The chapter discusses about the two improvements that are made for traditional LMSM viz, the implementation of advanced time integration schemes for overall enhancement of the stability of the numerical scheme and second is the inclusion of bending rigidity in the cable formulation. A modified beam model which uses only translational degrees of freedoms for solving beam equation has been implemented in this particular study. This would avoid the assemblage of rotational degrees of freedom for the solution of equation of motion. The time discretization procedure is discussed first.
4.1 Time Discretisation Procedure

Numerical models for towing systems require the discretisation of the physical system in time and space. The resulting equations are typically written as a non-linear matrix equation known as the semi-discrete equation of motion, because the time derivatives of the vector of dependent variables are left as continuous functions.

\[
M\ddot{x} + C\dot{x} + Kx = F(t)
\]  

(4.1a)

Where \(x\) denotes three dimensional space and \(F\) is the external force vector. The equations of motion for LMSM are most often presented in matrix form as a system of second-order ordinary differential equations with time as one of the independent variable. Therefore, the solution needs to progress over time with the help of a suitable time integration scheme. Most temporal integration schemes in use today have their roots in the method developed by Buckham and Nahon [Buckham, et.al, 1999]. These methods are broadly classified into single step and multi-step methods. Single step methods are attractive because of their computational simplicity and are selected throughout of this study. The main objective of these schemes is to provide an optimum blend of numerical diffusion to damp out high frequency oscillations in the solution and preserve low frequency modes without loss of accuracy. Hughes and Belytschko [Ted Belytschko, et.al, 1983] provide a summary of the development of these types of methods in the context of linear finite element structural dynamics. Most classical methods can now be cast into unified multi-parameter integration schemes where an adjustment in the parameters leads to one of several different methods with different numerical properties. Thomas [Thomas, 1994] studied the three “classic” methods (Newmark, Houbolt, and Wilson) and their applicability to the mooring dynamics problem.

In addition to Newmark and its variants which are widely employed with finite element based models, researchers in the cable dynamics field have employed a variety of different schemes for the temporal integration problem. Chiou and Leonard [Chiou, 1979] use simple backward finite difference method. Sun et al. [Sun Y,1998] use the generalized trapezoidal rule which is a first-order variant of the Newmark method. Paulling and

The most popular time/space discretisation scheme used in the solution of underwater towing problems is the ‘Box’ method, in which the governing equations are discretised on the half-grid point in both space and time. This method was first employed for the solution of tow- cable dynamics by Ablow and Schechter [Ablow, 1983]. Since then it has been widely employed in both towing and mooring applications [Howell, 1992]. Single step family of time integration methods were widely used in the solution of towing/mooring dynamics problems [Gobat, 2000], though, most of them are less accurate than LMS (linear multi-step) methods such as Adam-Badsforth one. The reason may be the requirement of more computational storage of the historical time step data.

4.2 Characteristics of Time Integration Schemes

While, there is not yet a universally accepted “perfect” time integration method, Hilber and Hughes (1978) gave a list of characteristics that a marching scheme should possess in order to be competitive and efficient:

1. Unconditional stability when applied to linear problems: Unconditional stability allows the time step to be chosen based on accuracy and resolution concerns, without regard for purely numerical issues.

2. No more than one set of implicit equations to be solved at each time step: This minimizes computational expense compared to schemes which may achieve a high order of accuracy at a significant computational cost.

3. At least second-order accuracy: This is a reflection of the constraints imposed by Dahlquist’s theorem which states that a third-order accurate method with the most appropriate stability conditions does not exist. Again, without a significant increase in computational effort, second-order accuracy is the best we can do.

4. Controllable algorithmic dissipation in the higher modes: In some cases with sufficiently small temporal and spatial discretisation, it may be desirable to have less high frequency numerical dissipation.

5. Self-starting, no information is needed prior to time step zero: Accuracy at time step zero (and thus accurate algorithm starting information) is critical in transient analysis
applications. It is less important in cases where we can slowly ramp up a forcing scenario and are not concerned with start-up transients.

4.2.1 Stability
An integration scheme is said to be stable if the numerical solution, under any initial conditions, does not grow without bound [Bathe, 1998]. An algorithm is unconditionally stable for linear problems if the convergence of the solution is independent of the size of the time step Δt. Otherwise the algorithm is conditionally stable for values of Δt less than a critical value Δt_{cr}. The value of the critical time step is equal to a constant multiplying the smallest natural period of the system and it depends also on the material damping of the system. Therefore, unconditionally stable schemes are generally preferred, as in that case the size of the time step is determined only by the accuracy of the solution. Furthermore, all integration schemes can be classified as either explicit or implicit methods. The great advantage of explicit schemes is that the solution does not involve the inversion of the stiffness matrix. However, Dahlquist [Dahlquist,1973] demonstrated that all explicit methods are conditionally stable with respect to the size of the time step. On the other hand most implicit integration methods are unconditionally stable, but the inversion of the stiffness matrix at each time step makes them computationally expensive.

4.2.2 Number of Implicit Systems to be Solved
Hilber and Hughes [Hughes, 1987] suggest that the algorithm should not require the solution of more than one implicit system, of the size of the mass and stiffness matrices, at each time step. Although, algorithms that require two or more implicit systems of the size of the mass and stiffness matrices to be solved at each time step possess improved properties (e.g. [Argyris et al,2000]), they require at least twice the storage and computational effort of simpler methods.

4.2.3 Accuracy
Another parameter that comes next to stability in terms of importance is the accuracy of the numerical scheme. In general, the accuracy depends on the size of the time step. The smaller the time step, the more accurate is the solution. An integration scheme is convergent if the numerical solution approaches the exact solution as Δt tends to zero. According to Hilber and Hughes [Hughes, 1987] the second order accurate methods are immensely
superior to the first order accurate methods. Furthermore, Dahlquist theorem suggests that a third order accurate unconditionally stable linear multistep method does not exist [Hughes, 1987].

4.2.4 Numerical dissipation
The necessity for time integration algorithms to possess numerical damping is widely recognized. Due to poor spatial discretization, the most of the approximate methods like finite difference one, cannot represent accurately at high-frequency modes. Strang and Fix [Strang, 1986], and others, showed that modes corresponding to higher frequencies become more and more inaccurate. Thus, the role of the numerical damping is to eliminate spurious high frequency oscillations. Specifically in underwater towing problems, the highest modes of the system do not have to be represented in an accurate way, since the excitation from external disturbances such as waves etc, are operating at low frequency modes. Therefore, a desirable property of an algorithm is the preferential numerical damping (“filtering”) of the inaccurate high frequency modes and the preservation of the important low frequency modes.

Another way to filter the higher modes could be the use of viscous damping. However, Hughes [Hughes, 1987] argues that the use of viscous damping affects a middle band of frequencies, not the inaccurate higher frequency modes. Therefore, the only adequate way to damp out the spurious modes is the use of controllable algorithmic damping.

4.2.5 Self-starting
Integration schemes which are not self-starting require data from more than two time steps to proceed the solution. In this case, the standard practice is to assume the initial conditions. Thus, apart from the algorithm, a starting procedure should be implemented and analysed. Obviously, this requires additional computational effort and storage. Thus self-starting algorithms are generally preferred.

4.2.6 Overshooting
The term overshooting describes the tendency of an algorithm (for large time steps) to exceed heavily the exact solution in the first few time steps, but eventually to converge to the exact solution. This peculiar phenomenon was first discovered by Goudreau and Taylor [Goudreau, 1992] as a property of the Wilson θ-method and is not related to the stability
and accuracy characteristics of the algorithms discussed so far. The overshoot in displacement and velocity is a common phenomenon. Low overshoot is preferred particularly when accurate transient dynamic analysis is to be carried out.

4.3 Implementation of Time integration schemes

While numerous time marching schemes are available, only some of the most popular are presented and implemented in this thesis. These are implicit form of Houbolt, Newmark, HHT-α and Euler [Hughes, 1987]. All these schemes are implemented in LMSM formulation and studied their performance in terms of accuracy and compared with experimental values from literature [Wu, 2002]. (See Chapter 6 section 6.3)

4.3.1 Houbolt Scheme

Houbolt method was one of the earliest algorithms to include numerical dissipation in the equation of motion for the benefit of asymptotic annihilation in which the high-frequency response is nearly annihilated in one time step [Hughes, 1987]. The Houbolt method has been available in numerous commercial finite element codes because of its asymptotic annihilation property which has been found to very useful to stabilize computations involving highly nonlinear phenomena. While the original scheme is a two-step procedure, the present study focused on single step Houbolt (SSH) method discussed in literature [Hughes, 1987]. The method is outlined by equations A2.1-A2.5 (Appendix –II).

The recommended value for controlling parameter of the scheme, $\gamma$ is 3/2 to minimize velocity error and 1/2 to avoid velocity overshoot [Hughes, 1999]. From a stability and accuracy point of view, it is unconditionally stable, second order accurate and is not suitable for high frequency dynamic problems [Zhou, 2004].

4.3.2 NewMark’s method

Newmark [Hughes, 1987] proposed what has become one of the most popular family of algorithms for the solution of problems in structural dynamics. The method is outlined by equations A2.6-A2.8 (Appendix –II). $\beta$ and $\gamma$ are the controlling parameters in the scheme. The method is implicit, and unconditional stability is guaranteed for $\beta$=1/4 and $\gamma$ =1/2 [Hughes, 1987]. The trapezoidal rule is a particular case of this family, for which $\beta$=1/4 and $\gamma$=1/2. This case also corresponds to the assumption that the acceleration is constant over
the time interval \([t_n, t_{n+1}]\) and equal to \((a_n + a_{n+1})/2\). This method is also known as the average acceleration method. The method is discussed in appendix-II.

4.3.3 Box Method

Box method is the most widely used scheme for the solution of underwater towing and mooring dynamics which uses finite difference discretisation for the equation of motion. In the box method, the governing equations are discretised on the half-grid point in both space and time. This method was first employed for the solution of tow cable dynamics by Ablow and Schechter [Ablow, 1983]. The box method has got unconditional stability for the case of linear problems, but has subjected to phenomenon known as Crank-Nicolson noise [Gobat, 2000], whereby the high frequency components of the solution oscillate with every time step. In a linear problem, this noise can be removed by computing step-to-step averages after the solution is completed. For a nonlinear problem, however, the noise can be a source of instability and hence should be eliminated as the solution progresses. Given the stability problems associated with the box method, a new solution method is sought for the dynamic analysis using lumped mass spring formulation.

4.3.4 HHT-\(\alpha\) method

Hilber, Hughes and Taylor (1977) introduced generalization of Newmark method in order to achieve controllable algorithmic dissipation of the high frequency modes. A slightly modified version of the HHT scheme, which was suggested by Hughes, is examined in the present study. The method employs Newmark equations for displacement and velocity variations. The scheme is discussed in appendix-II (eq A2.10-2.12). The main parameter which controls the scheme is \(\alpha\). If \(\alpha=0\) the equations reduces to that of Newmark’s method. It has been found that if the parameters are selected such that \(\alpha=[-1/3, 0]\) and \(\gamma = (1-2\alpha)/2\) and \(\beta= (1- \alpha^2)/4\) an unconditional stable, second order accurate scheme results. Decreasing \(\alpha\) may result in the increase of the numerical dissipation.

4.4 Implementation of HHT-\(\alpha\) Method in LMSM

By neglecting fluid damping and considering force balance equations for the \(i^{th}\) node in a cable segment in two dimensional spaces as an example, lumped mass formulation gives

\[
m_i \ddot{y}_i - (T_{i+1}\cos\phi_{i+1} - T_{i-1}\cos\phi_{i-1}) = F_{yi}
\]

(4.13)
\[ m_i \ddot{z}_i - (T_{i-1}\sin\varphi_{i-1} - T_{i-1}\sin\varphi_{i-1}) = F_{zi} \]  

This can be reduced into more general form

\[ M \ddot{a}^{n-1} + (1 + \alpha)Kd^{n-1} - \alpha Kd^n = F(t_{n+1}) \]

Where M, K and F are mass, stiffness matrix and external force vector respectively (See Appendix-II).

Assuming fluid damping not present the stiffness matrix becomes

\[
K^n = \begin{bmatrix}
T_{i-1}\cos\alpha_{i-1} & T_{i-1}\cos\alpha_{i-1} \\
y_{i-1} & z_{i-1} \\
T_{i-1}\sin\alpha_{i-1} & T_{i-1}\sin\alpha_{i-1} \\
y_{i-1} & z_{i-1}
\end{bmatrix}
\]

\[
K^{n-1} = \begin{bmatrix}
T_{i-1}\cos\alpha_{i-1} & T_{i-1}\cos\alpha_{i-1} \\
y_{i-1} & z_{i-1} \\
T_{i-1}\sin\alpha_{i-1} & T_{i-1}\sin\alpha_{i-1} \\
y_{i-1} & z_{i-1}
\end{bmatrix}
\]

\[
T_{i+1} = \left(\frac{\left(y_{i+1} - y_i\right)^2 + \left(z_{i+1} - z_i\right)^2}{l_{i+1}}\right)^{\frac{1}{2}} - 1 \]

\[
T_{i-1} = \left(\frac{\left(y_{i} - y_{i-1}\right)^2 + \left(z_{i} - z_{i-1}\right)^2}{l_{i-1}}\right)^{\frac{1}{2}} - 1 \]

Where

\[
M = \begin{bmatrix}
m_i & 0 \\
0 & m_i
\end{bmatrix}
\]

The right hand side of equation 4.15 consists of forces like gravitational, drag, buoyancy forces etc. Out of these gravitational and buoyancy force are constant over time. But drag forces depend on the instantaneous velocity.

The time domain is divided into a set of discrete steps \( t = j\delta t \) \((j = 1, 2, 3, \ldots)\). Assuming solution parameters are known at the previous time step \( t = j\delta t \), the question is how to find out the unknowns at the next time step \( t = (j+1) \delta t \). The equation 4.15 is solved for instantaneous acceleration first by assembling the stiffness and mass matrices. Subsequently the displacement and velocity vectors are corrected using equation 4.11 and 4.12. This procedure was repeated till the last time step.

### 4.5 Improved Bending Rigidity formulation for LMSM

Out of the various LMSM formulations for the cable, the method proposed by Shan Huang [Shan, 1994] is the most promising. But the limitation is that it does not have bending...
rigidity in the cable formulation. Therefore the model is best suited for long cables having small diameter (i.e. negligible bending rigidity). But in the case of two-part towing, this cannot be ignored as secondary and depressor cables are considered to be very short compared to primary cable. Hence the present study attempts to include the bending rigidity in the cable formulation using LMSM.

Figure 4.1 shows a small segment of the cable. In segmental formulation the independent variables to be considered are rotational (moment $M$ and slope at the two ends $\theta_1$ and $\theta_2$) and translational (space, tension $T$) ones. Subsequently, it is necessary to assemble translational and rotational equilibrium equations to get real-time position and slope of the cable segments. In the modified beam model all the degrees of freedom are of translational type. Thus, the equations to be assembled remain to be that of translational degrees of freedom rather than rotational and translational mix. The formulation has been discussed in detail in the literature [Wasfy, 2000] where it was mainly used to simulate the flexible dynamics of spatial mechanisms.

Suppose we have an originally undeformed straight beam of length $l$ (figure 4.2). Let $x$ be the co-ordinate along the neutral axis of the undeformed beam and $y$ be the co-ordinate of the transverse distance between the original undeflected shape of the beam and the deformed shape. The total strain energy neglecting the shear deformation of the beam according to the Euler beam theory is given by
Strain energy, \[ SE = \frac{1}{2EI} \int_0^l M^2 dx \] (4.19)

Where \( E \) is the Young’s modulus, \( I \) is the moment inertia of the cross-section in the transverse direction and \( M \) is the bending moment. To obtain the axial response of the modified element, two truss elements are inserted between nodes 1 and 2 and nodes 2 and 3 (Figure 4.4). Subsequently, a torsion spring was inserted at node 2 to provide bending rigidity. Hence the improved cable model is shown in the figure 4.3.

Wasfy [Wasfy, 2000] has shown that

\[ SE = \frac{EI}{6} \left( 3la^2 + 3abl^2 + b^2l^3 \right) \] (4.20)

The coefficients \( a \) and \( b \) are

\[ a = \frac{2}{l^3} (\alpha_1 + \alpha_2)(l_1 - 2l_2) \] (4.21)

\[ b = \frac{6}{l^3} (\alpha_1 + \alpha_2)(l_2 - l_1) \] (4.22)

The symbols \( \alpha_1 \) and \( \alpha_2 \) represents the inclination of cable segment 1-2 and 2-3 with respect to horizontal as shown in figure 4.4.
Where \( l_1 \) is the distance between nodes 1 and 2. Similarly \( l_2 = \) distance 2-3

Assuming \( l_1 = l_2 = l/2 \) and \( \alpha_1 = \alpha_2 = \alpha/2; \)

\[
SE = \frac{1}{2} \left( \frac{EI}{l} \right) \alpha^2
\]

(4.23)

This equation represents strain energy stored in a torsion spring with stiffness \( k_b \) given by

\[
k_b = \frac{EI}{l}
\]

(4.24)
The free body diagram with equivalent nodal forces is shown in figure 4.5. These nodal forces are attributed to rotational moments and are assembled on the right hand side of the translational equations of motion for the cable.