CHAPTER 1

INTRODUCTION
1.1 Introduction

Various derivations of the word "algebra" have been given by different writers. The first mention of the word is to be found in the title of a book “Hidab al-jabrwal-muqubala” written in Baghdad about 825 A.D. by the Arab mathematician Mohammed ibn-Musa al-Khowarizmi. The words jabr (JAH-ber) and muqubalah (moo-KAH-ba-lah) were used by al-Khowarizmi to designate two basic operations in solving equations. Jabr was to transpose subtracted terms to the other side of the equation. Muqabalah was to cancel like terms on opposite sides of the equation. In fact, the title has been translated to mean "science of restoration and opposition" or "science of transposition and cancellation" and "The Book of Completion and Cancellation" or "The Book of Restoration and Balancing". Jabr is used in the step where \( x - 3 = 10 \) becomes \( x = 13 \). The left-side of the first equation, where \( x \) is lessened by 3, is "restored" or "completed" back to \( x \) in the second equation. Muqabalah takes us from \( x + y = y + 4 \) to \( x = 4 \) by "cancelling" or "balancing" the two sides of the equation. Eventually the muqabalah was left behind, and this type of math became known as algebra in many languages.

Other writers have derived the word from the Arabic particle al (the definite article), and gerber, meaning "man." Since, however, Geber happened to be the name of a celebrated Moorish philosopher who flourished in about the 11\(^{th}\) or 12\(^{th}\) century, it has been supposed that he was the founder of algebra, which has since perpetuated his name. The evidence of Peter Ramus (1515-1572) on this point is interesting, but he gives no authority for his singular statements. In the preface to his Arithmeticaelibri duo ettotidem Algebrae (1560), he says: "The name Algebra is Syriac, signifying the art or doctrine of an excellent man. For Geber, in Syriac, is a name applied to men, and is sometimes a term of honour, as master or doctor among us. The term "algebra" is now in universal use.

Early Indian and Chinese geometrical problems involved algebraic equations and their solutions similar to those of the Greeks who solved many comparatively difficult algebraic problems in a purely geometrical way. While the Greek algebra was developed by Diophantus in his Arithmatica, in the 3rd century A.D., the algebra in Babylon was developed much earlier in a more advanced form including problems.
on cubic and biquadratic equations as shown by Neugebauer and others. Once
cannot help wondering whether this Babylonian algebra could have been
transmitted in seminal forms to lay the foundation of Indian and Chinese algebra on
the one hand and for the Hellenistic development on the other. During the decay of
Western Science in the early Middle Age, the algebra of the Diophantine period was
forgotten and when the great Arab Scientific Movement took place, Arabic algebra
very probably derived its inspiration from India rather than from Greece.

In India, the geometrical methods of solving algebraic problems have been
traced to the various Sulba Sutras. The Shulba Sutras are part of the larger corpus of
texts called the Shrauta Sutras considered to be appendices to the Vedas. They are
the only sources of knowledge of Indian mathematics from the Vedic period. The
four major Shulba Sutras, which are mathematically the most significant, are those
composed by Baudhayana, Manava, Apastamba and Katyayana, about whom very
little is known. Pythagoras theorem and Pythagorean triples, as found in the Sulba
Sutras. The rope stretched along the length of the diagonal of a rectangle makes an
area which the, vertical and horizontal sides make together, in other words:
\[ a^2 = b^2 + c^2 \]. Examples of Pythagorean triples given as the sides of right angled
triangles:

\[
\begin{align*}
3^2 + 4^2 &= 5^2 \\
9 + 16 &= 25
\end{align*}
\]

\[
\begin{align*}
5^2 + 12^2 &= 13^2 \\
25 + 144 &= 169
\end{align*}
\]

\[
\begin{align*}
9^2 + 40^2 &= 41^2 \\
81 + 1600 &= 1681
\end{align*}
\]

Figure 1.1 Right angled triangles

These include solutions of linear, simultaneous and even indeterminate equations,
arose in connection with the construction of different types of sacrificial altars and
arrangements for laying bricks into them. In the development of early mathematics,
when the symbols for operation began to be used in the computations a new branch
evolved being separated from arithmetic and geometry which is known as algebra.
The differentiation of algebra as a distinct branch from mathematics in general took
place, from about the time of the Brahmagupta (598 A.D.), following the technique of indeterminate analysis. In fact, Brahmagupta used the term kuttaka-ganita or simply kuttaka for algebra. The term kuttaka meaning “pulverize refers to a branch of the science of algebra dealing particularly with the subject of linear equation, quadratic equation and indeterminate equations of first, second and higher degree. It is interesting to find that this subject was considered so important by the Hindus that the whole science of algebra was named after it in the beginning of the seventh century.” Algebra is also called avyakta - ganita or “the science of calculation with unknowns” (avyakta = unknown) in contradistinction to the science of calculation with known (vyakta = known) for arithmetic including geometry and mensuration. The term bijaganita meaning science of calculation with elements or unknown quantities (bij) was hinted by Prthudakasvami (850 A. D.) and used with definition by Bhaskara - II (1150 A.D.).

It is widely acknowledged that algebra is an essential part of undergraduate mathematical learning and yet it is also known for its high level of difficulty at the collegiate level. Many undergraduate and graduate students, including prospective teachers, struggle to grasp even the most fundamental concepts of algebra. For most of the students experience mathematical abstraction and formal proof. Now, it is often the first time in which teachers expect students to “go beyond learning ‘imitative behavior patterns’ for mimicking the solution of a large number of variations on a small number of problems ³by requiring proofs to explain abstract theories and ideas. In particular, students are expected to mentally construct new objects based on a list of properties and then operate on these objects. However, simply being exposed to these abstract concepts does not imply the development of mathematical meaning. Students must take an active role in the learning process by building on their past mathematical knowledge to make sense of abstract concepts.

Cook (2012) asserted in his dissertation that the difficulty students experience in abstract algebra is due to the lack of established connections between undergraduate mathematics and school mathematics. He affirmed that prospective teachers “do not build upon their elementary understandings of algebra, leaving

³Dubinsky et al., 1994, p. 268
them unable to communicate traces of any deep and unifying ideas that govern the subject”. These conjectures imply that undergraduate professors must be able to not only convey an abstract idea to students but also provide students the opportunity to build mathematical meaning upon these abstractions. If teachers do not know how to translate those abstractions into a form that enables learners to relate the mathematics to what they already know, they will not learn with understanding. Thus, we can only expect undergraduate students to really access the benefits of this study through complete comprehension by connecting abstract theory to past knowledge and ideas to aid in the construction of mathematical meaning.

1.2 Role and Importance of Study

Algebra is one of the main areas of pure mathematics that uses mathematical statements such as term, equations, or expressions to relate relationships between objects that change over time. We use Algebra in finances, engineering, and many scientific fields. It is actually quite common for an average person to perform simple Algebra without realizing it. For example, if you go to a store and have twenty rupees to spend on two rupees candy bars. This gives us the equation $2x = 10$ where $x$ is the number of candy bars you can buy. Many people don't realize that this sort of calculation is Algebra; they just do it. Whether you like it or not, Algebra is actually needed in your everyday life.

In recent years there has been growing interest in the role of history of mathematics in improving the teaching and learning of mathematics. Educators throughout the world have been formulating and conducting research on the use of history of mathematics in mathematics education. Some of the results of this research have been communicated at meetings of interested organisations, and through papers in various journals. A research programme is beginning to emerge, with contributions from many places over the globe. Such a programme involves a consolidated critical bibliography of work that has been done, and a programme for developing a deeper understanding of the factors involved in the relations between history and pedagogy of mathematics, in different areas of mathematics, and with pupils and students at different stages and with different environments and
backgrounds. It also involves the identification and spreading of information and good practice in learning and teaching situations.

The present research work sketches out some of the concerns in the hope that many people across the world will wish to contribute to the international discussions and the growing understandings reached in and about this area. The overall intention is to study the role of history of mathematics, in its many dimensions, at all the levels of the educational system: in its relations to the teaching and the learning of mathematics as well as with regard to teacher training and in educational research. History of mathematics as a component of the teaching of mathematics is, as any educational project, directed towards more or less explicit expectations in terms of better learning of some mathematics. Research on the use of history of mathematics in teaching is thus an important part of research in mathematical education.

To study such a large and multi-faceted theme we propose to analyse it in a number of inter-related questions which together will give insight into the whole process. The order in which the questions are put down here carries no implication about their relative importance or significance.

1. How does the educational level of the learner bear upon the role of history of mathematics or Algebra?

The way history of mathematics can be used, and the rationale for its use, may vary according to the educational level of the class: children at elementary school and students at university do have different needs and possibilities. Questions arise about the ways in which history can address these differences. This may, again, be reflected in different training needs for teachers at these levels. To speak about the "use" of the history of mathematics stand out that history of mathematics is something external to mathematics. This assumption would not be universally agreed, however.
2. At what level does history of mathematics/Algebra as a taught subject become relevant?

In analysing the role of history of mathematics, it is important to distinguish issues around using history of mathematics in a situation whose immediate purpose is the teaching of mathematics, and teaching the history of mathematics as such, in a course or a shorter session. It could be that courses in the history of mathematics, and its classroom use, should be included in a teacher training curriculum. There is also a third area, related but separate, namely the history of mathematics education, which is a rather different kind of history.

3. What are the particular functions of a history of mathematics course or component for teachers?

History of mathematics may play an especially important role in the training of future teachers, and also teachers undergoing in-service training. There are a number of reasons for including a historical component in such training, including the promotion of enthusiasm for mathematics, enabling trainees to see pupils differently, to see mathematics differently, and to develop skills of reading, library use and expository writing which can be neglected in mathematics courses. It may be useful here to distinguish the training needs for primary, secondary and higher levels. A related issue is what kinds of history of mathematics is appropriate in teacher training and why: for example, it could be that the history of the foundations of mathematics and ideas of rigor and proof are especially important for future secondary and tertiary teachers. This issue is also relevant for other categories than future teachers, and is picked up again in question 5.

4. What is the relation between historians of mathematics and those whose main concern is in using history of mathematics in mathematics education?

This question focuses on the professional base from which practitioners emerge, and relates to the social fabric of today's mathematics education community as well as to issues about the nature of history. There are, gratifyingly, a number of leading
historians of mathematics with an interest in educational issues, as there are leading mathematicians and mathematics educators with an interest in history. But as well as minor misapprehensions of the nature of the others' activities, there may be deeper tensions and conflicting aims which it is important to bring to the surface. For example, historians may underestimate the difficulty of transmuting the historical knowledge of the teacher into a productive classroom activity for the learner. It is important that historians and mathematics educators work co-operatively, since historical learning and classroom experience at the appropriate level do not always co-exist in the same person.

5. Should different parts of the curriculum involve history of mathematics in a different way?

Already research is taking place to investigate the particularities of the role of history in the teaching of algebra, compared with the role of history in the teaching of geometry. Different parts of the syllabus make reference, of course, to different aspects of the history of mathematics, and it may be that different modes of use are relevant. Looking at the curriculum in a broad way, we may note that the histories of computing, of statistics, of core "pure" mathematics and of the interactions between mathematics and the world are all rather different pursuits. Even for the design of the curriculum historical knowledge may be valuable. A survey of recent trends in research could lead to suggestions for new topics to be taught.

6. Does the experience of learning and teaching mathematics in different parts of the world make different demands on the history of mathematics?

A historical dimension to mathematics learning helps bring out two contrary perceptions in a dialectical way. One is that mathematical developments take place within cultural contexts and it is valid to speak of Islamic mathematics, Greek mathematics and so on, as developments whose style is characteristic of the generating culture. The antithesis to this is the realisation that all human cultures have given rise to mathematical developments which are now the heritage of everyone; this therefore acts against a narrow ethnocentric view within the
educational system. The Study should explore the benefit to learners of realising both that they have a local heritage from their direct ancestors—in the way in which Moslem children in countries where they are in a minority are known to derive pride and strength from learning about Islamic mathematical achievements—but also that every culture in the world has contributed to the knowledge and experience base made available to today’s learners. There are many detailed studies of the interplay between history of mathematics and culture in educational contexts throughout the world, notably in Brazil, the Maghreb, Mozambique, China, Portugal etc, which should be drawn upon in analysing and responding to this question.

7. What role can history of mathematics play in supporting special educational needs?

The experience of teachers with responsibility for a wide variety of special educational needs is that history of mathematics can empower the students and valuably support the learning process. Among such areas are experiences with mature students, with students attending numeracy classes, with students in particular apprenticeship situations, with hitherto low-attaining students, with gifted students, and with students whose special needs arise from handicaps. Here the many different experiences need to be researched, their particular features drawn out, and an account provided in an overall framework of analysis and understanding.

8. What are the relations between the roles we attribute to history and the ways of using it in education?

This question has been the focus of considerable attention over recent decades. Every time someone reports on a classroom experience of using history and what it achieved they have been offering a response to this question. So a search of the literature is a fundamental part of researching the response to this question. The question also involves also a listing of ways of introducing or incorporating a historical dimension, then one would draw attention to the range of educational aims served by each mode of incorporation: the way that historical anecdotes are intended to change the image of mathematics and humanize it, for example. Or
again, the way that mathematics is not, historically, a relentless surge of progress but can be a study in twists, turns, false paths and dead-ends both humanizes the subject and helps learners towards a more realistic appreciation of their own endeavors.

There are rich issues for discussion and research in, for example, the use of primary sources in mathematics classrooms at appropriate levels. This question is a very broad one that could involve a large number of people: it may be wise to distinguish the taxonomic question --the range of different classroom aims and modes of activity-- from the further exploration of each issue.

9. What are the consequences for classroom organisation and practice?

The consequences of integrating history are far-reaching. In particular, there are wider opportunities for modes of assessment. Assessment can be broadened to develop different skills (such as writing and project activity), and consequences for students' interest and enjoyment have been noted. Teachers may well need practical guidance and support both in fresh areas of assessment, and in aspects of classroom organisation. This in turn may have consequences for teacher training as well as curriculum design.

10. How can history of mathematics be useful for the mathematics education researcher?

This question provides an opportunity for an exploration of the relations between the subject of this study and researchers in the mathematical education community (whose aims are, in turn, to provide insights into the processes of learning and teaching). One example is the use of history of mathematics to help both teacher and learner understand and overcome epistemological breaks in the development of mathematical understanding. A constructive critical analysis of the view that 'ontogeny recapitulates phylogeny' that the development of an individual's mathematical understanding follows the historical development of mathematical ideas-- may be appropriate. Another example is of research on the development of
mathematical concepts. In this case the researcher applies history as possible 'looking glasses' on the mechanisms that put mathematical thought into motion. Such combinations of historical and psychological perspectives deserve serious attention.

These issues could be studied in teaching experiments in which the above questions are addressed, and also questions like: What is good for the learner? How do you know it is good for the learner? and so on. Even if a teaching experiment does not use history of mathematics explicitly, the elaboration of the teaching project may have made use of the results of history of mathematics. For instance, such a question as 'is it good for the learner?' may be better understood in the light of the history of mathematics. So the question here is: how can research in mathematics education profit from historical knowledge? The answer to this question might deal with themes such as the historical genesis of a concept and an epistemological analysis of the interplay between history and the teaching of a subject. Moreover, history of mathematics helps to understand the distance between the way in which concepts function in the mathematics community and the way they function in the school. There are also fundamental questions about the style and evaluation of research in this area. Different styles which have been used in the past range from the anecdotal (in effect) to quasi-scientific surveys with questionnaires and statistical apparatus. A process of such considerable complexity evidently calls for a research methodology of some sophistication. Fortunately the wider mathematics education community has been studying this problem for some time: it is indeed the subject of an earlier ICMI Study (What is research in mathematics education and what are its results?). So a group could be encouraged to draw upon the wider community experience and consider its application to our area of concern.

11. What are the national experiences of incorporating history of mathematics in national curriculum documents and central political guidance?

This is not so much a question for discussion as a fairly straightforward empirical question, needing input from knowledgeable people in as many countries and states
as possible. But of course it has policy implications too, and could lead to a sharing of experience among members of the community about how they have reached the policy-making level in their countries to influence the content or rhetoric of public documents. Perhaps this study could be carried on in parallel with the more discursive questions, organised by a small group who could put the results (in the sense of public documents or quotations from them as well as brief historical accounts of national curriculum change) on the WorldWideWeb as they are collected. In some parts of the world a different relationship between history and mathematics may have been developed. For example, in Denmark and Sweden history of mathematics is regarded as an intrinsic part of the subject itself. There are also differences in styles of examination and assessment. If everyone with access to examples of such different approaches, from different countries and states, could pool their experience it would be a most valuable input to the Study.

12. What work has been done on the area of this Study in the past?

The answer is: quite a lot. But it is all over the place and needs to be gathered together and referenced analytically. A major annotated critical bibliographical study of the field, which might well take up a sizable proportion of the final publication, would be an enormously valuable contribution.

1.3 The link between historical and conceptual levels of learning

In many history texts, algebra is considered to have three stages in its historical development: the rhetorical stage, the syncopated stage, and the symbolic stage. By the rhetorical, we mean the stage where all statements and arguments are made in words and sentences. In the syncopated stage, some abbreviations are used when dealing with algebraic expressions. And finally, in the symbolic stage, there is total symbolization – all numbers, operations, relationships are expressed through a set of easily recognized symbols, and manipulations on the symbols take place according to well-understood rules. These three stages are certainly one way of looking at the history of algebra. But I want to argue that, besides these three
stages of expressing algebraic ideas, there are conceptual stages that have happened alongside of these changes in expressions. The conceptual stages are the geometric stage, where most of the concepts of algebra are geometric; the static equation-solving stage, where the goal is to find numbers satisfying certain relationships; the dynamic function stage, where motion seems to be an underlying idea; and finally the abstract stage, where structure is the goal. Naturally, neither these stages nor the earlier three are disjoint from one another; there is always some overlap. I will consider both of these sets of stages to see how they are sometimes independent of one another and at other times work together. But because the first set of stages is well known and discussed previously by so many persons, we will concentrate on the conceptual ones.

The term algebraic reasoning has been used to describe mathematical processes of generalizing a pattern and modeling problems with various representations. Driscoll (1999) defined algebraic reasoning as the "capacity to represent quantitative situations so that relations among variables become apparent". For Langrall and Swafford (1997) algebraic reasoning is "the ability to operate on an unknown quantity as if the quantity is known". Vance (1998) characterized algebraic reasoning as a way of reasoning involving variables, generalizations, different modes of representation, and abstracting from computations. Kaput (1993) viewed algebraic reasoning as a process of construction and representation of patterns and regularities, deliberate generalization, and active exploration and conjecture. These definitions will serve as a basis to explain conceptual understanding in algebra in this paper. Understanding is a logical power manifested by abstract thought. Piaget\(^2\) suggested that understanding in general and in mathematics in particular is a highly complex process of abstraction. He proposed the term reflective abstraction to explain the process of developing conceptual understanding. It can be said that those who have a conceptual understanding grasp the full meaning of knowledge, and can discern, interpret, compare and contrast related ideas of the subtle distinctions among a variety of situations. Conceptual understanding in algebra can be characterized as the ability to recognize functional

\(^2\)Piaget, 1970, p. 221
relationships between known, and unknown, independent and dependent variables, and to distinguish between and interpret different representations of the algebraic concepts. It is manifested by competency in reading, writing, and manipulating both number symbols and algebraic symbols used in formulas, expressions, equations, and inequalities. Fluency in the language of algebra demonstrated by confident use of its vocabulary and meanings and flexible operation upon its grammar rules (i.e., mathematical properties and conventions) are indicative of conceptual understanding in algebra, as well.

Algebra is considered to have three stages in its historical development: the rhetorical stage, the syncopated stage, and the symbolic stage. But besides these three stages of expressing algebraic ideas, there are four more conceptual stages which have happened alongside of these changes in expressions. These stages are the geometric stage, where most of the concepts of algebra are geometric ones; the static equation-solving stage, where the goal is to find numbers satisfying certain relationships; the dynamic function stage, where motion seems to be an underlying idea, and finally, the abstract stage, where mathematical structure plays the central role. The stages of algebra are, of course not entirely disjoint from one another; there is always some overlap. We discuss here high points of the development of the stages and reflect on the use of these historical stages in the teaching of algebra. We don't know of any other subject which is taught in such an anti-historical way as mathematics. Although mathematicians are often fairly scrupulous in giving credit to the original discoverers of theorems, they also are energetic in restating these theorems in terms of concepts which the original discoverers would have been completely unfamiliar with. The teaching of mathematics gives students little way of understanding where mathematical ideas have come from and what the original motivation for the development of various mathematical topics was.

Graduate students learn all sorts of high-powered concepts and theorems about Banach spaces, for instance, before they ever have any idea of why mathematicians ever got interested in such spaces or what the theory they are learning is good for. (Many students never do learn this.)
In our opinion this has a lot to do with the fact that today we see a splintering of mathematics into zillions of tiny little subspecialties, many of whose practitioners know almost nothing about any mathematics except their own little splinter.

We are not a historian of mathematics by any means. Here, we simply present a brief sketch of the development of modern Algebra, sometimes called Abstract Algebra.