Chapter 3

Different option pricing formulas
An option is a financial derivative, which represents a contract sold by one party to another party. The contract gives the buyer the right but not the obligation, to buy or sell any asset.

There are some reasons, why one should use an option? The main advantage of using option is that, one can also make profit, when the market goes down.

3.1 Types of option:

We have already discussed two basic types of options in chapter-1.

1. A call option
2. A put option

A call option holder has a right to buy an asset for certain price, at a certain date, while a put option holder has a right to sell an asset for certain price, at a certain date. The specified date is called an expiration date or maturity date and a specified price is called the exercise price or a strike price. According to the style, there are two types of options:

1. American option
2. European option

The only difference between these two options is that, American option can be exercised at any time till the expiration date, while European option can be exercised only at the expiration date. Generally most of the exchanges tread American option.

In fact European option are comparatively easy to analyze than American option.
3.2 Pricing an option:

In this section we will consider only European type of option.

First, we will see, what are the factors that affect pricing of stock(asset) option.

Here, we have given some of the factors, which affects the stock(asset) option price [12]:

1. The current stock(asset) price, $S_0$
2. The Strike price, $X$
3. The expiration time, $T$
4. The Volatility, $\sigma$
5. The risk free interest rate, $r$
6. The dividends that are expected to be paid.

We will also assume that:

1. There are no transaction costs involved.
2. All treading profits are subject to the same tax rate.
3. Borrowing and lending are possible only at the risk free interest rate, also we will consider that, there is no arbitrage opportunity.
Some useful notations:

- $S_0$ is current stock(asset) price
- $S_T$ is stock(asset) price at expiration
- $X$ is Exercise price or strike price
- $T$ is Total period of time
- $C$ is the value of call option to buy one unit of share(asset)
- $P$ is the value of put option to sell one unit of share(asset)

We will consider $r > 0$.

### 3.3 Option Pricing Models:

Option pricing theory is very important topic in Finance since 1972, When Black and Scholes published a paper in which they have given a model for valuing a European option using “replication” portfolio.

In fact, their derivation is mathematically complicated but using some logic, there are also simple models for valuing options.

In this section, we will consider two models namely:

1. Binomial Option Pricing Model (BOPM).
2. Black Scholes Merton Model.
3.3.1 Binomial Option Pricing Model (BOPM):

This is very useful and popular model to value an option using tree.

This method can be visualized through diagram, in which the stock(asset) price has only two possibilities, either it can go up by a certain percentage by a certain probability or it can go down.

It is based on the Binomial Tree Model for changes in price of the stock(asset).

In this model, we divide total time interval $[0,T]$ in to $n$ equal sub intervals, and assume that, the asset price will move either up by some amount or down by some amount.

**Single Step BOPM**

Let us assume $n = 1$.

Consider Stock(asset), having initial price $S_0$, and option on the stock(asset) having initial premium $C$. Over an interval $[0,T]$, the stock price moves from $S_0$ to either $S_0u$ or $S_0d$, where $u > 1 > d$.

Similarly, for call premium, either it is $C_u = \max \{S_0u - X, 0\}$ or $C_d = \max \{S_0d - X, 0\}$. 
Now, we will consider the portfolio which consists of buying $\Delta$ shares (assets) and selling one option. We will find the value of $\Delta$ in such a way so that the portfolio becomes risk less.

If we assume that, the movement is up then the value of the portfolio at time $T$ is

$$S_0u\Delta - C_u$$

If we assume that, the movement is down then the value of the portfolio at time $T$ is

$$S_0d\Delta - C_d$$

These two are equal when,

$$S_0u\Delta - C_u = S_0d\Delta - C_d$$
Therefore,

\[ \Delta = \frac{C_u - C_d}{S_0u - S_0d} \]

This equation shows that \( \Delta \) is the ratio of the change in the option price to the change in the stock(asset) price as we move between the nodes at time \( T \).

Now, present value of the portfolio is \((S_0u\Delta - C_u)e^{-rT}\), where \( r \) is the risk free interest rate.

The actual cost of the portfolio is \( S_0\Delta - C \)

We have,

\[ S_0\Delta - C = (S_0u\Delta - C_u)e^{-rT} \]

and therefore,
\[ C = S_0 \Delta \left( 1 - u e^{-rT} \right) + C_u e^{-rT} \]
\[ = S_0 \left( \frac{C_u - C_d}{S_0 u - S_0 d} \right) \left( 1 - u e^{-rT} \right) + C_u e^{-rT} \]
\[ = \frac{C_u \left( 1 - d e^{-rT} \right) + C_d \left( u e^{-rT} - 1 \right)}{u - d} \]
\[ = e^{-rT} \left( p C_u + (1 - p) C_d \right) \]

where,
\[ p = \frac{e^{rT} - d}{u - d} \]

**Example 3.3.1.** Suppose we have single step BOPM with \( S = 100, X = 100, u = 1.1, d = 0.9, r = 10\% \) and \( T = 1 \). Find the value of \( C \).

**Solution 3.3.1.** Here we have
\[ C = e^{-rT} \left( p C_u + (1 - p) C_d \right) \]

First, we will find the value of \( p \),
\[ p = \frac{e^{rT} - d}{u - d} \]
\[ = \frac{e^{0.1} - 0.9}{1.1 - 0.9} \]
\[ = \frac{1.0258 - 0.9}{1.1 - 0.9} \]
\[ = 1.0258 \]

Here,
\[ C_u = 10 \quad \text{and} \quad C_d = 0 \]
Therefor, $C = 9.2818$

### 3.4 Black Scholes Merton Model:

In 1970s, Fisher Black, Myron Scholes and Robert Merton have derived the model for pricing of European stock option. The model has become famous with the name Black Scholes Merton Model. The use of this model is to find mathematical value of an option. Robert Merton and Myron Scholes were also awarded by the Nobel prize in 1997 in Economics for this contribution to the society. Unfortunately, Fisher Black died in 1995, other wise, he too would be one of the prize recipients.

Now, we will derive BSM differential equation.

We will use the following assumption in the derivation [12]:

1. The stock(asset) price follows the GBM with constant $\mu$ and $\sigma$.
2. Short selling is permitted.
3. There are no transaction cost.
4. No dividends are paid throughout the life of the derivative.
5. No arbitrage opportunity.
7. $r$ is constant.

**Derivation of BSM differential equation**
Consider a price of derivative at a time \( t \). Suppose \( T \) is the expiry time, then the total time of maturity is \( T - t \).

As we have assumed the stock(asset) price follows GBM, we have:

\[
 dS = \mu S dt + \sigma S dZ \tag{3.3}
\]

Suppose that \( C \) is the price of a call option or any other derivative on \( S \).

The variable \( C \) must be a function of \( S \) and \( t \).

By Ito’s lemma (See Chapter-2) we have,

\[
 dC = \left( \frac{\partial C}{\partial S} \mu S + \frac{\partial C}{\partial t} + \frac{1}{2} \frac{\partial^2 C}{\partial S^2} \sigma^2 S^2 \right) dt + \frac{\partial C}{\partial S} \sigma S dZ \tag{3.4}
\]

Equation 3.3 and 3.4 can also be written as

\[
 \Delta S = \mu S \Delta t + \sigma S \Delta Z \tag{3.5}
\]

and

\[
 \Delta C = \left( \frac{\partial C}{\partial S} \mu S + \frac{\partial C}{\partial t} + \frac{1}{2} \frac{\partial^2 C}{\partial S^2} \sigma^2 S^2 \right) \Delta t + \frac{\partial C}{\partial S} \sigma S \Delta Z \tag{3.6}
\]

Where, \( \Delta S \) and \( \Delta C \) are the changes in \( S \) and \( C \) in a small time interval \( \Delta t \).
Consider a portfolio, which consists of buying \( \frac{\partial C}{\partial S} \) shares (assets) and selling 1 option then the value \( \Pi \) of the portfolio,

\[
\Pi = \frac{\partial C}{\partial S} S - C
\] (3.7)

The change in the value of the portfolio in the time interval \( \Delta t \) is given by,

\[
\Delta \Pi = \frac{\partial C}{\partial S} \Delta S - \Delta C
\] (3.8)

Substituting, 3.5 and 3.6 in 3.8, we get:

\[
\Delta \Pi = \left( -\frac{\partial C}{\partial t} - \frac{1}{2} \frac{\partial^{2} C}{\partial S^{2}} \sigma^{2} S^{2} \right) \Delta t
\] (3.9)

Since the portfolio does not involve \( \Delta Z \), it must be risk free during time \( \Delta t \).

As we have assumed short selling is permitted, the portfolio must earn the same return as other securities.

If it earns more than this then there is an arbitrage opportunity, which is not possible.

If it earns less than this then they make risk free profit by selling the portfolio and buy other securities (assets), which means we have,
\[ \Delta \Pi = r \Pi \Delta t. \] (3.10)

Where, \( r \) is the risk free interest rate.

Substituting, 3.8 and 3.9 in 3.10, we get:

\[
\left( \frac{\partial C}{\partial t} + \frac{1}{2} \frac{\partial^2 C}{\partial S^2} \sigma^2 S^2 \right) \Delta t = r \left( C - \frac{\partial C}{\partial S} S \right) \Delta t
\]

Therefore,

\[
\frac{\partial C}{\partial t} + r S \frac{\partial C}{\partial S} + \frac{1}{2} \frac{\partial^2 C}{\partial S^2} \sigma^2 S^2 = r C
\]

Which is known as BSM differential equation. The equation has many solutions corresponding to all the different derivatives that can be defined with \( S \) as the underlying variable.

The particular solution is obtained, when the equation is solved using some boundary conditions, which are known as Payoff functions.

In the next chapter, we have solved BSM equation for different payoff functions.