysis of variable thickness, curved planform prismatic structures that may have components which have curved cross-sections.

4.3.2 Basic Formulation:

The FS method uses a combination of FEs and Fourier series to analyze structures in which the geometrical properties are invariant in a particular direction. The theory of a family of curved, variable thickness FSs are presented for prismatic folded plates and shells that are curved in plan. Consider the MR curved shell strip shown in Fig. 4.3.1
Fig. 4.3.1 Primatic Shells with Curved Planform

Displacement components $u_i, v_i$ and $w_i$ are associated with movements in the local $\ell, \eta$ and $n$ directions respectively. Note that $\eta$ varies from an angle zero to $\beta$ along a curve of radius $r$. The displacement components $u_i$ and $w_i$ may be written in terms of global displacements $u$ and $w$ in the $r$ and $z$ directions as

$$u_i = u \cos \alpha + w \sin \alpha$$  \hspace{1cm} (4.1)$$

$$w_i = -u \sin \alpha + w \cos \alpha$$  \hspace{1cm} (4.2)$$

where $\alpha$ is the angle between the $r$ and $\ell$ axes.
The radius of curvature $R$ may be obtained from the expression

$$\frac{d\alpha}{d\ell} = -\frac{1}{R} \quad (4.3)$$

The total potential energy for a typical curved MR strip spanning over an angle $\beta$ resting on an elastic Winkler-type foundation of modulus $k$ is given in terms of the global displacements $u$, $v$, $w$ and the rotations $\phi$ and $\psi$ of the midsurface normal in the $\ell_n$ and $\eta_n$ planes respectively by the expression

$$I(u,v,w,\phi,\psi) = \frac{1}{2} \beta \int_0^1 \left( \varepsilon_m^T D_m \varepsilon_m + \varepsilon_b^T D_b \varepsilon_b + \varepsilon_s^T D_s \varepsilon_s + kw_\ell^2 \right) r d\ell d\eta - \int u^T g r d\ell d\eta - \int u g r d\eta$$

$$\quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad (4.32)$$

where the membrane strains $\varepsilon_m$ are given by $\varepsilon_m = \left[ \varepsilon_\ell, \varepsilon_\eta, \gamma_{\ell\eta} \right]^T$ \quad (4.33)

in which the strain in the $\ell$ direction may be expressed in terms of the local displacements as $\varepsilon_\ell = \frac{\partial u_\ell}{\partial \ell} + \frac{w_\ell}{R}$ \quad (4.6.1)

or in terms of the global displacement as

$$\varepsilon_\ell = \frac{\partial u}{\partial \ell} \cos \alpha + \frac{\partial w}{\partial \ell} \sin \alpha \quad (4.6.2)$$

The longitudinal strain is expressed as

$$\varepsilon_\eta = \left( u_\ell \cos \alpha - w_\ell \sin \alpha + \frac{\partial v}{\partial \eta} \right) \frac{1}{r} \quad (4.34.1)$$

or

$$\varepsilon_\eta = \left( u + \frac{\partial v}{\partial \eta} \right) \frac{1}{r} \quad (4.34.2)$$
and the shear strain is expressed as

\[
\gamma_{\ell n} = \left( \frac{\partial u_\ell}{\partial \eta} - \nu \cos \alpha \right) \frac{1}{r} + \frac{\partial \nu}{\partial \ell}
\]  

(4.35.1)

or

\[
\gamma_{\ell n} = \left( \frac{\partial u}{\partial \eta} \cos \alpha + \frac{\partial w}{\partial \eta} \sin \alpha - \nu \cos \alpha \right) \frac{1}{r} + \frac{\partial \nu}{\partial \ell}
\]  

(4.35.2)

The bending strains or curvatures \( \varepsilon_b \) are given by

\[
\varepsilon_b = \begin{bmatrix} K_{\ell \ell} & K_{\ell \eta} & K_{\eta \eta} \end{bmatrix}^T
\]  

(4.36)

where the curvature in the \( \ell \) direction is

\[
k_\ell = -\frac{\partial \phi}{\partial \ell}
\]  

(4.37)

The longitudinal curvature is given as

\[
k_{\eta} = -\left( \frac{\partial \psi}{\partial \eta} + \phi \cos \alpha \right) \frac{1}{r}
\]  

(4.38)

and the twisting curvature is given as

\[
k_{\ell \eta} = \left( \frac{\partial \phi}{\partial \eta} + \psi \cos \alpha + \frac{\partial \nu}{\partial \ell} \sin \alpha \right) \frac{1}{r} - \frac{\partial \psi}{\partial \ell} + \frac{\partial u_\ell}{\partial \eta} \frac{1}{rR}
\]  

(4.39.1)

or

\[
k_{\ell \eta} = \left( \frac{\partial \phi}{\partial \eta} + \psi \cos \alpha + \frac{\partial \nu}{\partial \ell} \sin \alpha \right) \frac{1}{r} - \frac{\partial \psi}{\partial \ell} - \frac{d\alpha}{d\eta} \left( \frac{\partial u}{\partial \eta} + \frac{\partial w}{\partial \eta} \sin \alpha \right) \frac{1}{r}
\]  

(4.39.2)

The transverse shear strains are given by

\[
\varepsilon_s = \begin{bmatrix} \gamma_{\ell n}, \gamma_{\eta n} \end{bmatrix}^T
\]  

(4.40)

in which the transverse shear strain in the \( \ell n \) plane is

\[
\gamma_{\ell n} = \frac{\partial w_\ell}{\partial \ell} - \phi - \frac{u_\ell}{R}
\]  

(4.41.1)

or

\[
\gamma_{\ell n} = -\frac{\partial u}{\partial \ell} \sin \alpha + \frac{\partial w}{\partial \ell} \cos \alpha - \phi
\]  

(4.41.2)

The longitudinal transverse shear strain in the \( \eta n \) plane is
\[ y_{\eta\eta} = \left( \frac{\partial w}{\partial \eta} + \nu \sin \alpha \right) \frac{1}{r} - \psi \]  

(4.42.1) 

or \[ y_{\eta\eta} = \left( -\frac{\partial u}{\partial \eta} \sin \alpha + \frac{\partial w}{\partial \eta} \cos \alpha + \nu \sin \alpha \right) \frac{1}{r} - \psi \]  

(4.42.2) 

For an isotropic material of elastic modulus \( E \), Poisson’s ratio \( \nu \) and thickness \( t \), the matrix of membrane rigidities has the form 

\[
D_m = \frac{Et}{(1 - \nu^2)} \begin{pmatrix}
1 & \nu & 0 \\
\nu & 1 & 0 \\
0 & 0 & (1 - \nu)/2
\end{pmatrix}
\]  

(4.43) 

The matrix of flexural rigidities may be expressed as 

\[
D_f = \frac{Et^3}{12(1 - \nu^2)} \begin{pmatrix}
1 & \nu & 0 \\
\nu & 1 & 0 \\
0 & 0 & (1 - \nu)/2
\end{pmatrix}
\]  

(4.44) 

and the matrix of shear rigidities is given as 

\[
D_s = \frac{kEt}{2(1 + \nu)} \begin{pmatrix}
1 & 0 \\
0 & 1
\end{pmatrix}
\]  

(4.45) 

where \( \kappa \) is the shear modification factor and is taken as 5/6 for an isotropic material. 

The displacement components \( u \) are listed as \( u = [u, v, w, \phi, \psi]^T \) and the corresponding distributed loadings \( g \) may be written as 

\[
g = [g_u, g_v, g_w, g_\phi, g_\psi]^T
\]

The distributed line loadings are \( g = [F_u, F_v, F_w, F_\phi, F_\psi]^T \) in which the line forces are \( F_u, F_v, \) and \( F_w \). The distributed line couples are \( F_\phi \) and \( F_\psi \). These loadings are applied at \( \ell = \tilde{\ell} \), where the corresponding displacements are \( u = [\tilde{u}, \tilde{v}, \tilde{w}, \tilde{\phi}, \tilde{\psi}]^T \).
4.3.3 Illustrative Example

4.3.3.1 Problem Definition:

The example problem is taken from Reference\textsuperscript{46}. The analysis and optimization of a box girder bridge is carried out using the finite element method. The bridge spans over an angle of $\alpha = 1$ rad. with an inner radius of $r_i = 23.77$m. The box girder bridge is simply supported at each radial end and free along the curved edges. The box girder bridge is modeled using 58 key points. The geometrical model of two cell box girder bridge is shown in Fig. 4.3.2(b). The cross sectional view, plan view and the position of design variables are shown in Fig. 4.3.4(a), (b) and (c) respectively. The box girder bridge is discretized using 4578 Nodes and 2310 Elements. The finite element model of two cell box girder bridge is shown in Fig. 4.3.3.

4.3.3.2 Validation of Results:

The cross sectional shape of the box girder bridge is modeled using nine segments and eight key points by Ozakca and Taysi\textsuperscript{46}. The initial volume of the box girder bridge obtained by them is 192.5 m\textsuperscript{3}. The initial volume of the box girder bridge obtained in the present study is 190.1 m\textsuperscript{3}. Comparing the results with the benchmark problem, it is observed that an excellent agreement is found, thus validating the present concept.
Fig. 4.3.2 Geometrical model of a two cell box girder bridge

Fig. 4.3.3 Finite element model of a two cell box girder bridge
Fig. 4.3.4 Two cell Box girder bridge curved in plan

[All dimensions are in m. (x 0.3048)]

(a) Cross sectional view  (b) Plan view

(c) Position of design variables.
4.4 Structural Optimization of Axisymmetric and Prismatic Shells:

The structural shape and thickness optimization of axisymmetric and prismatic shell structures is presented through some numerical examples. These shells appear in many practical forms, such as water tanks and other liquid-containing structures, roof structures (domes), silos and pressure vessels.

4.4.1 Problem definition:

The mathematical statement of an optimization has been defined earlier in Chapter 3. Before any optimization process can be started, the objective function, the constraint functions and bounds on the design variables have to be specified.

4.4.1.1 Selection of objective function:

The objective function \( F(s) \) to be minimized is the strain energy which is a nonlinear function of the design variable \( s \)

\[
F(s) = \int_{\Omega} \sigma^T D^{-1} \sigma \, d\Omega \quad (4.46)
\]

where \( \sigma \) is the stress resultant vector and \( D \) is the matrix of rigidities. It can be shown that minimizing the strain energy of a structure is equivalent to increasing the rigidity of the structure. Such rigid structures have higher resistance against deformations and may therefore be considered structurally more efficient.

In this research work, volume or weight minimization of a structure has also been considered in which
$$F(s) = \sum_{\ell=1}^{n_e} \rho_\ell \Omega_\ell (s) \quad \text{or} \quad F(s) = \sum_{\ell=1}^{n_e} \Omega_\ell (s) \quad (4.47)$$

where $\rho_\ell$ is the density and $\Omega_\ell$ is the volume of the $\ell$th FE and $n_e$ is the number of elements. Volume or weight minimization is considered more important in the aerospace industry, where structural designs are often based on weight considerations rather than on cost and fabrication considerations.

4.4.1.2 Selection of constraints :

In order to complete the formulation of the problem, some restrictions must be imposed on the values of the design variables for the mathematical model to be meaningful. Such restrictions are termed the constraints of the problem. In the optimization problem the constraints are classified as inequality, equality and side constraints.

Inequality constraints are of the type $g_j(s) \leq 0$. This type of constraint can be imposed by putting a limitation on the stresses (stress constrained) or displacements (displacement constrained) at some specified points.

Stress constrained:  
$$g^\sigma = 1 - \frac{\sigma}{\sigma_{all}} \geq 0 \quad (4.48)$$

Displacement constrained:  
$$g^d = 1 - \frac{d}{d_{all}} \geq 0 \quad (4.49)$$

where $\sigma$ and $d$ are the stress and displacement values at a point and $\sigma_{all}$ and $d_{all}$ are the allowable stress and displacement values.
prescribed for the whole structure. The above class of constraints is also referred to as nonlinear constraints.

Another class of constraints is the equality constraints of the type \( h_k(s) = 0 \). These constraints are also known as linear or geometric constraints. This type of constraint can be imposed by specifying that the volume or weight of the structure should remain constant.

\[
V = V_{\text{initial}} \quad (4.50)
\]

The other important constraint is the side constraint of the type \( s_i^l \leq s_i \leq s_i^u \). These constraints are introduced to allow for technological limitations on the design variables and to deal with various geometrical requirements. They impose precise conditions to the optimization problem that must be satisfied for the design to be considered feasible. They are usually bounded from below and above. These bounds can usually be based on construction and analysis considerations. The bounds on the design variable can be imposed on the shell problems as \( 0 \leq s_i \leq b \)

where \( b \) is the given dimension. These types of constraints limit the region of search for the optimum solution.

4.5 Optimization of Axisymmetric shells

4.5.1 Problem Definition:

The problem which is analyzed in the section 4.1.3 is optimized here. A total of four shape variables and one thickness variable is considered for the optimization. Many researchers dealt with structural shape optimization with different objective functions and geometric constraints. However, in many of the cases, the optimum
shapes of the structures were obtained by prescribing the movement directions of the design variables in radial direction only.

In this research work, the structural shape optimization of spherical shell is investigated by prescribing the movement directions of shape design variables in horizontal, vertical and both horizontal & vertical directions. In case-1, the shape design variables \( S_2, S_3, S_4, S_5 \), and \( t \) move in horizontal direction (X – direction). In case-2, the shape design variables \( S_2, S_3, S_4, S_5 \) and \( t \) move in vertical direction (Y – direction). In case-3, the shape design variables move in both horizontal and vertical directions (X,Y – directions). The movement directions of shape design variables are shown in Fig 4.5.1(a),(b) & (c).

- **Design Variables**: Co-ordinates of key points
- **Constraints**: Volume
- **Objective Function**: Strain energy, Stress leveling index

### 4.5.2 Results and Discussion:

The initial strain energy of the spherical shell is 40.41 kN.m. The initial stress leveling index is \( 0.594 \times 10^{12} \) kN\(^2\)/m\(^2\). The initial maximum deflection is 0.162 m. The initial volume of the spherical shell is 15.19 m\(^3\).

#### 4.5.2.1 Strain energy minimization:

In case-1, i.e. when the shape design variables moves in the horizontal direction, the maximum deflection is 0.0091 m. The strain energy of the shell after optimization is 7.42 kN.m. The deflected shape is shown in Fig 4.5.2(a). The von Mises stress intensities are shown in Fig 4.5.2(b). The deflection was reduced by 94.4 %. The
strain energy was reduced by 81.6 %. The iteration history of the optimization is shown in Fig 4.5.2(c). The optimum shape of the spherical shell is shown in Fig. 4.5.2(d).

In case-2, i.e. when the shape design variables moves in the vertical direction, the maximum deflection is 0.0137 m. The strain energy of the shell after optimization is 4.55 kN.m. The deflected shape is shown in Fig 4.5.3(a). The von Mises stress intensities are shown in Fig 4.5.3(b). The deflection was reduced by 91.5 % The strain energy was reduced by 88.7 %. The iteration history of the optimization is shown in Fig 4.5.3(c). The optimum shape of the spherical shell is shown in Fig. 4.5.3(d).

In case-3, i.e. when the shape design variables moves in both the horizontal and vertical directions, the maximum deflection is 0.0051 m. The strain energy of the shell after optimization is 3.92 kN.m. The deflected shape is shown in Fig 4.5.4(a). The von Mises stress intensities are shown in Fig 4.5.4(b). The deflection was reduced by 96.9 %. The strain energy was reduced by 90.3 %. The iteration history of the optimization is shown in Fig 4.5.4(c). The optimum shape of the spherical shell is shown in Fig. 4.5.4(d).

The summary of the design variables for the three different cases are shown in Table 4.1.1 and the summary of the structural optimization results are shown in Table 4.1.3

4.5.2.2 Stress leveling index minimization:

In case-1, i.e. when the shape design variables moves in the horizontal direction, the maximum deflection is 0.0052 m. The stress
leveling index of the shell after optimization is $0.105 \times 10^{12} \text{kN}^2/\text{m}^2$. The deflected shape is shown in Fig 4.5.5(a). The von Mises stress intensities are shown in Fig 4.5.5(b). The deflection was reduced by 96.8 %. The stress leveling index was reduced by 82.3 %. The iteration history of the optimization is shown in Fig 4.5.5(c). The optimum shape of the spherical shell is shown in Fig. 4.5.5(d).

In case-2, i.e. when the shape design variables moves in the vertical direction, the maximum deflection is 0.0266 m. The stress leveling index of the shell after optimization is $0.156 \times 10^{12} \text{kN}^2/\text{m}^2$. The deflected shape is shown in Fig 4.5.6(a). The von Mises stress intensities are shown in Fig 4.5.6(b). The deflection was reduced by 83.6 %. The stress leveling index was reduced by 73.7 %. The iteration history of the optimization is shown in Fig 4.5.6(c). The optimum shape of the spherical shell is shown in Fig. 4.5.6(d).

In case-3, i.e. when the shape design variables moves in both the horizontal and vertical directions, the maximum deflection is 0.0045 m. The stress leveling index of the shell after optimization is $0.059 \times 10^{12} \text{kN}^2/\text{m}^2$. The deflected shape is shown in Fig 4.5.7(a). The von Mises stress intensities are shown in Fig 4.5.7(b). The deflection was reduced by 97.2%. The stress leveling index was reduced by 90.1 %. The iteration history of the optimization is shown in Fig 4.5.7(c). The optimum shape of the spherical shell is shown in Fig. 4.5.7(d). The summary of the design variables for the three different cases are shown in Table 4.1.2 and the summary of the structural optimization results are shown in Table 4.1.4.
Fig. 4.5.1 Prescribed movement directions of shape design variables
(a) Deflected shape

(b) von Mises stresses
Fig. 4.5.2 Responses for movement of shape design variables in X direction – Strain energy minimization
(a) Deflected shape

(b) von Mises stresses
Fig. 4.5.3  Responses for movement of shape design variables in Y direction – Strain energy minimization
(a) Deflected shape

(b) von Mises stresses

ANSYS 9.0

- 5677
- 9130
- 12583
- 16036
- 19489
- 22942
- 26395
- 29848
- 33301
- 36754
Fig. 4.5.4 Responses for movement of shape design variables in X,Y directions – Strain energy minimization
(a) Deflected shape

(b) von Mises stresses
Fig. 4.5.5 Responses for movement of shape design variables in X direction – Stress leveling index minimization

(c) Iteration history

(d) Optimum shape
(a) Deflected shape

(b) von Mises stresses
Fig. 4.5.6 Responses for movement of shape design variables in Y direction – Stress leveling index minimization
(a) Deflected shape

(b) von Mises stresses
Fig. 4.5.7 Responses for movement of shape design variables in X,Y directions – Stress leveling index minimization
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**Table – 4.1.2 : Summary of design variables - Stress levelling index minimization**

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<td>Strain energy (kN.m)</td>
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<th>% reduction in</th>
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<td>Stress levelling index (kN²/m²)</td>
<td>Maximum deflection (m)</td>
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4.6 Optimization of Prismatic Shells with Rectangular Planform

4.6.1 Problem Definition:

The problem which is analyzed in the section 4.2.3 is optimized here. Many researchers dealt with structural shape optimization with different objective functions and geometric constraints. However, in many of the cases, the optimum shapes of the structures were obtained by prescribing the movement directions of the design variables in radial direction only.

Here, the structural shape optimization of cylindrical shell is investigated by prescribing the movement directions of shape design variables in horizontal, vertical and both horizontal & vertical directions. In case-1, the shape design variables $S_1, S_3, S_4, S_5, S_6$ move in horizontal direction (X – direction). In case-2, the shape design variables $S_1, S_2, S_3, S_4, S_5, S_6$ move in vertical direction (Y – direction). In case-3, the shape design variables moves in both horizontal and vertical directions (X, Y – directions). The movement directions of shape design variables for the above three cases are shown in Fig 4.6.1(a), (b) and (c).

Design Variables : Co-ordinates of key points
Constraints : Volume
Objective Function : Strain energy, Stress leveling index

4.6.2 Results and Discussion:

The initial total strain energy of the cylindrical shell is 0.0393 kN.m. The initial stress leveling index is $10.08 \times 10^6$ kN$^2$/m$^2$. The
The initial maximum deflection is 0.0069 m. The initial volume of the cylindrical shell is 3.704 m$^3$.

4.6.2.1 Strain energy minimization:

In case-1, i.e. when the shape design variables move in the horizontal direction, the maximum deflection is 0.0011 m. The strain energy of the shell after optimization is 0.0052 kN.m. The deflected shape is shown in Fig 4.6.2(a). The von Mises stress contour is shown in Fig 4.6.2(b). The deflection is reduced by 84.1%. The strain energy is reduced by 86.8%. The iteration history of the optimization is shown in Fig 4.6.2(c). The optimum shape of the cylindrical shell is shown in Fig. 4.6.2(d).

In case-2, i.e. when the shape design variables move in the vertical direction, the maximum deflection is 0.0014 m. The strain energy of the shell after optimization is 0.0055 kN.m. The deflected shape is shown in Fig 4.6.3(a). The von Mises stress contour is shown in Fig 4.6.3(b). The deflection is reduced by 79.7%. The strain energy is reduced by 86.0%. The iteration history of the optimization is shown in Fig 4.6.3(c). The optimum shape of the cylindrical shell is shown in Fig. 4.6.3(d).

In case-3, i.e. when the shape design variables move in both horizontal and vertical directions, the maximum deflection is 0.0007 m. The strain energy of the shell after optimization is 0.0019 kN.m. The deflected shape is shown in Fig 4.6.4(a). The von-Mises stress contour is shown in Fig 4.6.4(b). The deflection is reduced by 89.9%. The strain energy is reduced by 95.2%. The
iteration history of the optimization is shown in Fig 4.6.4(c). The optimum shape of the cylindrical shell is shown in Fig. 4.6.4(d).

The summary of the design variables for the three different cases is presented in Table 4.2.1. The summary of the structural optimization results is shown in Table 4.2.3.

**4.6.2.2 Stress leveling index minimization:**

In case-1, i.e. when the shape design variables moves in the horizontal direction, the maximum deflection is 0.0027 m. The stress leveling index of the shell after optimization is $3.26 \times 10^6$ kN/m$^2$. The deflected shape is shown in Fig. 4.6.5(a). The von Mises stress contour is shown in Fig. 4.6.5(b). The deflection is reduced by 60.9%. The stress leveling index is reduced by 67.7%. The iteration history of the optimization is shown in Fig. 4.6.5(c). The optimum shape of the cylindrical shell is shown in Fig. 4.6.5(d).

In case-2, i.e. when the shape design variables moves in the vertical direction, the maximum deflection is 0.0019 m. The stress leveling index of the shell after optimization is $4.42 \times 10^6$ kN/m$^2$. The deflected shape is shown in Fig. 4.6.6(a). The von Mises stress contour is shown in Fig 4.6.6(b). The deflection is reduced by 72.5%. The stress leveling index is reduced by 56.2%. The iteration history of the optimization is shown in Fig 4.6.6(c). The optimum shape of the cylindrical shell is shown in Fig. 4.6.6(d).

In case-3, i.e. when the shape design variables moves in both the horizontal and vertical directions, the maximum deflection is 0.0013 m. The stress leveling index of the shell after optimization is...
2.73 \times 10^6 \text{ kN}^2/\text{m}^2. The deflected shape is shown in Fig. 4.6.7(a). The von Mises stress contour is shown in Fig. 4.6.7(b). The deflection is reduced by 81.2 %. The stress leveling index is reduced by 72.9 %. The iteration history of the optimization is shown in Fig. 4.6.7(c). The optimum shape of the cylindrical shell is shown in Fig. 4.6.7(d).

The summary of the design variables for the three different cases is shown in Table 4.2.2. The summary of the structural optimization results is presented in Table 4.2.4.
(a) X direction (one – quarter of the shell) - case 1

(b) Y direction (case 2)

(c) X,Y directions (case 3)

Fig. 4.6.1 Prescribed movement directions of shape design Variables
(a) Deflected shape

(b) von Mises stresses (x 0.0488 kN/m$^2$)
Fig. 4.6.2 Responses for movement of shape design variables in X direction - Strain energy minimization
(a) Deflected shape

(b) von Mises stresses (x 0.0488 kN/m²)
Fig. 4.6.3  Responses for movement of shape design variables in Y direction - Strain energy minimization
(a) Deflected shape

(b) von Mises stresses (x 0.0488 kN/m²)
Fig. 4.6.4 Responses for movement of shape design variables in X,Y directions - Strain energy minimization
(a) Deflected shape

(b) von Mises stresses (x 0.0488 kN/m²)
Fig. 4.6.5  Responses for movement of shape design variables in X direction – Stress levelling index minimization
(a) Deflected shape

(b) von Mises stresses ($\times 0.0488$ kN/m$^2$)
Fig. 4.6.6 Responses for movement of shape design variables in Y direction – Stress levelling index minimization
(a) Deflected shape

(b) von Mises stresses (x 0.0488 kN/m²)
Fig. 4.6.7 Responses for movement of shape design variables in X,Y directions – Stress levelling index minimization
### Table – 4.2.1 Summary of design variables - Strain energy minimization

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Table – 4.2.2 Summary of design variables - Stress levelling index minimization

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<th>Design variables (x 0.3048 m)</th>
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<tr>
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<td>s3x</td>
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<tr>
<td></td>
<td>s4x</td>
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<td>s5x</td>
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<td></td>
<td>s6x</td>
</tr>
<tr>
<td>2 Vertical / Y direction</td>
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</tr>
<tr>
<td></td>
<td>s2y</td>
</tr>
<tr>
<td></td>
<td>s3y</td>
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<td>s5y</td>
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<td>s6y</td>
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<td>3 Horizontal, Vertical / X,Y directions</td>
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### Table – 4.2.3 Summary of structural optimization results

**Strain energy minimization:**

<table>
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<tr>
<th>Case No.</th>
<th>Initial</th>
<th>After optimization</th>
<th>% reduction in</th>
<th>Strain energy (kN.m)</th>
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<td>Strain energy (kN.m)</td>
<td>Maximum deflection (m)</td>
<td>Strain energy (kN.m)</td>
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<td>0.0393</td>
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<td>0.0393</td>
<td>0.0007</td>
<td>0.0019</td>
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</table>

### Table – 4.2.4 Summary of structural optimization results

**Stress levelling index minimization**

<table>
<thead>
<tr>
<th>Case No.</th>
<th>Initial</th>
<th>After optimization</th>
<th>% reduction in</th>
<th>Stress levelling index (kN²/m²)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Maximum deflection (m)</td>
<td>Stress levelling index (kN²/m²)</td>
<td>Maximum deflection (m)</td>
<td>Stress levelling index (kN²/m²)</td>
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4.7 Optimization of Prismatic shells with curved planform:

4.7.1 Problem Definition:

The problem which is analyzed in the section 4.3.3 is optimized here for the two load cases. A concentrated load of 4.448 kN is applied on the top flange, which is above the (i) Inner web and (ii) Middle web. Three shape and five thickness design variables are considered. The shape design variables are length of segment 2 (i.e. $S_1$) and the total length of segment 3-6 (i.e. $S_2$) and 4-7 (i.e. $S_3$). Thickness design variables are the thickness of the top flange cantilever segments ($t_1$), the top and bottom flanges ($t_3$, $t_4$), the middle and outer webs ($t_5$, $t_2$).

- **Design Variables**: Length Variables, Thickness Variables
- **Constraints**: Volume
- **Objective Function**: Strain Energy

4.7.2 Results & Discussion:

4.7.2.1 Case-1: Load above the inner web

In case-1, the box girder bridge is optimized for the minimization of strain energy subjected to the constraint that volume of the structure should remain constant. The iteration history for the minimization of the strain energy is shown in Fig. 4.7.1(a). The optimum shape of the box girder bridge is shown in Fig. 4.7.1(b). The strain energy of the structure before & after optimization are found to be 44.91 kN.m. & 2.42 kN.m. respectively. This results into a reduction of 94.6 % in strain energy (as against 62.2 % reported in Ref.46) which is a significant improvement over the previous authors. The summary of the design variables for the minimization of strain.
energy is given in Table-4.3.1. The structural optimization results are given in Table-4.3.2.

4.7.2.2 Case-2: Load above the middle web

In case-2, the box girder bridge is optimized for the minimization of strain energy subjected to the constraint that volume of the structure should remain constant. The iteration history for the minimization of the strain energy is shown in Fig. 4.7.2(a). The optimum shape of the box girder bridge is shown in Fig. 4.7.2(b). The strain energy of the structure before & after optimization are found to be 65.34 kN.m. & 16.19 kN.m. respectively. This results into a reduction of 75.2 % in strain energy (as against 65.5 % reported in Ref.46) which is a considerable improvement over the previous authors. The summary of the design variables for the minimization of strain energy is given in Table-4.3.3. The structural optimization results are given in Table-4.3.4.
(a) Iteration history

(b) Optimum shape

Fig. 4.7.1 Load above the inner web - Minimization of strain energy
Fig. 4.7.2 Load above the middle web - Minimization of strain energy
### Table - 4.3.1 Summary of design variables for minimization of strain energy - Load above the inner web

<table>
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<th>Type</th>
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<th>Maximum (x 0.3048 m)</th>
<th>Initial (x 0.3048 m)</th>
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### Table – 4.3.2 Structural optimization results for minimization of strain energy - Load above the inner web

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**Table – 4.3.3** Summary of design variables for minimization of strain energy - Load above the middle web

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**Table – 4.3.4** Structural optimization results for minimization of strain energy - Load above the middle web

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