A three species Syn-Eco system Consisting of Prey-Predator, Host-Comensal and Enemy-Amensal pairs with a cover for the prey species

4.1 Introduction

Prey-predation interaction involves attacks on prey by the predator species. And also, the prey species naturally tries to escape from the attacks of the predator. In the absence of a cover (protection), the prey hardly succeeds in its attempts to escape and becomes vulnerable target for the predator and eventually faces the risk of becoming extinct. Therefore a protection/cover for the prey is essential for its existence. On the other hand, as a result of the cover provided for prey, the predator species faces a possible shortage of resources which needs to be supplemented from resources other than the prey.

Some popular examples of cover for the prey are given below:

- Fish ponds are covered with nets to protect the fish from prospective predators like eagles and other birds.
- Chicken are protected from predators like birds, dogs etc. by way of being covered with nets.

This chapter deals with a three species eco system considered in chapter 2 with the inclusion of a cover/protection proportional to its population size $N_1$ of the prey ($S_1$).

![Three species Eco system](image-url)

Fig. 4.1: Three species Eco system

The steady states of the system are identified, an analysis of the local asymptotic stability is carried out for each of the states and criteria for stability have been established.
4.2 Notation

\( N_1 \): The population strength of \( S_1 \) (prey to \( S_2 \)/ commensal to \( S_3 \)).

\( N_2 \): The population strength of \( S_2 \) (predator to \( S_1 \)/ ammensal to \( S_3 \)).

\( N_3 \): The population strength of \( S_3 \) (host to \( S_1 \)/ enemy to \( S_2 \)).

\( a_i \): The Natural growth rate of \( S_i \), \( i=1,2,3 \).

\( a_{ii} \): Self inhibition coefficient of \( S_i \), \( i=1,2,3 \). (The rate of decrease of \( N_i \) due to insufficient natural resources of \( S_i \)).

\( a_{12} \): The rate of decrease of prey species (\( S_1 \)) due to inhibition by predator species (\( S_2 \)).

\( a_{13} \): The rate of increase of the commensal (\( S_1 \)) due to its promotion by its host \( S_3 \).

\( a_{21} \): The rate of increase of the predator (\( S_2 \)) due to its attacks on its prey (\( S_1 \)).

\( a_{23} \): The rate of decrease of the ammensal (\( S_2 \)) due to the harm caused by its enemy (\( S_3 \)).

\( K_i (=a_i/a_{ii}) \): Carrying capacity of \( S_i \), \( i=1,2,3 \).

\( p (=a_{12}/a_{11}) \): Coefficient of prey /commensal (\( S_1 \)) inhibition of the predator (\( S_2 \)).

\( q (=a_{13}/a_{11}) \): Coefficient of commensalism.

\( r (=a_{21}/a_{22}) \): Coefficient of predator (\( S_2 \)) consumption of the prey (\( S_1 \)).

\( s (=a_{23}/a_{22}) \): Coefficient of Ammensalism.

\( c \): Cover proportional to the population of \( S_1 \).

\( t_{ij}^* \): Dominance reversal time between \( S_i \) and \( S_j \).

Further, the three variables \( N_1 \), \( N_2 \) and \( N_3 \) are non-negative and the model parameters \( a_i, a_{ii}, i=1,2,3 \) and \( a_{12}, a_{21}, a_{13}, a_{23} \) are all assumed to be non-negative constants.

4.3 Basic Growth rate Equations of the Model

Employing the notation from section 4.2, the model equations for the three species multi reactive ecosystem are given by the following system of non-linear ordinary differential Equations.
1. Equation for the growth rate of $S_1$(with Population strength $N_1$):

\[
\frac{dN_1}{dt} = a_1 N_1 - a_{12} N_1 N_2 + a_{13} N_1 N_3 - a_{11} N_1^2 - a_{12} (1-c) N_1 N_2
\]  

\[\cdots (4.3.1)\]
2. Equation for the growth rate of $S_2$ (with population strength $N_2$):

\[
\frac{dN_2}{dt} = a_2 N_2 - a_{22} N_2^2 + a_{21} (1 - c) N_1 N_2 - a_{23} N_2 N_3
\]

\[\cdots (4.3.2)\]
3. **Equation for the growth rate of \( S_3 \) (with population strength \( N_3 \)).**

<table>
<thead>
<tr>
<th>Rate of change of the species ( S_3 )</th>
<th>Natural growth rate of ( S_3 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \frac{dN_3}{dt} ) = ( (a_3N_3) )</td>
<td>Reduction in growth rate of ( S_3 ) due to limitations of its natural resources (Self inhibition) ( (a_{33}N_3^2) )</td>
</tr>
</tbody>
</table>

i.e., \( \frac{dN_3}{dt} = a_3N_3 - a_{33}N_3^2 \) \( \ldots \) (4.3.3)

In terms of the notation adopted in 4.2, equations (4.3.1), (4.3.2) and (4.3.3) can be rewritten as

\[
\frac{dN_1}{dt} = a_{11}N_1[K_1 - N_1 - p(1-c)N_2 + qN_3] \quad \ldots \quad (4.3.4)
\]

\[
\frac{dN_2}{dt} = a_{22}N_2[K_2 - N_2 + r(1-c)N_1 - sN_3] \quad \ldots \quad (4.3.5)
\]

\[
\frac{dN_3}{dt} = a_{33}N_3[K_3 - N_3] \quad \ldots \quad (4.3.6)
\]

**4.4 Equilibrium States**

The Equilibrium states (\( \overline{N}_1, \overline{N}_2, \overline{N}_3 \)) for the present model are obtained by considering \( \frac{dN_i}{dt} = 0, i = 1, 2, 3 \).

The System under investigation has **Eight** Equilibrium states denoted by \( E_i \), \( i = 1, 2, 3, \ldots, 8 \) which are spanned over four distinct classes. Criteria for the asymptotic stability of each of these states have been derived.
A. Fully washed out state

\[ E_1 : \overline{N}_1 = 0, \overline{N}_2 = 0, \overline{N}_3 = 0 \]

B. States in which two of the three species are washed out and third exists.

\[ E_2 : \overline{N}_1 = 0, \overline{N}_2 = 0, \overline{N}_3 = K_3 \]
\[ E_3 : \overline{N}_1 = K_1, \overline{N}_2 = 0, \overline{N}_3 = 0 \]
\[ E_4 : \overline{N}_1 = 0, \overline{N}_2 = K_2, \overline{N}_3 = 0 \]

C. States in which only one of the three species is washed out while the other two co-exist.

\[ E_5 : \overline{N}_1 = 0, \overline{N}_2 = K_2 - sK_3 (K_2 > sK_3), \overline{N}_3 = K_3 \]
\[ E_6 : \overline{N}_1 = K_1 + qK_3, \overline{N}_2 = 0, \overline{N}_3 = K_3 \]
\[ E_7 : \overline{N}_1 = \frac{K_1 - pK_2 (1-c)}{1 - rp(1-c)^2} (K_1 > pK_2 (1-c)), \overline{N}_2 = \frac{K_2 - rK_1 (1-c)}{1 - rp(1-c)^2}, \overline{N}_3 = 0 \]

D. The Co-existent state or normal steady state

\[ E_8 : \overline{N}_1 = \frac{K_1 + qK_3 - p(1-c)(K_2 - sK_3)}{1 + rp(1-c)^2}, \overline{N}_2 = \frac{r(1-c)(K_1 + qK_3) + K_2 - sK_3}{1 + rp(1-c)^2}, \overline{N}_3 = K_3 \]

4.5 Stability analysis of the Equilibrium States

We consider slight deviations \( U_1(t), U_2(t) \) and \( U_3(t) \) over the steady state \( \overline{N}_1, \overline{N}_2, \overline{N}_3 \):

\[ i.e., \overline{N}_1 = \overline{N}_1 + U_1(t), \overline{N}_2 = \overline{N}_2 + U_2(t), \overline{N}_3 = \overline{N}_3 + U_3(t) \]

Where \( U_1(t), U_2(t) \) and \( U_3(t) \) are so small that their second and higher powers and products are negligible.

A. Fully washed out state

4.5.1: \( E_1 \): Fully washed out equilibrium state

\[ \overline{N}_1 = 0, \overline{N}_2 = 0, \overline{N}_3 = 0 \]

Substituting in (4.3.4), (4.3.5) and (4.3.6) we get,
\[
\begin{align*}
\frac{dU_1}{dt} &= K_1 a_{11} U_1, \\
\frac{dU_2}{dt} &= K_2 a_{22} U_2, \\
\frac{dU_3}{dt} &= K_3 a_{33} U_3.
\end{align*}
\]

The characteristic matrix for the system (4.5.1.1) is
\[
A = \begin{bmatrix}
K_1 a_{11} & 0 & 0 \\
0 & K_2 a_{22} & 0 \\
0 & 0 & K_3 a_{33}
\end{bmatrix}
\]

and the characteristic equation, \( |A - \lambda I| = 0 \) is
\[
(\lambda - K_1 a_{11})(\lambda - K_2 a_{22})(\lambda - K_3 a_{33}) = 0
\]

... (4.5.1.2)

The roots of (4.5.1.2) are \( \lambda_1 = K_1 a_{11}, \lambda_2 = K_2 a_{22}, \lambda_3 = K_3 a_{33} \) which are all positive.

So the present state is \textit{Unstable}.

The equations (4.5.1.1) yield the solution curves
\[
\begin{align*}
U_1 &= U_{10} e^{K_1 a_{11} t} \\
U_2 &= U_{20} e^{K_2 a_{22} t} \\
U_3 &= U_{30} e^{K_3 a_{33} t}
\end{align*}
\]

where \( U_{10}, U_{20} \) and \( U_{30} \) are initial values of \( U_1, U_2 \) and \( U_3 \) respectively.

\textbf{Case 4.5.1.1:} \( U_{30} > U_{20} > U_{10} \) and \( K_3 a_{33} > K_2 a_{22} > K_1 a_{11} \)

![Graph showing the solution curves](image)

\textit{Fig. 4.2:} \( U_{30} > U_{20} > U_{10} \) and \( K_3 a_{33} > K_2 a_{22} > K_1 a_{11} \)
Under this case, the natural growth rate of $S_3$ is highest followed by that of $S_2$ and $S_1$ in that order. And all the three perturbation curves of the species $S_1$, $S_2$ and $S_3$ diverge away from the equilibrium. Hence the present state is *Unstable* as shown in Figure 4.2.

**Case 4.5.1.2:** $U_{30} > U_{20} > U_{10}$ and $K_1a_{11} > K_2a_{22} > K_3a_{33}$

![Graph showing perturbation curves](image)

**Figure 4.3:** $U_{30} > U_{20} > U_{10}$ and $K_1a_{11} > K_2a_{22} > K_3a_{33}$

Here, the initial value of the perturbation $U_3$ of the species $S_3$ is more than that of $S_2$ and $S_1$ as shown in Figure 4.3. This dominance of $U_3$ over $U_2$ continues up to the time instant $t_{23}^* = \frac{1}{K_2a_{22} - K_3a_{33}} \ln \left( \frac{U_{30}}{U_{20}} \right)$ from which the dominance reverses. Similarly the dominance between $U_3$ and $U_1$ reverses from $t_{13}^* = \frac{1}{K_1a_{11} - K_3a_{33}} \ln \left( \frac{U_{30}}{U_{10}} \right)$ and between $U_1$ and $U_2$ from $t_{12}^* = \frac{1}{K_1a_{11} - K_2a_{22}} \ln \left( \frac{U_{20}}{U_{10}} \right)$.

**B. States in which two of the three species are washed out and third exists**

**4.5.2: E2:** *The State in which $S_1$ and $S_2$ are washed out while $S_3$ exists*

$\bar{N}_1 = 0$, $\bar{N}_2 = 0$, $\bar{N}_3 = K_3$

The perturbation equations for the present state, upon substitution of $\bar{N}_1$, $\bar{N}_2$ and $\bar{N}_3$ in (4.3.4), (4.3.5) and (4.3.6) are
The characteristic matrix for the system of equations (4.5.2.1) is

\[
A = \begin{bmatrix}
    a_{11}(K_1 + qK_3) & 0 & 0 \\
    0 & a_{22}(K_2 - sK_3) & 0 \\
    0 & 0 & -a_{33}K_3
\end{bmatrix}
\]

and the characteristic equation, \(|A - \lambda I| = 0\) is

\[
[a_{11}(K_1 + qK_3)](\lambda - a_{22}(K_2 - sK_3))(\lambda - K_3a_{33}) = 0 \quad \text{... (4.5.2.2)}
\]

The roots of (4.5.2.2) are

\[\lambda_1 = a_{11}(K_1 + qK_3), \quad \lambda_2 = a_{22}(K_2 - sK_3) \quad \text{and} \quad \lambda_3 = -K_3a_{33},\]

Of these, \(\lambda_1\) is evidently positive. So the current state is \textit{Unstable}.

The solution curves of (4.5.2.1) are

\[
\begin{align*}
    U_1 &= U_{10}e^{a_{11}(K_1 + qK_3)t} \\
    U_2 &= U_{20}e^{a_{22}(K_2 - sK_3)t} \\
    U_3 &= U_{30}e^{-K_3a_{33}t}
\end{align*}
\]

Where \(U_{10}, U_{20}, \text{and } U_{30}\) are the initial values of \(U_1, U_2, \text{and } U_3\) respectively.

\textbf{Case 4.5.2.A: } \(K_2 - sK_3 > 0\)

Under this condition, the two of the Eigen values (\(\lambda_1\) and \(\lambda_2\)) are positive and the third one (\(\lambda_3\)) is negative. Hence the present state is \textit{Unstable}.
**Case 4.5.2.A (i):** $U_{30} > U_{20} > U_{10}$

As illustrated in Figure 4.4, the curves $U_1$ and $U_2$ of the species $S_1$ and $S_2$ respectively, diverge away from the equilibrium position while $U_3$ of $S_3$ converges to it asymptotically. Further, the perturbation $U_3$ initially dominates $U_1$ and $U_2$ and the dominance reverses between $U_2$ and $U_3$ from the time instant

$$t^*_2 = \frac{1}{a_{22}(K_2 - sK_3)} \ln\left(\frac{U_{30}}{U_{20}}\right)$$

and between $U_1$ and $U_3$ from

$$t^*_1 = \frac{1}{a_{11}(K_1 + qK_3)} \ln\left(\frac{U_{30}}{U_{10}}\right)$$

Further, the perturbation $U_2$ of the species $S_2$ dominates $U_1$ initially, and the dominance reverses from

$$t^*_2 = \frac{1}{a_{11}(K_1 + qK_3) - a_{22}(K_2 - sK_1)} \ln\left(\frac{U_{20}}{U_{10}}\right)$$

**Case 4.5.2.A (ii):** $U_{20} > U_{30} > U_{10}$

As illustrated in Figure 4.5, the curves $U_1$ and $U_2$ of the species $S_1$ and $S_2$ respectively, diverge away from the equilibrium position while $U_3$ of $S_3$ converges to it asymptotically. Further, the perturbation $U_3$ initially dominates $U_1$ and $U_2$ and the dominance reverses between $U_2$ and $U_3$ from the time instant

$$t^*_2 = \frac{1}{a_{22}(K_2 - sK_3)} \ln\left(\frac{U_{30}}{U_{20}}\right)$$

and between $U_1$ and $U_3$ from

$$t^*_1 = \frac{1}{a_{11}(K_1 + qK_3)} \ln\left(\frac{U_{30}}{U_{10}}\right)$$

Further, the perturbation $U_2$ of the species $S_2$ dominates $U_1$ initially, and the dominance reverses from

$$t^*_2 = \frac{1}{a_{11}(K_1 + qK_3) - a_{22}(K_2 - sK_1)} \ln\left(\frac{U_{20}}{U_{10}}\right)$$

 onwards.
Under this case, initially the perturbation $U_2$ of the species $S_2$ dominates the remaining two perturbations $U_1$ and $U_3$. But from the time

$$t_{12}^* = \frac{1}{a_{11}(K_1 + qK_3) - a_{22}(K_2 - sK_3)} \ln \left( \frac{U_{20}}{U_{10}} \right)$$

the dominance reverses between $U_1$ and $U_2$.

Also, from $t_{31}^* = \frac{1}{a_{11}(K_1 + qK_3) + K_3 a_{33}} \ln \left( \frac{U_{30}}{U_{10}} \right)$, the dominance reverses between $U_1$ and $U_3$. This is shown in Figure 4.5.

**Case 4.5.2.B: $K_2 - sK_3 < 0$**

Under this condition, two of the Eigen values ($\lambda_2$ and $\lambda_3$) are negative and the third ($\lambda_1$) is positive. Hence this state is *Unstable*.

**Case 4.5.2.B (i): $U_{10} > U_{20} > U_{30}$**

Here, the perturbations of the species $S_2$ and $S_3$ converge to the equilibrium position while that of $S_1$ diverges away from the equilibrium position as shown in Figure 4.6.

![Fig. 4.6: $U_{10} > U_{20} > U_{30}$](image-url)
Case 4.5.2.B (ii): \( U_{30} > U_{20} > U_{10} \)

Fig. 4.7: \( U_{30} > U_{20} > U_{10} \)

Under this condition, though the initial value of the perturbation \( U_3 \) of the species \( S_3 \) is highest, it reduces with time and falls below \( U_1 \) of \( S_1 \) from the time

\[
t_{31}^* = \frac{1}{a_{11}(K_1 + qK_3) + K_3a_{33}} \ln \left( \frac{U_{30}}{U_{10}} \right)
\]

onwards and below \( U_2 \) of \( S_2 \) from

\[
t_{32}^* = \frac{1}{a_{22}(K_2 - sK_3) + K_3a_{33}} \ln \left( \frac{U_{40}}{U_{20}} \right)
\]
onwards. Further, \( U_2 \) dominates \( U_1 \) initially but the dominance reverses from

\[
t_{12}^* = \frac{1}{a_{11}(K_1 + qK_3) - a_{22}(K_2 - sK_3)} \ln \left( \frac{U_{20}}{U_{10}} \right).
\]

This is shown in Figure 4.7.

4.5.3: \( E_3 \): The State in which \( S_2 \) and \( S_3 \) are washed out while \( S_1 \) exists

\[
\bar{N}_1 = K_1, \quad \bar{N}_2 = 0, \quad \bar{N}_3 = 0
\]

Substituting in (4.3.4), (4.3.5) and (4.3.6), the equations of perturbation for the present state are

\[
\begin{align*}
\frac{dU_1}{dt} &= -a_{11}K_1 U_1 - pa_{11}K_1 (1-c)U_2 + qa_{11}K_1 U_3 \\
\frac{dU_2}{dt} &= a_{22}K_2 + r(1-c)K_1 U_2 \\
\frac{dU_3}{dt} &= a_{33}K_3 U_3
\end{align*}
\]

\[\text{... (4.5.3.1)}\]
for which, the characteristic matrix is
\[
A = \begin{bmatrix}
-a_{11}K_1 & -pa_{11}K_1(1-c) & qa_{11}K_1 \\
0 & a_{22}[K_2 + r(1-c)K_1] & 0 \\
0 & 0 & a_{33}K_3 \\
\end{bmatrix}
\]

and the corresponding roots are
\[
\lambda_1 = -a_{11}K_1, \quad \lambda_2 = a_{22}[K_2 + r(1-c)K_1] \quad \text{and} \quad \lambda_3 = K_3a_{33}.
\]

Of these, \(\lambda_1\) is negative where as \(\lambda_2\) and \(\lambda_3\) are evidently positive. So the current state is \textit{Unstable}.

**Case 4.5.3.1: \(U_{10} > U_{30} > U_{20}\)**

Under this case, the curves corresponding to the perturbations of \(S_2\) and \(S_3\) diverge away from the equilibrium position while that of \(S_1\) moves towards it. Further, the dominance reversal times for various pairs of curves are given by \(t_{31}^*, t_{21}^*\) and \(t_{23}^*\) as shown in Figure 4.8.

![Figure 4.8: U_{10} > U_{30} > U_{20}](image-url)
Case 4.5.3.2: $U_{30} > U_{10} > U_{20}$

Here, the initial value of perturbation $U_3$ of the species $S_3$ is more than those of $S_1$ and $S_2$ and it continues so till the end. Among $U_1$ and $U_2$, initially the curve $U_1$ dominates $U_2$ but from the time $t_{12}$ onwards the dominance reverses as shown in Figure 4.9.

Case 4.5.4: $E_4$: The State in which $S_1$ and $S_3$ are washed out while $S_2$ is not

$\overline{N}_1 = 0, \overline{N}_2 = K_2, \overline{N}_3 = 0$

This state, upon substitution in (4.3.4), (4.3.5) and (4.3.6), will result in the perturbation equations given by

\[
\begin{align*}
\frac{dU_1}{dt} &= a_{11}[K_1 - pK_2 (1 - c)]U_1 \\
\frac{dU_2}{dt} &= a_{22}[rK_2 (1 - c)U_1 - K_2 U_2 - K_2 sU_3] \\
\frac{dU_1}{dt} &= a_{33}K_3 U_3
\end{align*}
\]

... (4.5.4.1)

The characteristic matrix for the system (4.5.4.1) is given by

\[
A = \begin{bmatrix}
a_{11}[K_1 - pK_2 (1 - c)] & 0 & 0 \\
a_{22}rK_2 (1 - c) & -a_{22}K_2 & -a_{22}K_2 s \\
0 & 0 & a_{33}K_3
\end{bmatrix}
\]

And the roots of the corresponding characteristic equation are

$\lambda_1 = a_{11}[K_1 - pK_2 (1 - c)], \quad \lambda_2 = -a_{22}K_2 \quad \text{and} \quad \lambda_3 = K_3 a_{33}.$
Of these, $\lambda_2$ is negative and $\hat{\lambda}_3$ is evidently positive. So the current state is *Unstable*.

**Case 4.5.4.A**: $\lambda_1 > 0$

Under this case, two Eigen values are positive and the third is negative.

**Case 4.5.4.A (i)**: $U_{20} > U_{10} > U_{30}$

As shown in Figure 4.10, under the condition of the present case, the perturbation curves corresponding to the species $S_1$ and $S_3$ diverge away from the equilibrium while that of $S_2$ approaches the equilibrium. Though the perturbation $U_2$ of the species $S_2$ is maximum initially, it comes below that of $U_1$ from the time $t_{12}^*$ and below $U_3$ from the time $t_{32}^*$. Further, among the curves $U_1$ and $U_3$, $U_1$ dominates $U_3$ until the time $t_{31}^*$ from which the dominance reverses.

![Figure 4.10: $U_{20} > U_{10} > U_{30}$](image)

**Case 4.5.4.A (ii)**: $U_{30} > U_{20} > U_{10}$

Here, the initial value of the perturbation of the species $S_3$ is more than those of $S_1$ and $S_2$ and the dominance continues till the end. But the curve $U_2$ corresponding to the species $S_2$ comes below the curve $U_1$ from the time $t_{12}^*$ onwards. This is illustrated in Figure 4.11.
Case 4.5.4.B: $\lambda_i < 0$

Under this case, two Eigen values are negative and the third is positive.

Case 4.5.4.B (i): $U_{30} > U_{20} > U_{10}$

Under the condition of this case, the perturbation curves corresponding to the species $S_1$ and $S_2$, i.e. $U_1$ and $U_2$ move towards the equilibrium while that of $S_3$ moves away from it. Dominance reversal takes place between $U_1$ and $U_2$ from $t_{12}^*$ onwards from where $U_1$ dominates from $U_2$ as shown in Figure 4.12.
Case 4.5.4.B (ii): $U_{20} > U_{30} > U_{10}$

Here, even though the initial value of $U_2$ is maximum, it falls below $U_3$ from $t_{32}^*$ onwards and below $U_1$ from $t_{12}^*$ onwards. This is shown in Figure 4.13.

C. States in which only one of the three species is washed out while the other two co-exist

4.5.5: $E_5$: The State in which $S_1$ only is washed out while $S_2$ and $S_3$ co-exist

$\bar{N}_1 = 0$,  $\bar{N}_2 = K_2 - sK_3$ ($K_2 > sK_3$),  $\bar{N}_3 = K_3$

The equations of perturbation for the present case from (4.3.4), (4.3.5) and (4.3.6)

are

$$\frac{dU_1}{dt} = a_{11}[K_1 - p(1-c)\bar{N}_2 + q\bar{N}_3]U_1$$

$$\frac{dU_2}{dt} = a_{22}[\bar{N}_2r(1-c)U_1 + (K_2 - 2\bar{N}_2 - s\bar{N}_3)U_2 - s\bar{N}_2U_3 + K_2\bar{N}_2 - \bar{N}_2^2 - s\bar{N}_2\bar{N}_3]$$

$$\frac{dU_3}{dt} = -a_{33}K_3U_3$$

The characteristic matrix for the system (4.5.5.1) is given by

$$A = \begin{bmatrix} a_{11}[K_1 - p(1-c)\bar{N}_2 + q\bar{N}_3] & 0 & 0 \\ a_{22}\bar{N}_2r(1-c) & a_{22}(K_2 - 2\bar{N}_2 - s\bar{N}_3) - a_{22}s\bar{N}_2 & 0 \\ 0 & 0 & -a_{33}K_3 \end{bmatrix}$$
for which the corresponding roots are

\[
\lambda_1 = a_{11}[K_1 - p(1 - c)\bar{N}_2 + q\bar{N}_3], \quad \lambda_2 = a_{22}(K_2 - 2\bar{N}_2 - s\bar{N}_3) \quad \text{and} \quad \lambda_3 = -a_{33}K_2.
\]

**Case 4.5.5.A:** \( \lambda_1 > 0 \) and \( \lambda_2 < 0 \)

In this case, two of the roots are negative and the third one is positive. Hence this state is *Unstable.*

**Case 4.5.5.A (i):** \( U_{10} > U_{20} > U_{30} \)

As shown in Figure 4.14, the perturbation curves corresponding to the species \( S_1 \) and \( S_2 \) diverge from the equilibrium position while that of \( S_3 \) converges to the equilibrium. Even though the initial value of \( U_3 \) is greater than that of \( U_2 \) the dominance continues up to the time \( t_{23}^* \) and there onwards, the dominance reverses.

**Case 4.5.5.A (ii):** \( U_{10} > U_{20} > U_{30} \)

Under the condition of this case, the curves \( U_1 \) and \( U_2 \) move away from the equilibrium while the curve \( U_3 \) moves towards it. Among the three quantities \( U_1, U_2 \) and \( U_3 \), \( U_3 \) has the largest initial value. But it decreases and falls below \( U_1 \) and below \( U_2 \) from the time instances \( t_{13}^* \) and \( t_{23}^* \) onwards respectively. This is shown in Figure 4.15.
Case 4.5.5.B: $\lambda_1 < 0$ and $\lambda_2 < 0 \Rightarrow \frac{a_2}{a_{23}} > K_3$ and $sK_3 < K_2$

Here all the characteristic roots are negative. Hence the present state is *Stable* under this case.

**Case 4.5.5.B (i):** \( U_{30} > U_{20} > U_{10} \)

![Fig. 4.16: U_{30} > U_{20} > U_{10}]()

Under this case, all the curves \( U_1, U_2 \) and \( U_3 \) corresponding to the three species \( S_1, S_2 \) and \( S_3 \) respectively, move towards the equilibrium state vide Figure 4.16.

**Case 4.5.5.B (ii):** \( U_{10} > U_{30} > U_{20} \)

![Fig. 4.17: U_{10} > U_{30} > U_{20}]()
In this case too, all the three curves $U_1$, $U_2$ and $U_3$ move towards the equilibrium position as shown in Fig. 4.17.

### 4.5.6: $E_6$: The State in which $S_2$ only is washed out while $S_1$ and $S_3$ co-exist

$$\bar{N}_1 = K_1 + qK_3, \quad \bar{N}_2 = 0, \quad \bar{N}_3 = K_3$$

Upon substitution of $N_1 = \bar{N}_1 + U_1(t)$, $N_2 = \bar{N}_2 + U_2(t)$ and $N_3 = \bar{N}_3 + U_3(t)$, we get the equations of perturbation for this case from (4.3.4), (4.3.5) and (4.3.6) as

$$\begin{align*}
\frac{dU_1}{dt} &= a_{11} \left[(K_1 - 2\bar{N}_1 + q\bar{N}_3)U_1 - p(1-c)\bar{N}_1U_2\right] + q\bar{N}_1U_3 + K_1 \bar{N}_1 - \bar{N}_1^2 + q\bar{N}_1\bar{N}_3 \\
\frac{dU_2}{dt} &= a_{22} \left[K_2 + r(1-c)\bar{N}_1 - s\bar{N}_1\right]U_2 \\
\frac{dU_1}{dt} &= -a_{33}K_3U_3
\end{align*}$$

... (4.5.6.1)

The characteristic matrix for the system (4.5.6.1) is given by

$$A = \begin{bmatrix}
a_{11}(K_1 - 2\bar{N}_1 + q\bar{N}_3) & -a_{11}p(1-c)\bar{N}_1 & a_{11}q\bar{N}_1 \\
0 & a_{22}[K_2 + r(1-c)\bar{N}_1 - s\bar{N}_3] & 0 \\
0 & 0 & -a_{33}K_3
\end{bmatrix}$$

and the corresponding roots are

$$\lambda_1 = a_{11}(K_1 - 2\bar{N}_1 + q\bar{N}_3), \quad \lambda_2 = a_{22}[K_2 + r(1-c)\bar{N}_1 - s\bar{N}_3], \quad \lambda_3 = -a_{33}K_3.$$

Of these, $\lambda_1, \lambda_3$ are evidently negative.

### Case 4.5.6.A: $\lambda_2 > 0$

Under this case, two characteristic roots are negative while the third is positive. Hence, this state is **Unstable**.
Case 4.5.6.A (i): $U_{10} > U_{30} > U_{20}$ 

When the condition above holds the perturbation curves $U_1$ and $U_3$ corresponding to the species $S_1$ and $S_3$ move towards the equilibrium while that of $U_2$ diverges from the equilibrium. The perturbation $U_1$ with initial strength more than that of the remaining two quantities $U_2$ and $U_3$, falls below that of $S_2$ from $t_{21}^*$ onwards and below that of $S_3$ from the time instant $t_{32}^*$ onwards. Further, the dominance between $S_2$ and $S_3$ reverses from $t_{23}^*$ onwards from where $S_2$ takes a dominating position. This is illustrated in Fig. 4.18.

Case 4.5.6.A (ii): $U_{20} > U_{30} > U_{10}$ 

As shown in the Figure 4.19, the present state is *Unstable* under this case too with the perturbation curve corresponding to the species $S_2$, i.e. $U_2$ diverging from the equilibrium while the remaining two curves $U_1$ and $U_3$ approaching it asymptotically.
Case 4.5.6.B: $\lambda_2 < 0$

Under this case, all the three characteristic roots are negative and hence the present state is *Stable* under the present case.

**Case 4.5.6.B (i):** $U_{10} > U_{20} > U_{30}$

![Graph showing perturbations with $U_{10} > U_{20} > U_{30}$]

Fig. 4.20: $U_{10} > U_{20} > U_{30}$

Here, all the three curves of perturbations $U_1$, $U_2$ and $U_3$ approach the equilibrium asymptotically. As illustrated in Figure 4.20, the dominance reversal time instant for the species $S_1$ and $S_2$ is $t_{21}^*$ and that for the species $S_1$ and $S_3$ is $t_{13}^*$.

**Case 4.5.6.B (ii):** $U_{30} > U_{20} > U_{10}$

![Graph showing perturbations with $U_{30} > U_{20} > U_{10}$]

Fig. 4.21: $U_{30} > U_{20} > U_{10}$

Under this case, all the curves corresponding to the three species converge to the equilibrium. Further, the species $U_1$, which is dominated by $U_2$ initially, dominates $U_2$ from the time instant $t_{12}^*$ onwards.
4.5.7: The State in which $S_3$ only is washed out while $S_1$ and $S_2$ co-exist

\[
\overline{N}_1 = \frac{K_1 - pK_2(1-c)}{1 + rp(1-c)^2}(K_1 > pK_2(1-c)), \quad \overline{N}_2 = \frac{K_2 + rK_1(1-c)}{1 + rp(1-c)^2}, \quad \overline{N}_3 = 0
\]

From (4.3.4), (4.3.5) and (4.3.6), the equations of perturbation for the present case are

\[
\begin{align*}
\frac{dU_1}{dt} &= a_{11} \left[ (K_1 - 2\overline{N}_1 - p(1-c)\overline{N}_2)U_1 - p(1-c)\overline{N}_1U_2 \right] \\
&\quad + q\overline{N}_1U_3 + K_1\overline{N}_1 - \overline{N}_1 - p(1-c)\overline{N}_1\overline{N}_2 \\
\frac{dU_2}{dt} &= a_{22} \left[ r(1-c)\overline{N}_2U_1 - (K_2 - 2\overline{N}_2 + r(1-c)\overline{N}_1)U_2 \right] \\
&\quad - s\overline{N}_2U_3 + K_2\overline{N}_2 - \overline{N}_2 + r(1-c)\overline{N}_1\overline{N}_2 \\
\frac{dU_3}{dt} &= a_{33}K_3U_3
\end{align*}
\]

... (4.5.7.1)

and the corresponding characteristic matrix is

\[
A = \begin{bmatrix}
  a_{11}(K_1 - 2\overline{N}_1 - p(1-c)\overline{N}_2) & -a_{11}p(1-c)\overline{N}_1 & a_{11}q\overline{N}_1 \\
  a_{22}r(1-c)\overline{N}_2 & a_{22}[K_2 - 2\overline{N}_2 + r(1-c)\overline{N}_1] & -a_{22}s\overline{N}_2 \\
  0 & 0 & a_{33}K_3
\end{bmatrix}
\]

The characteristic equation for the above matrix is

\[
(a_3 - \lambda)^3 \left[ a_{11}(K_1 - 2\overline{N}_1 - p(1-c)\overline{N}_2) \right] - \lambda^2 \left[ a_{22}(K_2 - 2\overline{N}_2 + r(1-c)\overline{N}_1) \right] - \lambda \left[ a_{11}p(1-c)\overline{N}_1 \right] + a_{11}q\overline{N}_1 = 0
\]

... (4.5.7.2)

One root of the equation (4.5.7.2) is, say $\lambda_1 = a_3$ which is positive. Hence the present state is Unstable.

**Case 4.5.7 (i): $U_{30} > U_{20} > U_{10}$**

In this case, the perturbation curves corresponding to species $S_1$ and $S_3$, i.e. $U_1$ and $U_3$ diverge away from the equilibrium position while that corresponding to $U_2$ converges to it. Even though the initial value of $U_2$ is more compared to that of $U_3$, the dominance reverses from the time $t_{12}^*$ onwards. This is shown in Figure 4.22.

Fig. 4.22: $U_{30} > U_{20} > U_{10}$
Case 4.5.7 (ii): $U_{20} > U_{30} > U_{10}$

In this case too, the perturbation curves corresponding to species $S_1$ and $S_3$, i.e. $U_1$ and $U_3$ diverge from the equilibrium position while that corresponding to $U_2$ moves towards it. Here, initially the value of $U_2$ is more compared to that of $U_1$ and $U_3$. But it falls below that of $S_3$ and below that of $S_1$ from the times $t_{32}^*$ and $t_{12}^*$ respectively as shown in Figure 4.23.

Fig. 4.23: $U_{20} > U_{30} > U_{10}$

D. The Normal steady state

4.5.8: $E_8$: The Co-existent state

$$
\bar{N}_1 = \frac{K_1 + qK_3 - p(1-c)(K_2 - sK_3)}{1 + rp(1-c)^2} > 0,
$$

$$
\bar{N}_2 = \frac{r(1-c)(K_1 + qK_3) + K_2 - sK_3}{1 + rp(1-c)^2} > 0,
$$

$$
\bar{N}_3 = K_3
$$

The equations of perturbation for the present state upon substitution of $\bar{N}_1, \bar{N}_2$ and $\bar{N}_3$ in (4.3.4), (4.3.5) and (4.3.5) are

$$
\frac{dU_1}{dt} = a_{11} \left[ \frac{(K_1 - 2\bar{N}_1 - p(1-c)\bar{N}_2 + q\bar{N}_3)U_1 - p(1-c)\bar{N}_1U_2 + q\bar{N}_1U_3 + K_1\bar{N}_1}{-\bar{N}_1 - p(1-c)\bar{N}_2 + q\bar{N}_1\bar{N}_3} \right]
$$

$$
\frac{dU_2}{dt} = a_{22} \left[ \frac{r(1-c)\bar{N}_2U_1 - (K_2 - 2\bar{N}_2 + r(1-c)\bar{N}_1 - s\bar{N}_2)U_2 - s\bar{N}_2U_3 + K_2\bar{N}_2 - \bar{N}_2^2}{+r(1-c)\bar{N}_1\bar{N}_2 - s\bar{N}_2\bar{N}_3} \right] \quad \ldots (4.5.8.1)
$$

$$
\frac{dU_3}{dt} = -a_{33}K_3U_3
$$
The characteristic matrix for the system (4.5.8.1) is

\[
A = \begin{bmatrix}
    a_{11}(K_1 - 2\bar{N}_1 - p(1-c)\bar{N}_2 + q\bar{N}_3) & -a_{11}p(1-c)\bar{N}_1 & a_{11}q\bar{N}_1 \\
    a_{22}r(1-c)\bar{N}_2 & a_{22}[K_2 - 2\bar{N}_2 + r(1-c)\bar{N}_1 - s\bar{N}_3] & -a_{22}s\bar{N}_2 \\
    0 & 0 & -a_{33}K_3 
\end{bmatrix}
\]

Let \(a_{11}(K_1 - 2\bar{N}_1 - p(1-c)\bar{N}_2 + q\bar{N}_3) = \alpha_1\); 
\(-a_{11}p(1-c)\bar{N}_1 = \beta_1\); 
\(a_{22}r(1-c)\bar{N}_2 = \alpha_2\); 
and \(a_{22}[K_2 - 2\bar{N}_2 + r(1-c)\bar{N}_1 - s\bar{N}_3] = \beta_2\).

Then the characteristic equation of \(A\), \(|A - \lambda I| = 0\) is

\[
(-a_{33}K_3 - \lambda)[(\alpha_1 - \lambda)(\beta_2 - \lambda) - \beta_1\alpha_2] = 0
\]

Clearly one root of the equation (4.5.8.2) is \(-a_{33}K_3 = \lambda_1\) (say) is negative. For the remaining two roots, say, \(\lambda_2, \lambda_3\) consider

\[
[(\alpha_1 - \lambda)(\beta_2 - \lambda) - \beta_1\alpha_2] = 0
\]

\[\Rightarrow \lambda^2 - (\alpha_1 + \beta_2)\lambda + \alpha_1\beta_2 - \beta_1\alpha_2 = 0\]  \([4.5.8.3]\)

**Case 4.5.8.1**: \(\alpha_1 + \beta_2 > 0\) and \(\alpha_1\beta_2 - \beta_1\alpha_2 > 0\)

Under this case, \(\lambda_2\) and \(\lambda_3\) are positive. Hence one root is negative and the remaining two roots are positive. Therefore, this state is **Unstable**.
As illustrated in Figure 4.24, $U_1$ and $U_3$, the respective perturbation curves of the species $S_1$ and $S_3$, diverge away from the equilibrium while that corresponding to $S_2$, $U_2$, converges to it. Also, the initial value of $U_2$ is more than those of $U_1$ and $U_3$. But the dominance reverses between $U_2$ and $U_3$ from $t_{32}^*$ onwards and between $U_2$ and $U_1$ from $t_{12}^*$ onwards.

**Case 4.5.8.2:** $\alpha_1 + \beta_2 < 0$ and $\alpha_1 \beta_2 - \beta_1 \alpha_2 > 0$

Under this case, $\lambda_2$ and $\lambda_3$ are negative. Hence all the three roots are negative. Therefore, this state is *Stable*.

![Diagram showing $U_1$, $U_2$, and $U_3$ perturbation curves]

**Fig. 4.25:** $\alpha_1 + \beta_2 < 0$ and $\alpha_1 \beta_2 - \beta_1 \alpha_2 > 0$

Here, all the three perturbation curves approach the equilibrium asymptotically as shown in Figure 4.25.

**Case 4.5.8.3:** $\alpha_1 \beta_2 - \beta_1 \alpha_2 < 0$

Under this case, among $\lambda_2$ and $\lambda_3$, one is positive and the other is negative. Hence one root is positive and the remaining two roots are negative. Therefore, this state is *Unstable*.

![Diagram showing $U_2$, $U_1$, and $U_3$ perturbation curves]

**Fig. 4.26:** $\alpha_1 \beta_2 - \beta_1 \alpha_2 < 0$
Under this case, $U_1$, the perturbation curve of the species $S_1$ diverges away from the equilibrium position while those of $S_2$ and $S_3$ converge to it. This is shown in Figure 4.26.

### 4.6 CONCLUSION

In this chapter, a stability analysis is carried out for a three species Eco system with cover for prey species $S_1$ and an alternate food for predator species $S_2$ and the observations on criteria for stability are made as under:

i. $E_5: \bar{N}_1 = 0, \quad \bar{N}_2 = K_2 - sK_3 (K_2 > sK_3), \quad \bar{N}_3 = K_3$:

This state is *Stable* when $sK_3 < K_2$. This is illustrated in cases 4.5.5.B(i) and 4.5.5.B(ii).

ii. $E_6: \bar{N}_1 = K_1 + qK_3, \quad \bar{N}_2 = 0, \quad \bar{N}_3 = K_3$:

Here, the species $S_2$ is washed out and $S_1$ and $S_3$ co-exist. This state is Stable when $\frac{a_2}{a_{23}} < K_3$ as shown in cases 4.5.6.B (i) and 4.5.6.B (ii).

$$E_8: \bar{N}_1 = \frac{K_1 + qK_3 - p(1-c)(K_2 - sK_3)}{1 + rp(1-c)^2} > 0,$$

iii. $\bar{N}_2 = \frac{r(1-c)(K_1 + qK_3) + K_2 - sK_3}{1 + rp(1-c)^2} > 0$,

$$\bar{N}_3 = K_3$$

This co existent state of all the three species is *Stable* when $\alpha_1 + \beta_2 < 0$ and $\alpha_1\beta_2 - \beta_1\alpha_2 > 0$

where $a_{11}(K_1 - 2\bar{N}_1 - p(1-c)\bar{N}_2) + q\bar{N}_3 = \alpha_1$;

$-a_{12}p(1-c)\bar{N}_1 = \beta_1$;

$a_{22}(1-c)\bar{N}_2 = \alpha_2$;

and $a_{23}[K_2 - 2\bar{N}_2 + r(1-c)\bar{N}_1 - s\bar{N}_3] = \beta_2$ as mentioned in case 4.5.8.2.

***************

175