Chapter 2

Different Stock Price Models
Chapter 2. Different Stock Price Models


In this Chapter we will make more use of the Binomial, Normal and Lognormal random Variables. We have already discussed this in Chapter-1.

2.1 Lognormal Model:

Here we will find a way to model the change of stock prices over a time interval $[0, T]$, from an initial value $S$ to a final value $S_T$. The model is probabilistic.

Suppose $S$ is the stock price, which grows with risk-free rate $r$ and without any randomness then

$$S_T = Se^{rT}$$

If we introduce randomness into the rate of return, in that case we will consider the following model:

$$S_T = Se^{\mu T + CTZ}$$
where \( Z \) is the standard normal variable with mean 0 and variance 1, \( \mu \) and \( C_T \) are some constants (parameters of the stock).

Since \( C_T \) depends on \( T \), we assume it to grow with \( T \).

**Theorem 2.1.1.** [8] If \( Z \) is standard normal variable, then \( E\left[e^{CZ}\right] = e^{\frac{C^2}{2}} \).

**Proof:** Since \( Z \) is a standard normal variable then,

\[
f_Z(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2}
\]

\[
E\left[e^{CZ}\right] = \int_{-\infty}^{\infty} e^{Cz} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} \, dz \\
= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{Cz - \frac{C^2}{2}} \, dz \\
= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{\frac{C^2}{2} - Cz + \frac{C^2}{2}} \, dz \\
= \frac{e^{\frac{C^2}{2}}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(z+C)^2} \, dz \\
= \frac{e^{\frac{C^2}{2}}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{x^2}{2}} \, dx \\
= \frac{e^{\frac{C^2}{2}}}{\sqrt{2\pi}} \sqrt{2\pi} = e^{\frac{C^2}{2}}
\]

\[\blacksquare\]
Therefor, in our model the random terms give a steady increase in the expected return, as $E[e^{CTZ}] = e^{\frac{C^2T}{2}}$.

Now, we want to find a model in which the randomness do not give any regular growth.

We therefore adjust it in the following way:

$$S_T = Se^{\mu T} e^{CTZ - \frac{C^2T}{2}}$$

Suppose $Z_1$ and $Z_2$ are two standard normal variables giving the random fluctuation in the two non overlapping interval $[0, T]$ and $[T, 2T]$, we have:

$$S_{2T} = Se^{\mu T} e^{CTZ_2 - \frac{C^2T}{2}} = Se^{\mu(2T)+CT(Z_1+Z_2)-C^2T}$$

Here $Z_1 + Z_2$ is again a normal random variable with mean 0 and variance 2,

$$\therefore S_{2T} = Se^{\mu 2T + \sqrt{2}CTZ - C^2T}$$

where $Z$ is standard normal variable.
Chapter 2. Different Stock Price Models

Suppose if we treat $[0, 2T]$ as a single interval, we get:

$$S_{2T} = S_0 e^{\mu 2T + C_{2T} Z - \frac{C_{2T}^2}{2}}$$

Comparing the above two expressions, we get:

$$C_{2T} = \sqrt{2} C_T$$

Now set, $C_T^2 = \sigma^2 T$, where $\sigma^2$ is the positive constant.

Therefore, our final model gives the following expression for the spot price at $T$ as:

$$S_T = S_0 e^{\left(\mu - \frac{\sigma^2}{2}\right)T + \sigma \omega_T}$$

where $\omega_T$ is normal with mean 0 and variance $T$. The parameters $\mu$ and $\sigma$ represent drift and volatility respectively and the model is called Lognormal Model because it is based on the lognormal distribution.

2.2 GBM Model:

In the price modeling of stock, Brownian Motion plays an important role in building a statistical model.
Before starting the discussion on Brownian Motion, first we will see some basic concepts required to understand the same.

**Definition 2.2.1.** Any variable whose value changes over time in an uncertain way is said to follow a **Stochastic Process**.

**Definition 2.2.2.** Markov process is a particular type of stochastic process where only present value of a variable relevant for predicting the future.

**Definition 2.2.3.** Wiener process is a particular type of Markov stochastic process with mean 0 and variance 1.

A generalized wiener process for a variable $x$ can be defined as:

$$dx = adt + bdz$$

where $a$ and $b$ are constants and a variable $z$ follows the Wiener process.

### 2.2.1 Ito Process:

In this subsection we will discuss about an important result called Ito’s Lemma [14]. An Ito process is a generalized Wiener process in which the parameters $a$ and $b$ are functions of the value of the underlying variable $x$ and time $t$. An Ito process can be written as:

$$dx = a(x,t)dt + b(x,t)dz$$

Here both drift rate and variance rate of an Ito process change over time.
Theorem 2.2.4 (Ito Lemma). Suppose the variable \( x \) follows an Ito process

\[
dx = a(x,t)dt + b(x,t)dz
\]

where \( dz \) is a wiener process and \( a \) and \( b \) are functions of \( x \) and \( t \). the variable \( x \) has drift rate \( a \) and variance rate \( b^2 \) then a continuously differentiable function \( G(x,t) \) follows the process

\[
dG = \left( \frac{\partial G}{\partial x} + \frac{\partial G}{\partial t} + \frac{1}{2} \frac{\partial^2 G}{\partial x^2} b^2 \right).
\]

Brownian Motion is also known as Wiener Process. Norbert Wiener (1896 − 1964) was a child prodigy from Boston, who graduated from high school at 11, from university at 14 and received his Ph.D. at 18. He had a wide range of interest in pure and applied Mathematics, Theoretical Physics, Communications and Philosophy. First time he initiated the subject “Cybernetics”.

Definition 2.2.5. [21] Brownian Motion: A standard one-dimensional Brownian Motion (on the time interval \([0, T]\)) is a one-dimensional process \( W = W_t, t \in [0, T] \) such that,

1. \( W_0 = 0 \) almost every where;
2. all of the sample paths of \( W \) are continuous;
3. \( W \) has independent increments: \( W_{t_1} - W_{t_0}, ..., W_{t_n} - W_{t_{n-1}} \) are independent for any \( 0 \leq t_0 < t_1 ... < t_n < \infty, n = 1, 2, 3, ... \);
4. for \( 0 \leq s < t < \infty \), \( W_t - W_s \) is a normally distributed random variable with mean 0 and variance \( t - s \).
Let us first introduce some basic concepts to understand the process of Brownian Motion in the model.

The upper most is the Geometric Brownian Motion (GBM), which is a special case of Brownian Motion process. GBM is also called exponential Brownian Motion.

**Definition 2.2.6.** [21] Geometric Brownian Motion: A stochastic process $S_T$ is said to follow a GBM if it satisfies the following stochastic differential equation,

\[
dS_t = \mu S_t dt + \sigma S_t d\omega_t
\]

Where $\mu$ is Drift and $\sigma$ is Volatility.

First we will solve the differential equation using the method of separation of variables, we get:

\[
\frac{dS_t}{S_t} = \mu dt + \sigma d\omega_t
\]

Taking integration with respect to $t$ on both the sides we get,

\[
\ln \left( \frac{dS_t}{S_t} \right) = \left( \mu - \frac{\sigma^2}{2} \right) t + \sigma \omega t \tag{2.1}
\]

Using Ito calculus we get,

\[
S_t = S_0 e^{\left( \mu - \frac{\sigma^2}{2} \right) t + \sigma \omega t}
\]
Now we will find expectation and variance of GBM:

\[
E [S(t)] = E \left[ S_0 e^{(\mu - \frac{\sigma^2}{2})t + \sigma \omega t} \right] \\
= S_0 E \left[ e^{(\mu - \frac{\sigma^2}{2})t} \right] \\
= S_0 E \left[ e^{(\mu - \frac{\sigma^2}{2})t} \right] \\
= S_0 e^{(\mu + \frac{\sigma^2}{2})t}
\]

**Example 2.2.1.**

\[
E [e^{\mu X}] = \frac{1}{\sigma \sqrt{2\pi}} \int_{-\infty}^{\infty} e^{tx} e^{-\frac{1}{2}(\frac{x-\mu}{\sigma})^2} dx \\
= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{t(\mu+\sigma y)} e^{-\frac{y^2}{2}} dy \\
= \frac{e^{\mu t}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(y^2 - 2\sigma ty)} dy \\
= \frac{e^{\mu t + \frac{\sigma^2 t^2}{2}}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(y^2 - 2\sigma ty + \sigma^2 t^2)} dy \\
= \frac{e^{\mu t + \frac{\sigma^2 t^2}{2}}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(y-\sigma t)^2} dy \\
= e^{\mu t + \frac{\sigma^2 t^2}{2}}
\]

The variance is given by

\[
Var [S(t)] = E [S^2(t)] - E [S(t)]^2 \\
= S_0^2 e^{2\mu t + \sigma^2 t} (e^{\sigma^2 t} - 1)
\]
2.3 Binomial Tree Model:

Binomial Tree Model (BTM) is a simple discrete financial market model, which is also known as Cox-Ross-Rubinstein model (CRR).

In this model, we will assume the sample space is finite and we will consider only probability measure which gives the probability to each outcome.

The BTM gives stock price movements as a sequence of small ups and downs. This model some times also called Bernoulli Trials.

Suppose $S$ is the spot price of the stock and over a small time interval $\Delta t$, it either goes up by factor $U$ to $SU$ or goes down by a factor $D$ to $SD$, also assume that $p$ is the probability for up movement and $1 - p$ is the probability for down movement.

The basic branch for this is:

Now, divide the total time interval $[0, T]$ into $n$ equal parts, set $\Delta t = \frac{T}{n}$. For each sub interval we have one branch of having up and down moves. We assume that each step is independent of the previous step, and the quantity $p, U, D$ are same for every branch.
One can see this as:

\[ S \rightarrow SU \xrightarrow{p} SU^2 \]
\[ \quad \rightarrow SD \xrightarrow{1-p} SUD \]
\[ \quad \rightarrow SD \xrightarrow{1-p} SD^2 \]

The possible prices at time \( T \) are \( SU^k D^{n-k} \).

They follow a binomial distribution:

\[ P \left(S_T = SU^k D^{n-k}\right) = \binom{n}{k} p^k (1-p)^{n-k} \]

Here we want \( k \) many \( U \), that means there will be \( (n-k) \) many \( D \).

Now, probability of \( S \rightarrow SU \) is \( p \), but \( U \) occurs \( k \) times, so the total probability is \( p^k \). Similarly probability of \( S \rightarrow SD \) is \( (1-p)^{n-k} \), and out of \( n \) sub interval we can get anywhere up move, so \( \binom{n}{k} \).

Therefore, total probability is:

\[ \binom{n}{k} p^k (1-p)^{n-k} \]
To find $p, U, D$ in the model, we compare it with GBM model.

According to GBM model the spot price at time $\Delta t$ is,

$$S_{\Delta t} = S_0 e^{\left(\mu - \frac{\sigma^2}{2}\right)\Delta t + \sigma \omega \Delta t}$$

Where, $\omega \Delta t$ is normal with mean 0 and variance $\Delta t$.

Now,

$$\ln S_{\Delta t} = \ln (S_0) + \left(\mu - \frac{\sigma^2}{2}\right) \Delta t + \sigma \omega \Delta t$$

Here, we have:

$$E [\ln S_{\Delta t}] = \ln (S_0) + \left(\mu - \frac{\sigma^2}{2}\right) \Delta t$$

and,

$$Var [\ln S_{\Delta t}] = \sigma^2 \Delta t$$
Now, consider $u = \ln U$ and $d = \ln D$

Then, we have

$$E[\ln S_{\Delta t}] = pu + (1 - p)d$$

and,

$$Var[\ln S_{\Delta t}] = p(1 - p)(u - d)^2$$

The above follows from the following example 2.3.1

**Example 2.3.1.** [8] If $X$ is any discontinuous random variable, which takes only two values $a$ and $b$ respectively, then show that $E[X] = pa + (1 - p)b$ and $Var[X] = p(1 - p)(a - b)^2$.

**Solution 2.3.1.** Consider,

$$y = \frac{X - b}{a - b}$$

$y$ takes only two values either 0 or 1, so it is a binomial variable with mean $p$ and variance $1 - p$. 
Now,

\[ E[X] = E[b + (a - b)y] \]
\[ = b + (a - b)E[y] \]
\[ = b + (a - b)p \]
\[ = pa + (1 - p)b \] \hspace{1cm} (2.3)

Now, comparing the values for two models we have:

\[ pu + (1 - p) = \ln S_0 + \left( \mu - \frac{\sigma^2}{2} \right) \Delta t \] \hspace{1cm} (2.4)

and,

\[ p(1 - p)(u - d)^2 = \sigma^2 \Delta t \] \hspace{1cm} (2.5)

Here, we have three variables and two equations, so we have infinitely many solutions. Thus, we have some freedom to choose convenient solutions. For that we set, \( U = D^{-1} \). This will give us \( u = -d \). Then equation 2.4 and 2.5 becomes

\[ (2p - 1)u = \ln S_0 + \left( \mu - \frac{\sigma^2}{2} \right) \Delta t \] \hspace{1cm} (2.6)

\[ 4p(1 - p)(u)^2 = \sigma^2 \Delta t \] \hspace{1cm} (2.7)
Solving equation 2.6 and 2.7 we get:

\[ u^2 = \sigma^2 \Delta t + \left( \ln S_0 + \left( \mu - \frac{\sigma^2}{2} \right) \Delta t \right)^2 \]  

(2.8)

Also we have:

\[ U = e^{\sqrt{\sigma^2 \Delta t + \left( \ln S_0 + \left( \mu - \frac{\sigma^2}{2} \right) \Delta t \right)^2}} \]

\[ D = e^{-\sqrt{\sigma^2 \Delta t + \left( \ln S_0 + \left( \mu - \frac{\sigma^2}{2} \right) \Delta t \right)^2}} \]

and

\[ p = \frac{1}{2} \left( 1 + \frac{\ln S_0 + \left( \mu - \frac{\sigma^2}{2} \right) \Delta t}{\sqrt{\sigma^2 \Delta t + \left( \ln S_0 + \left( \mu - \frac{\sigma^2}{2} \right) \Delta t \right)^2}} \right) \]
Different option pricing formulas
Chapter 3. Different option pricing formulas

An option is a financial derivative, which represents a contract sold by one party to another party. The contract gives the buyer the right but not the obligation, to buy or sell any asset.

There are some reasons, why one should use an option? The main advantage of using option is that, one can also make profit, when the market goes down.

3.1 Types of option:

We have already discussed two basic types of options in chapter-1.

1. A call option
2. A put option

A call option holder has a right to buy an asset for certain price, at a certain date, while a put option holder has a right to sell an asset for certain price, at a certain date. The specified date is called an expiration date or maturity date and a specified price is called the exercise price or a strike price. According to the style, there are two types of options:

1. American option
2. European option

The only difference between these two options is that, American option can be exercised at any time till the expiration date, while European option can be exercised only at the expiration date. Generally most of the exchanges tread American option.

In fact European option are comparatively easy to analyze than American option.
3.2 Pricing an option:

In this section we will consider only European type of option.

First, we will see, what are the factors that affect pricing of stock(asset) option.

Here, we have given some of the factors, which affects the stock(asset) option price [12]:

1. The current stock(asset) price, $S_0$

2. The Strike price, $X$

3. The expiration time, $T$

4. The Volatility, $\sigma$

5. The risk free interest rate, $r$

6. The dividends that are expected to be paid.

We will also assume that:

1. There are no transaction costs involved.

2. All trading profits are subject to the same tax rate.

3. Borrowing and lending are possible only at the risk free interest rate, also we will consider that, there is no arbitrage opportunity.
Some useful notations:

- $S_0$ is current stock(asset) price
- $S_T$ is stock(asset) price at expiration
- $X$ is Exercise price or strike price
- $T$ is Total period of time
- $C$ is the value of call option to buy one unit of share(asset)
- $P$ is the value of put option to sell one unit of share(asset)

We will consider $r > 0$.

### 3.3 Option Pricing Models:

Option pricing theory is very important topic in Finance since 1972, When Black and Scholes published a paper in which they have given a model for valuing a European option using “replication” portfolio.

In fact, their derivation is mathematically complicated but using some logic, there are also simple models for valuing options.

In this section, we will consider two models namely:

1. Binomial Option Pricing Model (BOPM).
2. Black Scholes Merton Model.
3.3.1 Binomial Option Pricing Model (BOPM):

This is very useful and popular model to value an option using tree.

This method can be visualized through diagram, in which the stock(asset) price has only two possibilities, either it can go up by a certain percentage by a certain probability or it can go down.

It is based on the Binomial Tree Model for changes in price of the stock(asset).

In this model, we divide total time interval $[0,T]$ in to $n$ equal sub intervals, and assume that, the asset price will move either up by some amount or down by some amount.

**Single Step BOPM**

Let us assume $n = 1$.

Consider Stock(asset), having initial price $S_0$, and option on the stock(asset) having initial premium $C$. Over an interval $[0,T]$, the stock price moves from $S_0$ to either $S_0u$ or $S_0d$, where $u > 1 > d$.

Similarly, for call premium, either it is $C_u = \max\{S_0u - X, 0\}$ or $C_d = \max\{S_0d - X, 0\}$. 
Now, we will consider the portfolio which consists of buying $\Delta$ shares (assets) and selling one option. We will find the value of $\Delta$ in such a way so that the portfolio becomes risk less.

If we assume that, the movement is up then the value of the portfolio at time $T$ is

$$S_0 u \Delta - C_u$$

If we assume that, the movement is down then the value of the portfolio at time $T$ is

$$S_0 d \Delta - C_d$$

These two are equal when,

$$S_0 u \Delta - C_u = S_0 d \Delta - C_d$$
Therefore,

\[ \Delta = \frac{C_u - C_d}{S_0u - S_0d} \]

This equation shows that \( \Delta \) is the ratio of the change in the option price to the change in the stock(asset) price as we move between the nodes at time \( T \).

Now, present value of the portfolio is \((S_0u\Delta - C_u)e^{-rT}\), where \( r \) is the risk free interest rate.

The actual cost of the portfolio is \( S_0\Delta - C \)

We have,

\[ S_0\Delta - C = (S_0u\Delta - C_u)e^{-rT} \]

and therefore,
\[ C = S_0 \Delta \left( 1 - ue^{-rT} \right) + C_u e^{-rT} \]
\[ = S_0 \left( \frac{C_u - C_d}{S_0 u - S_0 d} \right) \left( 1 - ue^{-rT} \right) + C_u e^{-rT} \]
\[ = \frac{C_u \left( 1 - de^{-rT} \right) + C_d \left( ue^{-rT} - 1 \right)}{u - d} \]
\[ = e^{-rT} \left( pC_u + (1 - p)C_d \right) \]

where,
\[ p = \frac{e^{rT} - d}{u - d} \]

**Example 3.3.1.** Suppose we have single step BOPM with \( S = 100 \), \( X = 100 \), \( u = 1.1 \), \( d = 0.9 \), \( r = 10\% \) and \( T = 1 \). Find the value of \( C \).

**Solution 3.3.1.** Here we have
\[ C = e^{-rT} \left( pC_u + (1 - p)C_d \right) \]

First, we will find the value of \( p \),
\[ p = \frac{e^{rT} - d}{u - d} \]
\[ = \frac{e^{(0.1)(1)} - 0.9}{1.1 - 0.9} \]
\[ = 1.0258 \] (3.2)

Here,
\[ C_u = 10 \text{ and } C_d = 0 \]
Chapter 3. Different option pricing formulas

Therefor, $C = 9.2818$

3.4 Black Scholes Merton Model:

In 1970s, Fisher Black, Myron Scholes and Robert Merton have derived the model for pricing of European stock option. The model has become famous with the name Black Scholes Merton Model. The use of this model is to find mathematical value of an option. Robert Merton and Myron Scholes were also awarded by the Nobel prize in 1997 in Economics for this contribution to the society. Unfortunately, Fisher Black died in 1995, otherwise, he too would be one of the prize recipients.

Now, we will derive BSM differential equation.

We will use the following assumption in the derivation [12]:

1. The stock(asset) price follows the GBM with constant $\mu$ and $\sigma$.
2. Short selling is permitted.
3. There are no transaction cost.
4. No dividends are paid throughout the life of the derivative.
5. No arbitrage opportunity.
7. $r$ is constant.

Derivation of BSM differential equation
Consider a price of derivative at a time $t$. Suppose $T$ is the expiry time, then the total time of maturity is $T - t$.

As we have assumed the stock(asset) price follows GBM, we have:

$$dS = \mu S dt + \sigma S dZ \quad (3.3)$$

Suppose that $C$ is the price of a call option or any other derivative on $S$.

The variable $C$ must be a function of $S$ and $t$.

By Ito’s lemma (See Chapter-2) we have,

$$dC = \left( \frac{\partial C}{\partial S} \mu S + \frac{\partial C}{\partial t} + \frac{1}{2} \frac{\partial^2 C}{\partial S^2} \sigma^2 S^2 \right) dt + \frac{\partial C}{\partial S} \sigma S dZ \quad (3.4)$$

Equation 3.3 and 3.4 can also be written as

$$\Delta S = \mu S \Delta t + \sigma S \Delta Z \quad (3.5)$$

and

$$\Delta C = \left( \frac{\partial C}{\partial S} \mu S + \frac{\partial C}{\partial t} + \frac{1}{2} \frac{\partial^2 C}{\partial S^2} \sigma^2 S^2 \right) \Delta t + \frac{\partial C}{\partial S} \sigma S \Delta Z \quad (3.6)$$

Where, $\Delta S$ and $\Delta C$ are the changes in $S$ and $C$ in a small time interval $\Delta t$. 
Consider a portfolio, which consists of buying \( \frac{\partial C}{\partial S} \) shares (assets) and selling 1 option then the value \( \Pi \) of the portfolio,

\[
\Pi = \frac{\partial C}{\partial S} S - C \tag{3.7}
\]

The change in the value of the portfolio in the time interval \( \Delta t \) is given by,

\[
\Delta \Pi = \frac{\partial C}{\partial S} \Delta S - \Delta C \tag{3.8}
\]

Substituting, 3.5 and 3.6 in 3.8, we get:

\[
\Delta \Pi = \left( -\frac{\partial C}{\partial t} - \frac{1}{2} \frac{\partial^2 C}{\partial S^2} \sigma^2 S^2 \right) \Delta t \tag{3.9}
\]

Since the portfolio does not involve \( \Delta Z \), it must be risk free during time \( \Delta t \).

As we have assumed short selling is permitted, the portfolio must earn the same return as other securities.

If it earns more than this then there is an arbitrage opportunity, which is not possible.

If it earns less than this then they make risk free profit by selling the portfolio and buy other securities (assets), which means we have,
\[ \Delta \Pi = r \Pi \Delta t. \] (3.10)

Where, \( r \) is the risk free interest rate.

Substituting, 3.8 and 3.9 in 3.10, we get:

\[
\left( \frac{\partial C}{\partial t} + \frac{1}{2} \frac{\partial^2 C}{\partial S^2} \sigma^2 S^2 \right) \Delta t = r \left( C - \frac{\partial C}{\partial S} S \right) \Delta t
\] (3.11)

Therefore,

\[
\frac{\partial C}{\partial t} + r S \frac{\partial C}{\partial S} + \frac{1}{2} \frac{\partial^2 C}{\partial S^2} \sigma^2 S^2 = r C
\] (3.12)

Which is known as BSM differential equation. The equation has many solutions corresponding to all the different derivatives that can be defined with \( S \) as the underlying variable.

The particular solution is obtained, when the equation is solved using some boundary conditions, which are known as Payoff functions.

In the next chapter, we have solved BSM equation for different payoff functions.
Chapter 4

BSM Formulas For Different Payoff Functions
In this chapter first we convert BSM equation into the heat equation using some transformation, then we will solve Heat equation using Fourier Transforms Method.

Many people have solved same problem using the Method of separation of variables and Laplace Transform Method [20].

We will find the solution of Heat equation using different boundary conditions, which are known as Payoff Functions.

In Mathematical Finance, the BSM equation is a Partial Differential Equation to find the value of European Call/Put option. Suppose $C(S,t)$ is the value of European call option. The equation

$$\frac{\partial C}{\partial t} + rS \frac{\partial C}{\partial S} + \frac{\sigma^2 S^2}{2} \frac{\partial^2 C}{\partial S^2} - rC = 0$$

(4.1)

is known as a Black-Scholes-Merton Partial Differential Equation [8].

Where,

- $S$ is Spot price of asset (i.e. the price of asset at time $t = 0$)
- $X$ is Exercise price or strike price
- $T$ is Total period of time
- $r$ is Risk free interest rate
- $\sigma$ is Volatility
Chapter 4. BSM Formulas For Different Payoff Functions

\[ t \in [0, T] \text{ and } C(S, t) = 0 \text{ for all } t \]

\[ C(S, t) \to S \text{ as } S \to \infty. \]

Consider the European call option whose final payoff at the expiry time \( T \) is given by a function \( f \) of the final spot price \( S \) (we note that in the literature it is often denoted by many as \( S_T \)) which is assumed to be a continuous function, that need not be differentiable. Also we demand,

\[ \lim_{t \to T^-} C(S, t) = f(S) \]

We can convert the Black-Scholes-Merton Partial Differential Equation into the heat equation using the following substitutions [8]:

\[ y = T - t \]

\[ x = \ln \left( \frac{S}{X} \right) + \left( r - \frac{\sigma^2}{2} \right)(T - t) \]

\[ D(x, y) = e^{r(T-t)}C(S, t) \]

These substitutions also convert the above mentioned boundary condition,

\[ \lim_{t \to T^-} C(S, t) = f(S) \]

into the initial condition,
\[
\lim_{y \to 0^+} D(x, y) = f(Xe^x)
\]

Thus the Black-Scholes-Merton Partial Differential Equation gets converted into the following Heat equation with the stated initial condition:

\[
\frac{\partial D}{\partial y} = \frac{\sigma^2}{2} \frac{\partial^2 D}{\partial x^2} \quad \text{with} \quad \lim_{y \to 0^+} D(x, y) = f(Xe^x).
\]

Applying Fourier Transform on Heat equation we get,

\[
\frac{\partial}{\partial y} F(D) + \frac{\sigma^2 \lambda^2}{2} F(D) = 0
\]

\[
\therefore \quad F(D) = C_1 e^{-\frac{\sigma^2 \lambda^2}{2} y}
\]

Now we get,

\[
F(D(x, 0)) = G(\lambda) \quad \text{because} \quad D(x, 0) = f(Xe^x).
\]

Here \(G\) is the Fourier Transform of \(f\), so that \(C_1\) is determined and we now have:

\[
F(D) = G(\lambda)e^{-\frac{\sigma^2 \lambda^2}{2} y}
\]
Taking inverse Fourier Transform on both the sides we get,

\[ D(x, y) = F^{-1} \left( G(\lambda) e^{-\frac{\sigma^2 \lambda^2}{2} y} \right) \]

Now, using convolution theorem and the facts, that

\[ F^{-1}(G(\lambda)) = f(Xe^x) \text{ and } F^{-1} \left( e^{-\frac{\sigma^2 \lambda^2}{2} y} \right) = \frac{1}{\sigma \sqrt{y}} e^{-\frac{x^2}{2\sigma^2 y}} \]

we get,

\[ D(x, y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(\nu) \frac{1}{\sigma \sqrt{y}} e^{-\frac{(x-\nu)^2}{2\sigma^2 y}} \, d\nu \]

\[ \therefore D(x, y) = \frac{1}{\sigma \sqrt{2\pi y}} \int_{-\infty}^{\infty} f(\nu) e^{-\frac{(x-\nu)^2}{2\sigma^2 y}} \, d\nu \]
4.1 Solution of the problem using different payoff functions:

4.1.1 Plain Vanilla Payoff:

This is very basic and commonly used payoff function. Many financial organizations use this as a payoff. The very first BSM formula was derived using this [12].

Now we consider Plain Vanilla payoff function which is as:

\[
f(S) = \max\{S - X, 0\} = \begin{cases} 
S - X & \text{if } S \geq X \\
0 & \text{if } S \leq X 
\end{cases}
\]

\[
\therefore f(e^x) = \max\{X(e^x - 1), 0\} = \begin{cases} 
X(e^x - 1) & \text{if } x \geq 0 \\
0 & \text{if } x \leq 0 
\end{cases}
\]

\[
\therefore D(x, y) = \frac{1}{\sigma \sqrt{2\pi y}} \int_{-\infty}^{\infty} X(e^\nu - 1)e^{-\frac{(\nu-x)^2}{2\sigma^2 y}} d\nu
\]

\[
= \frac{X}{\sigma \sqrt{2\pi y}} \left[ \int_0^{\infty} e^{\nu} e^{-\frac{(\nu-x)^2}{2\sigma^2 y}} d\nu - \int_0^{\nu-x} e^{\nu} e^{-\frac{(\nu-x)^2}{2\sigma^2 y}} d\nu \right]
\]

Substituting,

\[
Z = \frac{\nu - x}{\sigma \sqrt{y}}
\]
we get

\[
D(x, y) = \frac{X}{\sqrt{2\pi}} e^{x + \frac{\sigma^2 y^2}{2}} \int_{-\infty}^{\infty} e^{-\frac{(z^2 - 2\sigma \sqrt{y} + \sigma^2 y^2)}{2}} dZ - \frac{X}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{z^2}{2}} dZ
\]

\[
= \frac{X}{\sqrt{2\pi}} e^{x + \frac{\sigma^2 y^2}{2}} \int_{-\infty}^{\infty} e^{-\frac{t^2}{2}} dt - \frac{X}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{z^2}{2}} dZ
\]

\[
= \frac{X}{\sqrt{2\pi}} e^{x + \frac{\sigma^2 y^2}{2}} \int_{-\infty}^{x + \frac{\sigma^2 y^2}{2}} e^{-\frac{t^2}{2}} dt - \frac{X}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{z^2}{2}} dZ
\]

\[
\therefore D(x, y) = X e^{x + \frac{\sigma^2 y^2}{2}} N(d_1) - X N(d_2)
\]

where, \( d_1 = \frac{x + \sigma^2 y}{\sigma \sqrt{y}} \), \( d_2 = \frac{x}{\sigma \sqrt{y}} \) and \( N(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{t^2}{2}} dt \)

\[
\therefore C(S, t) = SN(d_1) - X e^{-r(T-t)} N(d_2)
\]

(4.2)

where,

\[
d_1 = \frac{x + \sigma^2 y}{\sigma \sqrt{y}} = \ln \left( \frac{S}{X} \right) + \left( r + \frac{\sigma^2}{2} \right) \frac{(T-t)}{\sigma \sqrt{T-t}}
\]

and

\[
d_2 = d_1 - \sigma \sqrt{T-t} = \ln \left( \frac{S}{X} \right) + \left( r - \frac{\sigma^2}{2} \right) \frac{(T-t)}{\sigma \sqrt{T-t}}.
\]
4.1.2 Log Payoff:

Paul Wilmott has discussed BSM formula for Log payoff function [22]. Other than this several types of option pricing formulas have been derived with different payoff functions [11].

Now we consider the payoff function which is known as Log payoff, which is as:

\[ f(S) = \max \left\{ \ln \left( \frac{S}{X} \right), \ 0 \right\} = \begin{cases} \ln \left( \frac{S}{X} \right) & \text{if } S \geq X \\ 0 & \text{if } S \leq X \end{cases} \]

\[ \therefore f(Xe^x) = \max\{x, \ 0\} = \begin{cases} x & \text{if } x \geq 0 \\ 0 & \text{if } x \leq 0 \end{cases} \]

\[ \therefore D(x, y) = \frac{1}{\sigma \sqrt{2\pi y}} \int_{0}^{\infty} \nu e^{-\frac{(x-\nu)^2}{2\sigma^2 y}} \, d\nu \]

Taking,

\[ Z = \frac{\nu - x}{\sigma \sqrt{y}} \]
we get,

\[
D(x, y) = \frac{1}{\sqrt{2\pi}} \int_{-\frac{\sigma \sqrt{y}}{\sigma \sqrt{y}}}^{\infty} (x + \sigma \sqrt{y}Z) e^{-\frac{Z^2}{2}} dZ \\
= \frac{x}{\sqrt{2\pi}} \int_{-\frac{\sigma \sqrt{y}}{\sigma \sqrt{y}}}^{\infty} e^{-\frac{Z^2}{2}} dZ + \sigma \sqrt{\frac{y}{2\pi}} \int_{-\frac{\sigma \sqrt{y}}{\sigma \sqrt{y}}}^{\infty} Ze^{-\frac{Z^2}{2}} dZ \\
= \frac{x}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{t^2}{2}} dt + \sigma \sqrt{\frac{y}{2\pi}} e^{-\frac{x^2}{2\sigma^2 y}} \\
\therefore D(x, y) = xN(d) + \sigma \sqrt{\frac{y}{2\pi}} e^{-\frac{x^2}{2\sigma^2 y}}
\]

where, \( d = \frac{x}{\sigma \sqrt{y}} \)

Therefore,

\[
C(S, t) = e^{-r(T-t)} \left[ \ln \left( \frac{S}{X} \right) + \left( r - \frac{\sigma^2}{2} \right) (T - t) \right] N(d) + \frac{1}{\sqrt{2\pi}} e^{-r(T-t)} \sigma \sqrt{T-t} - t e^{-\frac{x^2}{2\sigma^2 y}}
\]

where,

\[
d = \frac{\ln \left( \frac{S}{X} \right) + \left( r - \frac{\sigma^2}{2} \right) (T - t)}{\sigma \sqrt{T-t}}
\]
4.1.3 Modified Log Payoff:

H. V. Dedania and S. J. Ghevariya [5] have discussed BSM formula for Modified Log Payoff function. Which is closely related to Log Payoff function and is close to the celebrated Plain Vanilla Payoff function. Other than this several types of option pricing formulas have been derived with different payoff functions [11].

Now we consider the payoff function which is known as Modified Log Payoff, which is as:

\[
f(S) = \max \left\{ S \ln \left( \frac{S}{X} \right), 0 \right\} = \begin{cases} 
S \ln \left( \frac{S}{X} \right) & \text{if } S \geq X \\
0 & \text{if } S \leq X
\end{cases}
\]

\[
\therefore f(Xe^x) = \max \{ xe^x, 0 \} = \begin{cases} 
x X e^x & \text{if } x \geq 0 \\
0 & \text{if } x \leq 0
\end{cases}
\]

\[
\therefore D(x, y) = \frac{X}{\sigma \sqrt{2\pi y}} \int_0^\infty \nu e^{\nu} e^{-\frac{(\nu - x)^2}{2\sigma^2 y}} d\nu
\]

Taking,

\[
Z = \frac{\nu - x}{\sigma \sqrt{y}}
\]
we get,

\[ D(x, y) = \frac{X}{\sqrt{2\pi}} e^{\frac{x^2}{2}} \int_{-\infty}^{\infty} \left( x + \sigma \sqrt{y} Z \right) e^{x + \sigma \sqrt{y} Z} e^{-\frac{x^2}{2}} dZ \]

\[ = \frac{x}{\sqrt{2\pi}} \left[ e^{x} \int_{-\infty}^{\infty} e^{x + \sigma \sqrt{y} Z} e^{-\frac{x^2}{2}} dZ + \sigma \sqrt{y} e^{x} \int_{-\infty}^{\infty} e^{x + \sigma \sqrt{y} Z} e^{-\frac{x^2}{2}} dZ \right] \]

\[ = \frac{x}{\sqrt{2\pi}} \left[ e^{x} \int_{-\infty}^{\infty} e^{-\frac{(2x^2 - 2x\sigma \sqrt{y} Z + \sigma^2 y)}{2}} dZ + \sigma \sqrt{y} e^{x} \int_{-\infty}^{\infty} e^{-\frac{(2x^2 - 2x\sigma \sqrt{y} Z + \sigma^2 y)}{2}} dZ \right] \]

\[ = \frac{x}{\sqrt{2\pi}} \left[ e^{x} e^{\frac{x^2}{2}} \int_{-\infty}^{\infty} e^{-\frac{(x - \sigma \sqrt{y} Z)^2}{2}} dZ + \sigma \sqrt{y} e^{x} e^{\frac{x^2}{2}} \int_{-\infty}^{\infty} e^{-\frac{(x - \sigma \sqrt{y} Z)^2}{2}} dZ \right] \]

\[ = \frac{x}{\sqrt{2\pi}} \left[ e^{x} \left( e^{\frac{x^2}{2} + \sigma^2 y} + \sigma^2 y \right) \right] \int_{-\infty}^{\infty} e^{-\frac{x^2}{2}} dZ + \sigma \sqrt{y} e^{x} e^{\frac{x^2}{2}} \int_{-\infty}^{\infty} te^{-\frac{t^2}{2}} dt \]

\[ = \frac{x}{\sqrt{2\pi}} e^{x + \frac{x^2}{2}} \left[ \left( x + \sigma^2 y \right) \int_{-\infty}^{\infty} e^{-\frac{x^2}{2}} dZ + \sigma \sqrt{y} e^{\frac{x^2}{2}} \left( \frac{x + \sigma^2 y}{\sigma \sqrt{y}} \right)^2 \right] \]

\[ \therefore D(x, y) = \frac{X}{\sqrt{2\pi}} e^{x + \frac{x^2}{2}} \left[ \left( x + \sigma^2 y \right) N(d) + \sigma \sqrt{y} e^{\frac{x^2}{2}} \right] \]

where, \( d = \frac{x + \sigma^2 y}{\sigma \sqrt{y}} \).

\[ \therefore C(S, t) = S \left[ \ln \left( \frac{S}{X} \right) + \left( r + \frac{\sigma^2}{2} \right) (T - t) N(d) + \frac{1}{\sqrt{2\pi}} \sigma \sqrt{T - t} e^{-\frac{d^2}{2}} \right] \] (4.3)

where,

\[ d = \frac{\ln \left( \frac{S}{X} \right) + \left( r + \frac{\sigma^2}{2} \right) (T - t)}{\sigma \sqrt{T - t}} \]
4.2 Averages of Different Payoff Function:

In this section we will discuss averages of different payoff functions and also we have observed and shown that averages of two payoff functions will give exactly the average of two solutions. We have also extended the result for general payoff [15–18].

4.2.1 Average of Plain Vanilla and Log:

Here, we consider average of two payoff functions Plain Vanilla and Log, i.e.

\[
f(S) = \begin{cases} 
\frac{\ln(\frac{S}{X}) + S - X}{2} & \text{if } S \geq X \\
0 & \text{if } S \leq X 
\end{cases}
\]

\[
\therefore f(Xe^x) = \begin{cases} 
\frac{X(e^x - 1) + x}{2} & \text{if } x \geq 0 \\
0 & \text{if } x \leq 0
\end{cases}
\]

\[
\therefore D(x, y) = \frac{1}{2\sigma\sqrt{2\pi y}} \int_{0}^{\infty} (X(e^\nu - 1) + \nu)e^{-\frac{(x-\nu)^2}{2\sigma^2 y}} d\nu
\]

\[
= \frac{X}{2\sigma\sqrt{2\pi y}} \left[ \int_{0}^{\infty} e^\nu e^{-\frac{(x-\nu)^2}{2\sigma^2 y}} d\nu - \int_{0}^{\infty} e^{-\frac{(x-\nu)^2}{2\sigma^2 y}} d\nu \right] + \frac{1}{2\sigma\sqrt{2\pi y}} \int_{0}^{\infty} \nu e^{-\frac{(x-\nu)^2}{2\sigma^2 y}} d\nu
\]

\[
= \frac{X}{2\sigma\sqrt{2\pi y}} \int_{0}^{\infty} e^\nu e^{-\frac{(x-\nu)^2}{2\sigma^2 y}} d\nu - \frac{X}{2\sigma\sqrt{2\pi y}} \int_{0}^{\infty} e^{-\frac{(x-\nu)^2}{2\sigma^2 y}} d\nu + \frac{1}{2\sigma\sqrt{2\pi y}} \int_{0}^{\infty} \nu e^{-\frac{(x-\nu)^2}{2\sigma^2 y}} d\nu
\]
Substituting,

\[ Z = \frac{\nu - x}{\sigma \sqrt{y}} \]

we get,

\[
D(x, y) = \frac{X}{\sqrt{2\pi}} e^{\frac{x^2}{2}} \int_{-\infty}^{\infty} e^{-\frac{z^2 - 2z(x + \sigma^2y)}{2\sigma^2y}} dz - \frac{X}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{z^2}{2}} dz
\]

\[
+ \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{z^2}{2}} dz \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{z^2}{2}} dz
\]

\[
= \frac{X}{\sqrt{2\pi}} e^{\frac{x^2}{2}} \int_{-\infty}^{\infty} e^{-\frac{z^2}{2}} dz - \frac{X}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{z^2}{2}} dz
\]

\[
+ \frac{\sigma \sqrt{y}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} Ze^{-\frac{z^2}{2}} dz
\]

\[
= \frac{X}{\sqrt{2\pi}} e^{\frac{x^2}{2}} \int_{-\infty}^{\infty} e^{-\frac{z^2}{2}} dz - \frac{X}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{z^2}{2}} dz
\]

\[
+ \frac{\sigma \sqrt{y}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} Ze^{-\frac{z^2}{2}} dz
\]

\[
\therefore D(x, y) = \frac{X}{2} e^{\frac{x^2}{2}} N(d_1) - \frac{(X - x)}{2} N(d_2) + \frac{\sigma \sqrt{y}}{2} \int_{-\infty}^{\infty} e^{-\frac{z^2}{2}} dz
\]

where, \( d_1 = \frac{x + \sigma^2y}{\sigma \sqrt{y}}, d_2 = \frac{x}{\sigma \sqrt{y}} \) and \( N(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{t^2}{2}} dt \)
Therefore

\[
C(S, t) = \frac{S}{2} N(d_1) - \frac{1}{2} e^{-r(T-t)} \left[ X - \ln \left( \frac{S}{X} \right) - \left( r - \frac{\sigma^2}{2} \right) (T - t) \right] N(d_2) + e^{-r(T-t)} \frac{\sigma}{2} \sqrt{\frac{T-t}{2\pi}} e^{-d_2^2/2}
\]

where,

\[
d_1 = \frac{x + \sigma^2 y}{\sigma \sqrt{y}} = \frac{\ln \left( \frac{S}{X} \right) + \left( r + \frac{\sigma^2}{2} \right) (T - t)}{\sigma \sqrt{T-t}}
\]

and

\[
d_2 = d_1 - \sigma \sqrt{T-t} = \frac{\ln \left( \frac{S}{X} \right) + \left( r - \frac{\sigma^2}{2} \right) (T - t)}{\sigma \sqrt{T-t}}.
\]

### 4.2.2 Average of Log and Modified Log:

Here, we will take average of two known payoff functions Log and Modified Log, which is as,

\[
f(S) = \begin{cases} 
\ln \left( \frac{S}{X} \right) + S \ln \left( \frac{S}{X} \right) & \text{if } S \geq X \\
0 & \text{if } S \leq X
\end{cases}
\]
\[
\therefore f(Xe^x) = \begin{cases} 
\frac{xe^x + x}{2} & \text{if } x \geq 0 \\
0 & \text{if } x \leq 0
\end{cases}
\]

\[
\therefore D(x, y) = \frac{1}{2\sqrt{2\pi}y} \int_{0}^{\infty} (X\nu e^{\nu} + \nu)e^{-\frac{(x-\nu)^2}{2\sigma^2y}} d\nu
\]

\[
= \frac{X}{2\sqrt{2\pi}y} \int_{0}^{\infty} \nu e^{\nu} e^{-\frac{(x-\nu)^2}{2\sigma^2y}} d\nu + \frac{1}{2\sqrt{2\pi}y} \int_{0}^{\infty} \nu e^{-\frac{(x-\nu)^2}{2\sigma^2y}} d\nu
\]

Substituting,

\[
Z = \frac{\nu - x}{\sigma \sqrt{y}}
\]

we get,
\[ D(x, y) = \frac{X}{2\sqrt{2\pi}} \int_{-\infty}^{\infty} \left( x + \sigma \sqrt{y}Z \right) e^{-\frac{y^2}{2}} dZ + \frac{1}{2\sqrt{2\pi}} \int_{-\infty}^{\infty} \left( x + \sigma \sqrt{y}Z \right) e^{-\frac{y^2}{2}} dZ \]

\[ = \frac{X}{2\sqrt{2\pi}} xe^x \int_{-\infty}^{\infty} e^{\sigma \sqrt{y}Z} e^{-\frac{y^2}{2}} dZ + \frac{X}{2\sqrt{2\pi}} \sigma \sqrt{y}e^x \int_{-\infty}^{\infty} Ze^{\sigma \sqrt{y}Z} e^{-\frac{y^2}{2}} dZ \]

\[ + \frac{x}{2\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{x^2}{2}} dZ + \frac{\sigma \sqrt{y}}{2\sqrt{2\pi}} \int_{-\infty}^{\infty} Ze^{-\frac{x^2}{2}} dZ \]

\[ = \frac{X}{2\sqrt{2\pi}} \left( x + \sigma \sqrt{y}Z \right) e^{-\frac{x^2}{2}} \int_{-\infty}^{\infty} e^{-\frac{y^2}{2}} dZ + \frac{X}{2\sqrt{2\pi}} \sigma \sqrt{y}e^{x+\frac{\sigma^2 y}{2}} \int_{-\infty}^{\infty} Ze^{-\frac{y^2}{2}} dZ \]

\[ + \frac{x}{2\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{x^2}{2}} dZ + \frac{\sigma \sqrt{y}}{2\sqrt{2\pi}} \int_{-\infty}^{\infty} Ze^{-\frac{x^2}{2}} dZ \]

\[ = \frac{X}{2\sqrt{2\pi}} \left( x + \sigma \sqrt{y}Z \right) e^{-\frac{x^2}{2}} \int_{-\infty}^{\infty} e^{-\frac{y^2}{2}} dZ + \frac{X}{2\sqrt{2\pi}} \sigma \sqrt{y}e^{x+\frac{\sigma^2 y}{2}} e^{\frac{1}{2}(\frac{x^2}{2} + \frac{\sigma^2 y}{2})} \int_{-\infty}^{\infty} Ze^{-\frac{y^2}{2}} dZ \]

\[ + \frac{x}{2\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{x^2}{2}} dZ + \frac{\sigma \sqrt{y}}{2\sqrt{2\pi}} \int_{-\infty}^{\infty} Ze^{-\frac{x^2}{2}} dZ \]

\[ \therefore D(x, y) = \frac{X}{2} \left( x + \sigma^2 y \right) e^{\frac{x^2}{2}} N(d_1) + \frac{X}{2\sqrt{2\pi}} e^{-\frac{d_1^2}{2}} + \frac{X}{2} N(d_2) + \frac{\sigma \sqrt{y}}{2\sqrt{2\pi}} e^{-\frac{d_2^2}{2}} \]

where, \( d_1 = \frac{x + \sigma^2 y}{\sigma \sqrt{y}} \), \( d_2 = \frac{x}{\sigma \sqrt{y}} \) and \( N(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{t^2}{2}} dt \)

\[ \therefore C(S, t) = \frac{S}{2} \left[ \ln \left( \frac{S}{X} \right) + \left( r + \frac{\sigma^2}{2} \right) (T - t) \right] N(d_1) + \frac{S}{2\sqrt{2\pi}} e^{-r(T-t)} \sqrt{T-t} e^{-\frac{d_1^2}{2}} \]

\[ + \frac{1}{2} e^{-r(T-t)} \left[ \ln \left( \frac{S}{X} \right) + \left( r - \frac{\sigma^2}{2} \right) (T - t) \right] N(d_2) + \frac{1}{2\sqrt{2\pi}} e^{-r(T-t)} \sqrt{T-t} e^{-\frac{d_2^2}{2}} \]
where,

\[ d_1 = \frac{x + \sigma^2 y}{\sigma \sqrt{y}} = \frac{\ln \left( \frac{S}{X} \right) + \left( r + \frac{\sigma^2}{2} \right) (T - t)}{\sigma \sqrt{T - t}} \]

and

\[ d_2 = d_1 - \sigma \sqrt{T - t} = \frac{\ln \left( \frac{S}{X} \right) + \left( r - \frac{\sigma^2}{2} \right) (T - t)}{\sigma \sqrt{T - t}}. \]

### 4.2.3 Average of Plain Vanilla and Modified Log:

Here, we consider average of most well known Plain Vanilla Payoff function and recently discussed Modified Log payoff function [5], i.e.

\[
f(S) = \begin{cases} 
\frac{(S - X) + S \ln \left( \frac{S}{X} \right)}{2} & \text{if } S \geq X \\
0 & \text{if } S \leq X 
\end{cases}
\]

\[
\therefore f(Xe^x) = \begin{cases} 
\frac{Xe^x(x+1) - X}{2} & \text{if } x \geq 0 \\
0 & \text{if } x \leq 0 
\end{cases}
\]
\begin{align*}
D(x, y) &= \frac{1}{2\sigma \sqrt{2\pi y}} \int_{0}^{\infty} (X e^{\nu}(\nu + 1) - X) e^{-\frac{(x-\nu)^2}{2\sigma^2 y}} \, d\nu \\
&= \frac{X}{2\sigma \sqrt{2\pi y}} \int_{0}^{\infty} e^{\nu}(\nu + 1) e^{-\frac{(x-\nu)^2}{2\sigma^2 y}} \, d\nu - \frac{X}{2\sigma \sqrt{2\pi y}} \int_{0}^{\infty} e^{-\frac{(x-\nu)^2}{2\sigma^2 y}} \, d\nu \\
&= \frac{X}{2\sigma \sqrt{2\pi y}} \left[ \int_{0}^{\infty} \nu e^{\nu} e^{-\frac{(x-\nu)^2}{2\sigma^2 y}} \, d\nu + \int_{0}^{\infty} e^{\nu} e^{-\frac{(x-\nu)^2}{2\sigma^2 y}} \, d\nu - \frac{X}{2\sigma \sqrt{2\pi y}} \int_{0}^{\infty} e^{-\frac{(x-\nu)^2}{2\sigma^2 y}} \, d\nu \right]
\end{align*}

Substituting,

\[ Z = \frac{\nu - x}{\sigma \sqrt{y}} \]

we get,
Chapter 4. BSM Formulas For Different Payoff Functions

\[ D(x, y) = \frac{X}{2 \sqrt{2\pi}} \int_{-\infty}^{\infty} (x + \sigma \sqrt{y} Z) e^{x+\sigma \sqrt{y} Z} e^{-\frac{Z^2}{2}} dZ \]

\[ + \frac{X}{2 \sqrt{2\pi}} e^{x+\frac{\sigma^2 y}{2}} \int_{-\infty}^{\infty} e^{-\frac{(z-2\sigma \sqrt{y} z+\sigma^2 y)}{2}} dZ - \frac{X}{2 \sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{Z^2}{2}} dZ \]

\[ = \frac{X}{2 \sqrt{2\pi}} e^{x+\frac{\sigma^2 y}{2}} \int_{-\infty}^{\infty} e^{-\frac{Z^2}{2}} dZ - \frac{X}{2 \sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{Z^2}{2}} dZ \]

\[ = \frac{X}{2 \sqrt{2\pi}} (x+1)e^{x+\frac{\sigma^2 y}{2}} \int_{-\infty}^{\infty} e^{-\frac{Z^2}{2}} \sigma \sqrt{y} e^{x+\frac{\sigma^2 y}{2}} \int_{-\infty}^{\infty} e^{-\frac{Z^2}{2}} dZ \]

\[ - \frac{X}{2 \sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{Z^2}{2}} dZ \]

\[ = \frac{X}{2 \sqrt{2\pi}} (x+1+\sigma^2 y)e^{x+\frac{\sigma^2 y}{2}} \int_{-\infty}^{\infty} e^{-\frac{Z^2}{2}} dZ \]

\[ - \frac{X}{2 \sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{Z^2}{2}} dZ \]

\[ \therefore D(x, y) = \frac{X}{2} (x+1+\sigma^2 y)e^{x+\frac{\sigma^2 y}{2}} N \left( d_1 \right) + \frac{X}{2 \sqrt{2\pi}} e^{x+\frac{\sigma^2 y}{2}} e^{-\frac{d_1^2}{2}} - \frac{X}{2} N \left( d_2 \right) \]

where, \( d_1 = \frac{x+\sigma^2 y}{\sigma \sqrt{y}}, d_2 = \frac{x}{\sigma \sqrt{y}} \) and \( N(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-t^2/2} dt \)

\[ \therefore C(S, t) = \frac{S}{2} \left[ 1 + \ln \left( \frac{S}{X} \right) + (r + \frac{\sigma^2}{2}) (T-t) \right] N \left( d_1 \right) + \frac{S}{2 \sqrt{2\pi}} \sigma \sqrt{T} - te^{-\frac{d_1^2}{2}} \]

\[ - \frac{X}{2} e^{-r(T-t)} N \left( d_2 \right) \]
where,
\[ d_1 = \frac{x + \sigma^2 y}{\sigma \sqrt{y}} = \frac{\ln \left( \frac{S}{X} \right) + \left( r + \frac{\sigma^2}{2} \right) (T - t)}{\sigma \sqrt{T - t}} \]

and
\[ d_2 = d_1 - \sigma \sqrt{T - t} = \frac{\ln \left( \frac{S}{X} \right) + \left( r - \frac{\sigma^2}{2} \right) (T - t)}{\sigma \sqrt{T - t}}. \]

One can see from expressions (4.2) and (4.3) that, this is exactly the average of two solutions, i.e. solution using Plain Vanilla payoff (4.2) and solution using Modified Log payoff function (4.3).

**Theorem 4.2.1.** Let \( C_f \) and \( C_g \) be the solutions of BSM equation with the boundary conditions
\[ \lim_{t \to T^-} C_f(S,t) = f(S) \]
and
\[ \lim_{t \to T^-} C_g(S,t) = g(S) \]
then \( \alpha C_f + \beta C_g \) is a solution of BSM with the boundary condition \( \alpha f + \beta g \).

**Proof:** Let \( C_{\alpha f+\beta g} \) be the solution of BSM with the boundary condition \( \alpha f + \beta g \).
Define, \( V(S,t) = \alpha C_f + \beta C_g - C_{\alpha f+\beta g} \).
Here \( V(S,t) \) is a solution of BSM with the boundary condition
\[ \lim_{t \to T^-} V(S,t) = 0. \]

\[ \therefore V(S,t) = 0 \Rightarrow \alpha C_f + \beta C_g - C_{\alpha f+\beta g} = 0 \Rightarrow \alpha C_f + \beta C_g = C_{\alpha f+\beta g}. \]
Chapter 5

Use of Mathematica
In this chapter, we have discussed how to use Mathematica and Mathematica tools for different mathematical computations. S. Ghevariya [7] has used Mathematica along with C++ programming language and given several tables for comparative study.

Our main focus here is in seeing that the solution related to the average of the two payoff functions turns out to be the average of the two solutions.

5.1 History of Mathematica:

Wolfram Mathematica (people call it Mathematica) is a mathematical software, where symbolic computations are possible. It is used in various fields. It was considered by Stephen Wolfram. The language used in Mathematica is one of the programming languages called “Wolfram Language”.

Mathematica has variety of features, like: Libraries of elementary Mathematical functions and special functions, Tools for visualizing and analyzing graphs, Tools for combinatorial problems, Tools for financial calculations, including bonds, derivatives, options etc... and many more.

In every new version they add some new good commands and functions to make the things more simple and live. There are many ways to outspread applications written in Mathematica, one of them is Wolfram CDF Player. The name of this program was suggested by apple co-founder Steve Jobs. The first version of Mathematica was released in June 1988 as “Mathematica 1.0” and the latest version, that has been launched recently in March 2017 is “Mathematica 11.1” with around 130 new functions across a wide range of application areas.
Mathematica (A system to do Mathematics) is a symbolic and algebraic manipulation package which can handle the things in a neat way. It has graphical interface through which one can input the notations exactly the way we write on our black boards or in our notebooks.

With the help of Basic Math Assistant under the Palettes menu on the home page of Mathematica, one pulls down:

$$\sum_{n=\square}^{\square}$$

When one fills in the blanks in the boxes e.g. for the well known Leibnitz Series,

$$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \cdots$$

One has to write:

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2n-1}$$

Or equivalently one can write:

$$\text{Sum}[(-1)^{n+1}/(2n-1), n, 1, \infty]$$

or

$$\text{Sum}[(-1)^{n+1}/(2n-1), n, 1, \text{Infinity}]$$
On asking Mathematica to find sum (Shift+Enter), one obtains the answer $\frac{\pi}{4}$.
Thus we have been able to find sum of the series using Mathematica. i.e.

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2n-1} = \frac{\pi}{4}.$$ 

The same way we can also find integrals as:

$$\int_{a}^{b} f(x) \, dx$$

Alternatively we can write:

$$\text{Integrate}[f, x, x_{\text{min}}, x_{\text{max}}]$$

Here, in this thesis, we are using Mathematica for mainly two purposes: one is to verify the solutions and the second is graphical representations of the solutions using different payoff functions.

### 5.2 Verification of Solutions using Mathematica:

First we will show that how we can verify the solution using Mathematica.
In chapter-4, we have solved BSM equation 4.1 for the payoff function \( \max\{S - X, 0\} \) and we got the solution:

\[
C(S, t) = SN(d_1) - Xe^{-r(T-t)}N(d_2)
\]

where,

\[
d_1 = \frac{x + \sigma^2 y}{\sigma \sqrt{y}} = \frac{\ln \left( \frac{S}{X} \right) + \left( r + \frac{\sigma^2}{2} \right) (T - t)}{\sigma \sqrt{T - t}}
\]

and

\[
d_2 = d_1 - \sigma \sqrt{T - t} = \frac{\ln \left( \frac{S}{X} \right) + \left( r - \frac{\sigma^2}{2} \right) (T - t)}{\sigma \sqrt{T - t}}.
\]

In the solution \( N \) is the normal distribution.

Here, we have verified the solution using Mathematica.

First, we will define the normal distribution function as:
\[
\ln[1] := M[X_] := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{X} e^{-\frac{t^2}{2}} \, dt
\]

Now, using the above, we will define the Mathematica function Plain Vanilla payoff as a function of \(\sigma, X, S, t\) as follows:

\[
\ln[2] := \text{Plain Vanilla}[\sigma_, X_, S_, t_] := S \ast M \left[ \frac{\log \left( \frac{S}{X} \right) + \left( r + \frac{1}{2} \sigma^2 \right) (T - t)}{\sigma \sqrt{T - t}} \right] - X \ast e^{-r(T-t)} \ast M \left[ \frac{\log \left( \frac{S}{X} \right) + \left( r - \frac{1}{2} \sigma^2 \right) (T - t)}{\sigma \sqrt{T - t}} \right]
\]

Further we define Mathematica BSM equation as:

\[
\ln[3] := \text{BSM}[f_] := \partial_t f + \frac{1}{2} \ast \sigma^2 \ast S^2 \ast \partial_S \partial_S f + r \ast S \ast \partial_S f - r \ast f
\]

Thus, if \(g\) is a solution of BSM equation then \(\text{BSM}[g]\) must be zero.

So we write

\[
\ln[4] := \text{BSM}[\text{Plain Vanilla}[\sigma, X, S, t]]
\]

Now, we press the keys “SHIFT+ENTER”, and the output is as:
Chapter 5. Use of Mathematica

\[
e^{-r(-t+T)rX} \left( \sqrt{\frac{-t+T}{2}} \sqrt{\frac{\pi \sqrt{-t+T \sigma}}{2}} \right) \left( \frac{\sqrt{2\pi}}{2} \sqrt{-t+T} \sigma \left( r - \frac{\sigma^2}{2} \right) \text{Erf} \left[ \frac{\sqrt{\frac{(t-T)\left( r - \frac{\sigma^2}{2} \right) + \log \left[ \frac{S}{X} \right])^2}{(-t+T)\sigma^2}}}{\sqrt{2}} \right] \right)
\]

Out[4] = 

\[
- \frac{1}{\sqrt{2\pi}} e^{-r(-t+T)rX} \frac{\sqrt{2\pi}}{2} \sqrt{-t+T} \left( \frac{\sqrt{\frac{-t+T}{2}} \sqrt{\frac{\pi \sqrt{-t+T \sigma}}{2}}}{\sqrt{2}} \right) \left( \frac{\sqrt{2\pi}}{2} \sqrt{-t+T} \sigma \left( r - \frac{\sigma^2}{2} \right) \text{Erf} \left[ \frac{\sqrt{\frac{(t-T)\left( r - \frac{\sigma^2}{2} \right) + \log \left[ \frac{S}{X} \right])^2}{(-t+T)\sigma^2}}}{\sqrt{2}} \right] \right)
\]

\[
- \frac{1}{\sqrt{2\pi}} e^{-r(-t+T)rX} \frac{\sqrt{2\pi}}{2} \sqrt{-t+T} \left( \frac{\sqrt{\frac{-t+T}{2}} \sqrt{\frac{\pi \sqrt{-t+T \sigma}}{2}}}{\sqrt{2}} \right) \left( \frac{\sqrt{2\pi}}{2} \sqrt{-t+T} \sigma \left( r - \frac{\sigma^2}{2} \right) \text{Erf} \left[ \frac{\sqrt{\frac{(t-T)\left( r - \frac{\sigma^2}{2} \right) + \log \left[ \frac{S}{X} \right])^2}{(-t+T)\sigma^2}}}{\sqrt{2}} \right] \right)
\]

\[
+ \frac{1}{\sqrt{2\pi}} e^{-r(-t+T)rX} \left( \frac{\sqrt{2\pi}}{2} \sqrt{-t+T} \sigma \left( r - \frac{\sigma^2}{2} \right) \text{Erf} \left[ \frac{\sqrt{\frac{(t-T)\left( r - \frac{\sigma^2}{2} \right) + \log \left[ \frac{S}{X} \right])^2}{(-t+T)\sigma^2}}}{\sqrt{2}} \right] \right)
\]

\[
2 \left( \frac{2 \left( r + \frac{\sigma^2}{2} \right) \left( \frac{\left( t - T \right) \left( r - \frac{\sigma^2}{2} \right) + \log \left[ \frac{S}{X} \right]}{(-t-T)\sigma^2} \right) + \left( \frac{(t-T)\left( r - \frac{\sigma^2}{2} \right) + \log \left[ \frac{S}{X} \right])^2}{(-t+T)\sigma^2} \right)}{\sqrt{-t+T\sigma}} \right)
\]

\[
+ \frac{1}{\sqrt{2\pi}} e^{-r(-t+T)rX} \frac{\sqrt{\pi}}{2} \sqrt{-t+T} \sigma \text{Erf} \left[ \frac{\sqrt{\frac{(t-T)\left( r - \frac{\sigma^2}{2} \right) + \log \left[ \frac{S}{X} \right])^2}{(-t+T)\sigma^2}}}{\sqrt{2}} \right]
\]

\[
2 \left( \frac{2 \left( r + \frac{\sigma^2}{2} \right) \left( \frac{\left( t - T \right) \left( r - \frac{\sigma^2}{2} \right) + \log \left[ \frac{S}{X} \right]}{(-t-T)\sigma^2} \right) + \left( \frac{(t-T)\left( r - \frac{\sigma^2}{2} \right) + \log \left[ \frac{S}{X} \right])^2}{(-t+T)\sigma^2} \right)}{\sqrt{-t+T\sigma}} \right)
\]

\[
2 \left( \frac{\left( r + \frac{\sigma^2}{2} \right) \left( \frac{\left( t - T \right) \left( r - \frac{\sigma^2}{2} \right) + \log \left[ \frac{S}{X} \right]}{(-t-T)\sigma^2} \right) + \left( \frac{(t-T)\left( r - \frac{\sigma^2}{2} \right) + \log \left[ \frac{S}{X} \right])^2}{(-t+T)\sigma^2} \right)}{\sqrt{-t+T\sigma}} \right)
\]
\[ + \frac{1}{\sqrt{2\pi}} S \left[ \frac{(-t + T) \left( r + \frac{\sigma^2}{2} \right) + \log \left( \frac{S}{X} \right)}{(-t + T)\sigma^2} \right] \]

\[ \left( \sqrt{\frac{\pi}{2}} \sqrt{-t + T}\sigma \left( -r - \frac{\sigma^2}{2} \right) \text{Erf} \left( \frac{\sqrt{\frac{(-t + T) \left( r + \frac{\sigma^2}{2} \right) + \log \left( \frac{S}{X} \right)}{(-t + T)\sigma^2}}}{\sqrt{2}} \right) \right) \]

\[ \left( \sqrt{\frac{\pi}{2}} \sigma \text{Erf} \left( \frac{\sqrt{\frac{(-t + T) \left( r + \frac{\sigma^2}{2} \right) + \log \left( \frac{S}{X} \right)}{(-t + T)\sigma^2}}}{\sqrt{2}} \right) \right) \]

\[ \left( \frac{2 (-r - \frac{\sigma^2}{2}) (\sqrt{-t + T} \left( r + \frac{\sigma^2}{2} \right) + \log \left( \frac{S}{X} \right))}{(-t + T)\sigma^2} \right) \]

\[ \left( \sqrt{\frac{\pi}{2}} \sqrt{-t + T}\sigma \text{Erf} \left( \frac{\sqrt{\frac{(-t + T) \left( r + \frac{\sigma^2}{2} \right) + \log \left( \frac{S}{X} \right)}{(-t + T)\sigma^2}}}{\sqrt{2}} \right) \right) \]

\[ \left( \frac{2 (-r - \frac{\sigma^2}{2}) (\sqrt{-t + T} \left( r + \frac{\sigma^2}{2} \right) + \log \left( \frac{S}{X} \right))}{(-t + T)\sigma^2} \right) \]

\[ \left( \frac{2 (-r - \frac{\sigma^2}{2}) (\sqrt{-t + T} \left( r + \frac{\sigma^2}{2} \right) + \log \left( \frac{S}{X} \right))}{(-t + T)\sigma^2} \right) \]
\[ + \frac{1}{2} \sigma \frac{1}{\sqrt{2\pi}} e^{-r(t+T)} X \frac{\frac{\left( -\left( -\frac{\sigma^2}{2} \right) + \log \left( \frac{X}{\sigma} \right) \right)^2}{\left( -t + T \right)^2}} \sqrt{-t + T} \]

\[ + \frac{1}{2\sqrt{2\pi}} e^{-r(t+T)} X \frac{\left( \frac{\left( -\frac{\sigma^2}{2} \right) + \log \left( \frac{X}{\sigma} \right) \right)^2}{\left( -t + T \right)^2} \right)^{3/2} \]

\[ \sqrt{\frac{1}{2}} \text{Erf} \left[ \frac{\left( \frac{\left( -\frac{\sigma^2}{2} \right) + \log \left( \frac{X}{\sigma} \right) \right)^2}{\left( -t + T \right)^2} \right] \right] \]

\[ \frac{\left( \frac{\left( -\frac{\sigma^2}{2} \right) + \log \left( \frac{X}{\sigma} \right) \right)^2}{\left( -t + T \right)^2} \right] \]

\[ \frac{\left( \frac{\left( -\frac{\sigma^2}{2} \right) + \log \left( \frac{X}{\sigma} \right) \right)^2}{\left( -t + T \right)^2} \right] \]

\[ \frac{\left( \frac{\left( -\frac{\sigma^2}{2} \right) + \log \left( \frac{X}{\sigma} \right) \right)^2}{\left( -t + T \right)^2} \right] \]

\[ \frac{\left( \frac{\left( -\frac{\sigma^2}{2} \right) + \log \left( \frac{X}{\sigma} \right) \right)^2}{\left( -t + T \right)^2} \right] \]

\[ \frac{\left( \frac{\left( -\frac{\sigma^2}{2} \right) + \log \left( \frac{X}{\sigma} \right) \right)^2}{\left( -t + T \right)^2} \right] \]

\[ \frac{\left( \frac{\left( -\frac{\sigma^2}{2} \right) + \log \left( \frac{X}{\sigma} \right) \right)^2}{\left( -t + T \right)^2} \right] \]

\[ \frac{\left( \frac{\left( -\frac{\sigma^2}{2} \right) + \log \left( \frac{X}{\sigma} \right) \right)^2}{\left( -t + T \right)^2} \right] \]
\[
\sqrt{\frac{\pi}{2}} \sqrt{-t + T} \sigma \text{Erf} \left[ \frac{\sqrt{\left(\frac{\left(-t+T\right)\left(r + \frac{\sigma^2}{2}\right) + \text{Log} \left[ \frac{S}{X} \right]}{\sigma^2} \right)^2}{\sqrt{2}}} \right]
\]

\[
- \frac{1}{\sqrt{2\pi}} S \sigma e^{-\frac{\left(-t+T\right)\left(r + \frac{\sigma^2}{2}\right) + \text{Log} \left[ \frac{S}{X} \right]}{2\sigma^2}} \sqrt{-t + T} \frac{\left(-t + T\right)^{3/2} \sigma}{\sqrt{2}} \left( -\frac{1}{\sqrt{2\pi}} S \left( -t + T \right) \left( r + \frac{\sigma^2}{2} \right) + \text{Log} \left[ \frac{S}{X} \right] \right)
\]

\[
\sqrt{\frac{\pi}{2}} \text{Erf} \left[ \frac{\sqrt{\left(\frac{\left(-t+T\right)\left(r + \frac{\sigma^2}{2}\right) + \text{Log} \left[ \frac{S}{X} \right]}{\sigma^2} \right)^2}}{\sqrt{2}} \right]
\]

\[
- \frac{1}{\sqrt{2\pi}} S \sigma \left( -\frac{1}{\sqrt{2\pi}} S \left( -t + T \right) \left( r + \frac{\sigma^2}{2} \right) + \text{Log} \left[ \frac{S}{X} \right] \right)
\]

\[
\sqrt{-t + T} \left( -\frac{1}{\sqrt{2\pi}} S \sigma \left( -\frac{1}{\sqrt{2\pi}} S \left( -t + T \right) \left( r + \frac{\sigma^2}{2} \right) + \text{Log} \left[ \frac{S}{X} \right] \right) \right)
\]

\[
- S \sigma \text{Erf} \left[ \frac{\sqrt{\left(\frac{\left(-t+T\right)\left(r + \frac{\sigma^2}{2}\right) + \text{Log} \left[ \frac{S}{X} \right]}{\sigma^2} \right)^2}}{\sqrt{2}} \right]
\]

\[
4\sqrt{-t + T} \left( -\frac{1}{\sqrt{2\pi}} S \sigma \left( -\frac{1}{\sqrt{2\pi}} S \left( -t + T \right) \left( r + \frac{\sigma^2}{2} \right) + \text{Log} \left[ \frac{S}{X} \right] \right) \right) \sqrt{-t + T} \text{Erf} \left[ \frac{\sqrt{\left(\frac{\left(-t+T\right)\left(r + \frac{\sigma^2}{2}\right) + \text{Log} \left[ \frac{S}{X} \right]}{\sigma^2} \right)^2}}{\sqrt{2}} \right]
\]

\[
+ S \sigma^2 \left( -\frac{1}{\sqrt{2\pi}} S \sigma \left( -\frac{1}{\sqrt{2\pi}} S \left( -t + T \right) \left( r + \frac{\sigma^2}{2} \right) + \text{Log} \left[ \frac{S}{X} \right] \right) \right)
\]

\[
\sqrt{-t + T} \text{Erf} \left[ \frac{\sqrt{\left(\frac{\left(-t+T\right)\left(r + \frac{\sigma^2}{2}\right) + \text{Log} \left[ \frac{S}{X} \right]}{\sigma^2} \right)^2}}{\sqrt{2}} \right]
\]

\[
- \frac{1}{2\sqrt{2\pi}} S \sigma \left( -\frac{1}{\sqrt{2\pi}} S \left( -t + T \right) \left( r + \frac{\sigma^2}{2} \right) + \text{Log} \left[ \frac{S}{X} \right] \right)
\]

\[
2 \left( -\frac{1}{\sqrt{2\pi}} S \sigma \left( -\frac{1}{\sqrt{2\pi}} S \left( -t + T \right) \left( r + \frac{\sigma^2}{2} \right) + \text{Log} \left[ \frac{S}{X} \right] \right) \right)
\]
\[ +S\sigma^2 \sqrt{\left( \frac{(-t + T) \left(r + \frac{\sigma^2}{2}\right) + \log \left[ \frac{S}{X} \right]}{(-t + T)\sigma^2} \right)^2} \]

\[ \sqrt{-t + T}\sigma \text{Erf} \left[ \frac{\sqrt{\left( \frac{(-t + T) \left(r + \frac{\sigma^2}{2}\right) + \log \left[ \frac{S}{X} \right]}{(-t + T)\sigma^2} \right)^2}}{\sqrt{2}} \right] \]

\[ 4 \left( (-t + T) \left(r + \frac{\sigma^2}{2}\right) + \log \left[ \frac{S}{X} \right] \right)^2 \]

\[ + \sqrt{\frac{2}{\pi}} \frac{1}{2} S\sigma e^{-\frac{\left((-t+T)\left(r+\frac{\sigma^2}{2}\right)+\log\left[\frac{S}{X}\right]\right)^2}{2(-t+T)\sigma^2}} \]

\[ + \sqrt{\frac{2}{\pi}} \text{Erf} \left[ \frac{\sqrt{\left( \frac{(-t + T) \left(r + \frac{\sigma^2}{2}\right) + \log \left[ \frac{S}{X} \right]}{(-t + T)\sigma^2} \right)^2}}{\sqrt{2}} \right] \]

\[ -\frac{1}{2} S\sigma^2 \sqrt{\left( (-t + T) \left(r + \frac{\sigma^2}{2}\right) + \log \left[ \frac{S}{X} \right] \right)^2} \]

\[ \sqrt{-t + T}\sigma \text{Erf} \left[ \frac{\sqrt{\left( \frac{(-t + T) \left(r + \frac{\sigma^2}{2}\right) + \log \left[ \frac{S}{X} \right]}{(-t + T)\sigma^2} \right)^2}}{\sqrt{2}} \right] \]

\[ \left( (-t + T) \left(r + \frac{\sigma^2}{2}\right) + \log \left[ \frac{S}{X} \right] \right)^2 \]

\[ -r \frac{1}{\sqrt{2\pi}} e^{-r(-t+T)X} e^{-\frac{\left((-t+T)\left(r+\frac{\sigma^2}{2}\right)+\log\left[\frac{S}{X}\right]\right)^2}{2(-t+T)\sigma^2}} \]

\[ \sqrt{\frac{2}{\pi}} \text{Erf} \left[ \frac{\sqrt{\left( \frac{(-t + T) \left(r - \frac{\sigma^2}{2}\right) + \log \left[ \frac{S}{X} \right]}{(-t + T)\sigma^2} \right)^2}}{\sqrt{2}} \right] \]

\[ -r \frac{1}{\sqrt{2\pi}} e^{-r(-t+T)X} \frac{\sqrt{\left( \frac{(-t + T) \left(r - \frac{\sigma^2}{2}\right) + \log \left[ \frac{S}{X} \right]}{(-t + T)\sigma^2} \right)^2}}{\sqrt{-t + T}\sigma} \]

\[ +r \frac{1}{\sqrt{2\pi}} e^{-r(-t+T)X} \sqrt{-t + T}\sigma \left( \frac{(-t + T) \left(r - \frac{\sigma^2}{2}\right) + \log \left[ \frac{S}{X} \right]}{(-t + T)\sigma^2} \right)^2 \]
\[
\sqrt{\frac{\pi}{2}} \sqrt{-t + T} \sigma \operatorname{Erf} \left[ \frac{\sqrt{\left(\frac{(-t+T)(r - \frac{\sigma^2}{2}) + \log \left(\frac{S}{X}\right)}{-(-t+T)\sigma^2}\right)^2}}{\sqrt{2}} \right]
\]

\[
\left( (-t + T) \left( r - \frac{\sigma^2}{2} \right) + \log \left(\frac{S}{X}\right) \right)^2
\]

\[
+ r S \frac{\left(\frac{(-t+T)(r + \frac{\sigma^2}{2}) + \log \left(\frac{S}{X}\right)}{-(-t+T)\sigma^2}\right)^2}{2\sqrt{-t + T} \sigma} - \frac{r S}{2}
\]

\[
\frac{\sqrt{-t + T} \sigma \operatorname{Erf} \left[ \frac{\sqrt{\left(\frac{(-t+T)(r + \frac{\sigma^2}{2}) + \log \left(\frac{S}{X}\right)}{-(-t+T)\sigma^2}\right)^2}}{\sqrt{2}} \right]}{2 \left( (-t + T) \left( r + \frac{\sigma^2}{2} \right) + \log \left(\frac{S}{X}\right) \right)^2} + \frac{r S}{2}
\]

\[
+ \frac{r S}{2} \sqrt{\frac{\left(\frac{(-t+T)(r + \frac{\sigma^2}{2}) + \log \left(\frac{S}{X}\right)}{-(-t+T)\sigma^2}\right)^2}{(-t + T)\sigma^2}}
\]

\[
\sqrt{\frac{\pi}{2}} \sqrt{-t + T} \sigma \operatorname{Erf} \left[ \frac{\sqrt{\left(\frac{(-t+T)(r + \frac{\sigma^2}{2}) + \log \left(\frac{S}{X}\right)}{-(-t+T)\sigma^2}\right)^2}}{\sqrt{2}} \right]
\]
This is a very long, frightening and bewildering expression, but if we simplify this using the command:

\[ \text{ln}[5] := \text{Simplify}[%] \]

then,

\[ \text{Out}[5] = 0 \]

Here, we have out put as “0” i.e. BSM equation is verified here for the solution obtained for the Plain Vanilla payoff that we and many others have derived using different methods.

Now, we will verify the solution for Modified Log Payoff function, for that we will define as before ModifiedLog as a function of \( \sigma, X, S, t \) as follows:

\[ \text{ln}[6] := \text{ModifiedLog}[\sigma_-, X_-, S_-, t_-] := \]

\[
S(\log \left[ \frac{S}{X} \right]) \times M \left[ \frac{\log \left[ \frac{S}{X} \right] + \left( r + \frac{1}{2} \sigma^2 \right) \times (T - t)}{\sigma \times \sqrt{T - t}} \right] \\
+ S \left( r + \frac{1}{2} \sigma^2 \right) \times (T - t) \times M \left[ \frac{\log \left[ \frac{S}{X} \right] + S \left( r + \frac{1}{2} \sigma^2 \right) \times (T - t)}{\sigma \times \sqrt{T - t}} \right]
\]
Chapter 5. Use of Mathematica

\[ + S \sigma \sqrt{T - t} * \frac{1}{\sqrt{2\pi}} e^{-\left( \frac{\log\left(\frac{S}{X}\right) + (r + \frac{\sigma^2}{2}) (T - t)}{\sigma \sqrt{T - t}} \right)^2} \]

\[ \ln[7] := \text{BSM}[\text{ModifiedLog}[\sigma, X, S, t]] \]

\[ \text{Out}[7] = -rS \frac{\left( -x + \frac{\sigma^2}{2} \right)^2}{2(-x + \frac{\sigma^2}{2})^2} \frac{1}{\sqrt{2\pi}} \frac{1}{\sqrt{-t + T \sigma}} \]

\[ - r \frac{S}{2}(2(-x + \frac{\sigma^2}{2})^2 + \log \left( \frac{S}{X} \right)) \]

\[ - r S \frac{\left( -x + \frac{\sigma^2}{2} \right)^2}{2(-x + \frac{\sigma^2}{2})^2} \frac{1}{\sqrt{2\pi}} \frac{1}{\sqrt{-t + T \sigma}} \]

\[ - \frac{S}{2} \log \left( \frac{S}{X} \right) \]

\[ - r \frac{S}{2}(2(-x + \frac{\sigma^2}{2})^2 + \log \left( \frac{S}{X} \right)) \]

\[ - \frac{S}{2} \log \left( \frac{S}{X} \right) \]

\[ - \frac{S}{2} \log \left( \frac{S}{X} \right) \]

\[ - \frac{S}{2} \log \left( \frac{S}{X} \right) \]

\[ - \frac{S}{2} \log \left( \frac{S}{X} \right) \]

\[ + S \frac{\left( -x + \frac{\sigma^2}{2} \right)^2}{2(-x + \frac{\sigma^2}{2})^2} \frac{1}{\sqrt{2\pi}} \frac{1}{\sqrt{-t + T \sigma}} \]

\[ - \frac{r}{2\sqrt{2\pi}} e^{-\left( \frac{\log\left(\frac{S}{X}\right) + (r + \frac{\sigma^2}{2}) (T - t)}{\sigma \sqrt{T - t}} \right)^2} \]

\[ - \frac{S}{\sqrt{2\pi}} e^{-\left( \frac{\log\left(\frac{S}{X}\right) + (r + \frac{\sigma^2}{2}) (T - t)}{\sigma \sqrt{T - t}} \right)^2} \frac{1}{\sqrt{-t + T \sigma}} \]
\[
\left(\left(-r - \frac{\sigma^2}{2}\right) \left(-t + T \left(r + \frac{\sigma^2}{2}\right) + \log\left(\frac{s}{X}\right)\right)\right) \left(\frac{-t + T}{\sigma^2}\right) \\
- S \sqrt{\frac{2}{\pi}} e^{-\frac{\left(-t + T \left(r + \frac{\sigma^2}{2}\right) + \log\left(\frac{s}{X}\right)\right)^2}{2(-t + T)\sigma^2}} \\
\left(\frac{\left(-t + T \left(r + \frac{\sigma^2}{2}\right) + \log\left(\frac{s}{X}\right)\right)^2}{2(-t + T)^2\sigma^2}\right) \\
- \log\left[\frac{S}{X}\right] S \sqrt{\frac{2}{\pi}} e^{-\frac{\left(-t + T \left(r + \frac{\sigma^2}{2}\right) + \log\left(\frac{s}{X}\right)\right)^2}{2(-t + T)\sigma^2}} \\
\sqrt{-t + T} e^{-\frac{\left(-t + T \left(r + \frac{\sigma^2}{2}\right) + \log\left(\frac{s}{X}\right)\right)^2}{2(-t + T)^2\sigma^2}} \\
\left[-\frac{1}{\sqrt{2\pi}} \log\left[\frac{S}{X}\right] \sqrt{\frac{2}{\pi}} \sigma \text{Erf} \left[\frac{\left(-t + T \left(r + \frac{\sigma^2}{2}\right) + \log\left(\frac{s}{X}\right)\right)^2}{\sqrt{2}}\right] \right] \\
2 \left(-t + T \left(r + \frac{\sigma^2}{2}\right) + \log\left(\frac{s}{X}\right)\right) \\
+ \frac{1}{\sqrt{2\pi}} \log\left[\frac{S}{X}\right] e^{-\frac{\left(-t + T \left(r + \frac{\sigma^2}{2}\right) + \log\left(\frac{s}{X}\right)\right)^2}{2(-t + T)^2\sigma^2}} \sqrt{-t + T} \sigma S \\
\left(\frac{2\left(-r - \frac{\sigma^2}{2}\right) \left(-t + T \left(r + \frac{\sigma^2}{2}\right) + \log\left(\frac{s}{X}\right)\right) + \left(-t + T \left(r + \frac{\sigma^2}{2}\right) + \log\left(\frac{s}{X}\right)\right)^2}{2\left(-t + T \left(r + \frac{\sigma^2}{2}\right) + \log\left(\frac{s}{X}\right)\right)}\right) \\
\left(\frac{-t + T}{\sigma^2}\right) \\
2 \left(-t + T \left(r + \frac{\sigma^2}{2}\right) + \log\left(\frac{s}{X}\right)\right) \\
\left[-\frac{1}{\sqrt{2\pi}} \log\left[\frac{S}{X}\right] \sqrt{\frac{2}{\pi}} \sqrt{-t + T} \sigma \text{Erf} \left[\frac{\left(-t + T \left(r + \frac{\sigma^2}{2}\right) + \log\left(\frac{s}{X}\right)\right)^2}{\sqrt{2}}\right] \right] \\
\left[-\frac{1}{\sqrt{2\pi}} \log\left[\frac{S}{X}\right] \sqrt{\frac{2}{\pi}} \sqrt{-t + T} \sigma \text{Erf} \left[\frac{\left(-t + T \left(r + \frac{\sigma^2}{2}\right) + \log\left(\frac{s}{X}\right)\right)^2}{\sqrt{2}}\right] \right]
\]

\[
\left( \frac{2\left(-r - \frac{\sigma^2}{2}\right) \left(\frac{T}{\sigma^2}\right) + \log\left[\frac{S}{X}\right]}{(-t + T)\sigma^2} \right) + \left( \frac{\left(-\left(-t + T\right)\sigma^2\right)^2}{(-t + T)\sigma^2} \right) + \left( \frac{\left(-\left(-t + T\right)\sigma^2\right)^2}{(-t + T)\sigma^2} \right)
\]

\[
2 \left(\frac{\left(-t + T\right) \left(r + \frac{\sigma^2}{2}\right) + \log\left[\frac{S}{X}\right]}{\sigma^2} \right) \left( \frac{\left(-\left(-t + T\right)\sigma^2\right)^2}{(-t + T)\sigma^2} \right)
\]

\[
+ \frac{1}{2} S^2 \sigma^2 \left(\frac{\left(-\left(-t + T\right)\sigma^2\right)^2}{2(-t + T)\sigma^2} \right) \frac{\sqrt{\frac{2}{\pi}} \left(\frac{\left(-t + T\right) \left(r + \frac{\sigma^2}{2}\right) + \log\left[\frac{S}{X}\right]}{\sigma^2} \right)}{S\sqrt{-t + T}} 
\]

\[
- \frac{1}{2} S^2 \sigma^2 \left(\frac{\left(-\left(-t + T\right)\sigma^2\right)^2}{2(-t + T)\sigma^2} \right) \frac{\sqrt{\frac{2}{\pi}} \left(\frac{\left(-t + T\right) \left(r + \frac{\sigma^2}{2}\right) + \log\left[\frac{S}{X}\right]}{\sigma^2} \right)}{S\sqrt{-t + T}} 
\]

\[
+ \frac{1}{2} S^2 \sigma^2 \left(\frac{\left(-\left(-t + T\right)\sigma^2\right)^2}{2(-t + T)\sigma^2} \right) \frac{\sqrt{\frac{2}{\pi}} \left(\frac{\left(-t + T\right) \left(r + \frac{\sigma^2}{2}\right) + \log\left[\frac{S}{X}\right]}{\sigma^2} \right)}{S\sqrt{-t + T}} 
\]

\[
\sqrt{-t + T} \sigma \operatorname{Erf} \left(\frac{\left(\frac{-\left(-t + T\right) \left(r + \frac{\sigma^2}{2}\right) + \log\left[\frac{S}{X}\right]}{\sigma^2} \right)}{\sqrt{2}} \right) + \frac{1}{2} S \sigma^2 
\]

\[
\sqrt{-t + T} \sigma \operatorname{Erf} \left(\frac{\left(\frac{-\left(-t + T\right) \left(r + \frac{\sigma^2}{2}\right) + \log\left[\frac{S}{X}\right]}{\sigma^2} \right)}{\sqrt{2}} \right) + \frac{1}{2} S \sigma^2 
\]

\[
- \frac{1}{2} S^2 \sigma^2 \left(\frac{\left(-\left(-t + T\right)\sigma^2\right)^2}{2(-t + T)\sigma^2} \right) \frac{\sqrt{\frac{2}{\pi}} \left(\frac{\left(-t + T\right) \left(r + \frac{\sigma^2}{2}\right) + \log\left[\frac{S}{X}\right]}{\sigma^2} \right)}{S\sqrt{-t + T}} 
\]

\[
\frac{\left(\frac{-\left(-t + T\right) \left(r + \frac{\sigma^2}{2}\right) + \log\left[\frac{S}{X}\right]}{\sigma^2} \right)}{\sqrt{2}} \right) + \frac{1}{2} S \sigma^2 
\]

\[
\sqrt{-t + T} \sigma \operatorname{Erf} \left(\frac{\left(\frac{-\left(-t + T\right) \left(r + \frac{\sigma^2}{2}\right) + \log\left[\frac{S}{X}\right]}{\sigma^2} \right)}{\sqrt{2}} \right) + \frac{1}{2} S \sigma^2 
\]

\[
- \frac{1}{2} S^2 \sigma^2 \left(\frac{\left(-\left(-t + T\right)\sigma^2\right)^2}{2(-t + T)\sigma^2} \right) \frac{\sqrt{\frac{2}{\pi}} \left(\frac{\left(-t + T\right) \left(r + \frac{\sigma^2}{2}\right) + \log\left[\frac{S}{X}\right]}{\sigma^2} \right)}{S\sqrt{-t + T}} 
\]

\[
\frac{\left(\frac{-\left(-t + T\right) \left(r + \frac{\sigma^2}{2}\right) + \log\left[\frac{S}{X}\right]}{\sigma^2} \right)}{\sqrt{2}} \right) + \frac{1}{2} S \sigma^2 
\]
\[-\frac{1}{2} S^2 \sigma^2 e^{-\frac{\left(\frac{-t+T}{\sigma^2} + \log \left[ \frac{S}{X} \right] \right)}{2(-t+T)\sigma^2}} \sqrt{-t + T} \left( r + \frac{\sigma^2}{2} \right) \]

\[+ \frac{1}{2} S^2 \sigma^2 \left( -t + T \right) \left( r + \frac{\sigma^2}{2} \right) + \log \left[ \frac{S}{X} \right] \]

\[e^{-\frac{\left(\frac{-t+T}{\sigma^2} + \log \left[ \frac{S}{X} \right] \right)}{2(-t+T)\sigma^2}} \sqrt{2\pi S\sigma} \]

\[-\frac{1}{2} S^2 \sigma^2 \left( r + \frac{\sigma^2}{2} \right) \left( -t + T \right) \left( r + \frac{\sigma^2}{2} \right) + \log \left[ \frac{S}{X} \right] \]

\[e^{-\frac{\left(\frac{-t+T}{\sigma^2} + \log \left[ \frac{S}{X} \right] \right)}{2(-t+T)\sigma^2}} \sqrt{2\pi S\sqrt{-t + T} \sigma^3} \]

\[+ \frac{1}{2} S^2 \sigma^2 \left( -t + T \right) \left( r + \frac{\sigma^2}{2} \right) + \log \left[ \frac{S}{X} \right] \]

\[e^{-\frac{\left(\frac{-t+T}{\sigma^2} + \log \left[ \frac{S}{X} \right] \right)}{2(-t+T)\sigma^2}} \sqrt{2\pi S(-t + T)^{3/2} \sigma^3} \]

\[-\frac{1}{2} S^2 \sigma^2 \frac{1}{\sqrt{2\pi}} \log \left[ \frac{S}{X} \right] e^{-\frac{\left(\frac{-t+T}{\sigma^2} + \log \left[ \frac{S}{X} \right] \right)}{2(-t+T)\sigma^2}} \frac{S \sqrt{-t + T} \sigma}{S(-t + T)^{3/2} \sigma^3} \]

\[+ \frac{S \sigma^2}{2\sqrt{2\pi}} \log \left[ \frac{S}{X} \right] \left( -t + T \right) \left( r + \frac{\sigma^2}{2} \right) + \log \left[ \frac{S}{X} \right] \]

\[\sqrt{\frac{\pi}{2}} \text{Erf} \left[ \frac{\left(\frac{-t+T}{\sigma^2} + \log \left[ \frac{S}{X} \right] \right)}{\sqrt{2}} \right] \]

\[\left( -t + T \right)^{3/2} \sigma^3 \left( \frac{\left(\frac{-t+T}{\sigma^2} + \log \left[ \frac{S}{X} \right] \right)}{(-t+T)\sigma^2} \right)^{3/2} \]
\[-\frac{S\sigma^2}{2\sqrt{2\pi}} \log \frac{S}{X} \div \sqrt{-t + T\sigma} \sqrt{\left(\frac{\left((-t + T) \left(\frac{S^2}{2} + \log \left(\frac{S}{X} \right)\right)^2}{(t + T)\sigma^2} - \frac{S\sigma^2}{2\sqrt{2\pi}} \log \frac{S}{X} \right)^2}{\sqrt{\left(\frac{\left((-t + T) \left(\frac{S^2}{2} + \log \left(\frac{S}{X} \right)\right)^2}{(t + T)\sigma^2} - \frac{S\sigma^2}{2\sqrt{2\pi}} \log \frac{S}{X} \right)^2}}\]
\[ \sqrt{-t + T}\sigma\text{Erf} \left[ \frac{\sqrt{\left((t - T)\left(r + \frac{\sigma^2}{2}\right) + \log\left[\frac{S}{X}\right]\right)^2}}{\sqrt{2}} \right] \]

\[ S \left(\begin{array}{c}
(t + T) \left(r + \frac{\sigma^2}{2}\right) + \log\left[\frac{S}{X}\right]
\end{array}\right)^2 \]

\[ - \frac{1}{4} S\sigma^2 + \frac{1}{2} S\sigma^2 \sqrt{\pi} \sqrt{\frac{2}{2}} \left(\begin{array}{c}
(t + T) \left(r + \frac{\sigma^2}{2}\right) + \log\left[\frac{S}{X}\right]
\end{array}\right)^2 \]

\[ \sqrt{-t + T}\sigma\text{Erf} \left[ \frac{\sqrt{\left((t - T)\left(r + \frac{\sigma^2}{2}\right) + \log\left[\frac{S}{X}\right]\right)^2}}{\sqrt{2}} \right] \]

\[ (t + T) \left(r + \frac{\sigma^2}{2}\right) + \log\left[\frac{S}{X}\right] \]

\[ + \frac{rS\sigma}{\sqrt{2\pi}} e^{-\frac{\left((t - T)\left(r + \frac{\sigma^2}{2}\right) + \log\left[\frac{S}{X}\right]\right)^2}{2\left(t - t\right)\sigma^2}} \sqrt{-t + T} \]

\[ + \frac{rS}{2} \left(t + T\right) \left(r + \frac{\sigma^2}{2}\right) \left(1 + \text{Erf} \left[ \frac{(t + T) \left(r + \frac{\sigma^2}{2}\right) + \log\left[\frac{S}{X}\right]}{\sqrt{2}\sqrt{t + T}} \right] \right) \]

\[ \frac{rS}{\sqrt{2\pi}} \log\left[\frac{S}{X}\right] \sqrt{\frac{\pi}{2}} \]

\[ \sqrt{\frac{\pi}{2}} \sqrt{-t + T}\sigma\text{Erf} \left[ \frac{\sqrt{\left((t - T)\left(r + \frac{\sigma^2}{2}\right) + \log\left[\frac{S}{X}\right]\right)^2}}{\sqrt{2}} \right] \]

\[ (t + T) \left(r + \frac{\sigma^2}{2}\right) + \log\left[\frac{S}{X}\right] \]

\[ + \frac{rSe^{-\frac{\left((t - T)\left(r + \frac{\sigma^2}{2}\right) + \log\left[\frac{S}{X}\right]\right)^2}{2\left(t - t\right)\sigma^2}} \sqrt{-t + T} \left(r + \frac{\sigma^2}{2}\right)}{\sqrt{2\pi}\sigma} \]

\[ \frac{rSe^{-\frac{\left((t - T)\left(r + \frac{\sigma^2}{2}\right) + \log\left[\frac{S}{X}\right]\right)^2}{2\left(t - t\right)\sigma^2}} \sqrt{-t + T} \left(r + \frac{\sigma^2}{2}\right)}{\sqrt{2\pi}\sigma} \]

\[ + \frac{rS}{\sqrt{2\pi}} \log\left[\frac{S}{X}\right] e^{-\frac{\left((t - T)\left(r + \frac{\sigma^2}{2}\right) + \log\left[\frac{S}{X}\right]\right)^2}{2\left(t - t\right)\sigma^2}} \frac{\sqrt{-t + T}}{\sqrt{2\pi}} \sigma \]
Chapter 5. Use of Mathematica

\[ + rS \frac{\log \left[ \frac{S}{X} \right]}{\sqrt{2\pi}} \sqrt{-t + T} \sigma \sqrt{\frac{\left(\left(\left(-t + T\right) \left(r + \frac{\sigma^2}{2}\right) + \log \left[ \frac{S}{X} \right]\right)^2\right)}{\left(-t + T\right)^2}} \]

\[ - \frac{rS}{2} \log \left[ \frac{S}{X} \right] \sqrt{\frac{\left(\left(-t + T\right) \left(r + \frac{\sigma^2}{2}\right) + \log \left[ \frac{S}{X} \right]\right)^2}{\left(-t + T\right)^2}} \frac{\sqrt{-t + T} \sigma \text{Erf} \left[ \frac{\left(\left(-t + T\right) \left(r + \frac{\sigma^2}{2}\right) + \log \left[ \frac{S}{X} \right]\right)^2}{\left(-t + T\right)^2} \right]}{\left(-t + T\right) \left(r + \frac{\sigma^2}{2}\right) + \log \left[ \frac{S}{X} \right]} \]

\[ \ln[8] := \text{Simplify}[\%] \]

\[ S (2r + \sigma^2) \text{Erf} \left[ \frac{\sqrt{\left(\left(-t + T\right) \left(r + \frac{\sigma^2}{2}\right) + \log \left[ \frac{S}{X} \right]\right)^2}}{\left(-t + T\right)^2} \right] \]

\[ \text{Out}[8] = \frac{-4 \sqrt{-t + T} \sigma \sqrt{-\left(\left(-t + T\right) \left(2r + \sigma^2\right) - 2 \log \left[ \frac{S}{X} \right]\right)^2}}{4 \sqrt{-t + T} \sigma} + \frac{S \left(2r + \sigma^2\right) 2 \log \left[ \frac{S}{X} \right]}{4 \sqrt{-t + T} \sigma \sqrt{-\left(\left(-t + T\right) \left(2r + \sigma^2\right) - 2 \log \left[ \frac{S}{X} \right]\right)^2}} \]

\[ S (2r + \sigma^2) \sqrt{-t + T} \sigma \text{Erf} \left[ \frac{\left(-t + T\right) \left(r + \frac{\sigma^2}{2}\right) + \log \left[ \frac{S}{X} \right]}{\sqrt{2} \sqrt{-t + T} \sigma} \right] \]

\[ - \frac{S (2r + \sigma^2) \sqrt{-t + T} \sigma \text{Erf} \left[ \frac{\left(-t + T\right) \left(r + \frac{\sigma^2}{2}\right) + \log \left[ \frac{S}{X} \right]}{\sqrt{2} \sqrt{-t + T} \sigma} \right]}{4 \sqrt{-t + T} \sigma} \]
Try to see yourself that the above algebraic expression is zero.

Here, if we simplify the last expression then we are not getting zero, it is because of the complexity of the expression and the limitation of Mathematica but it is interesting to see that the last expression is zero. This is how machines and humans differ!!!

At last we will verify the solution for Average of Plain Vanilla Payoff and Modified Log Payoff for that we will define as before AvgofPlainVanillaandMLogPayoff as a function of $\sigma, X, S, t$ as follows:

$$
\ln[9] := \text{AvgofPlainVanillaandMLogpayoff}[\sigma\_, X\_, S\_, t\_] := \\
\frac{S}{2} M \left[ \frac{\text{Log}\left[\frac{S}{X}\right] + \left(r + \frac{1}{2} \sigma^2\right) (T-t)}{\sigma \sqrt{T-t}} \right] \left( 1 + \text{Log}\left[\frac{S}{X}\right] + \left(r + \frac{1}{2} \sigma^2\right) (T-t) \right) \\
+ \frac{S}{2} \sigma \sqrt{T-t} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{\left(\text{Log}\left[\frac{S}{X}\right] + \left(r + \frac{1}{2} \sigma^2\right) (T-t)\right)^2}{\sigma^2 (T-t)}\right) \\
- \frac{X}{2} M \left[ \frac{\text{Log}\left[\frac{S}{X}\right] + \left(r - \frac{1}{2} \sigma^2\right) (T-t)}{\sigma \sqrt{T-t}} \right] \exp\left(-r (T-t)\right)
$$

$$
\ln[10] := \text{BSM}[\text{AvgofPlainVanillaandMLogpayoff}[\sigma, X, S, t]]
$$
\[
\text{Out}[10] = -e^{-\frac{\left(-t+t\right)^2 + \log\left[\frac{S}{X}\right]}{2\left(-t+t\right)^2 \sigma^2}} S\sigma \\
- \frac{1}{4} e^{-r\left(-t+T\right)} r X \left(1 + \text{Erf}\left[\frac{\left(-t+T\right)\left(r - \frac{\sigma^2}{2}\right) + \log\left[\frac{S}{X}\right]}{\sqrt{2}\sqrt{-t+T} \sigma}\right]\right) \\
+ \frac{1}{4} S \left(-r - \frac{\sigma^2}{2}\right) \left(1 + \text{Erf}\left[\frac{\left(-t+T\right)\left(r + \frac{\sigma^2}{2}\right) + \log\left[\frac{S}{X}\right]}{\sqrt{2}\sqrt{-t+T} \sigma}\right]\right) \\
- e^{-r\left(-t+T\right)} - \frac{\left(-t+T\right)^2 \left(r - \frac{\sigma^2}{2}\right) + \log\left[\frac{S}{X}\right]}{2\left(-t+t\right)^2 \sigma^2} S X \left(1 + \left(-t+T\right) \left(r + \frac{\sigma^2}{2}\right) \log\left[\frac{S}{X}\right]\right) \\
\frac{2\sqrt{\pi}}{\left(-t+T\right)^2 + \log\left[\frac{S}{X}\right]} \left(1 + (-t+T) \left(r + \frac{\sigma^2}{2}\right) \log\left[\frac{S}{X}\right]\right) \\
- e^{-r\left(-t+T\right)} - \frac{\left(-t+T\right)^2 \left(r - \frac{\sigma^2}{2}\right) + \log\left[\frac{S}{X}\right]}{2\left(-t+t\right)^2 \sigma^2} S\sqrt{-t+T} \sigma \\
\left(-\frac{\left(r - \frac{\sigma^2}{2}\right)}{\left(-t+T\right)^2} + \log\left[\frac{S}{X}\right]\right) - \frac{\left(-t+T\right)^2 \left(r + \frac{\sigma^2}{2}\right) + \log\left[\frac{S}{X}\right]}{2\left(-t+T\right)^2 \sigma^2} \left(-\frac{\left(r - \frac{\sigma^2}{2}\right)}{\left(-t+T\right)^2} + \log\left[\frac{S}{X}\right]\right) \\
\frac{2\sqrt{2\pi}}{\left(-t+T\right)^2 + \log\left[\frac{S}{X}\right]} X \\
- r S e^{-r\left(-t+T\right)} - \frac{\left(-t+T\right)^2 \left(r - \frac{\sigma^2}{2}\right) + \log\left[\frac{S}{X}\right]}{2\left(-t+t\right)^2 \sigma^2} \left(-\frac{\left(r - \frac{\sigma^2}{2}\right)}{\left(-t+T\right)^2} + \log\left[\frac{S}{X}\right]\right) \\
\frac{2\sqrt{2\pi}}{\left(-t+T\right)^2 + \log\left[\frac{S}{X}\right]} X
\]
Chapter 5. Use of Mathematica

\[ \frac{e^{-\frac{(t - \tau)^2}{2(\tau + T)^2}}}{\sqrt{-t + T \sigma}} \]
\[ + \frac{r S}{2\sqrt{2\pi}} e^{\frac{(-t + T)(r + \frac{\sigma^2}{2}) + \log \left( \frac{S}{X} \right)}{\sqrt{2\sqrt{-t + T \sigma}}} \]
\[ + \frac{1}{4} r S \left( 1 + \text{Erf} \left[ \frac{(-t + T)(r + \frac{\sigma^2}{2}) + \log \left( \frac{S}{X} \right)}{\sqrt{2\sqrt{-t + T \sigma}}} \right) \right) \]
\[ - \frac{r S}{2\sqrt{2\pi}} e^{\frac{(t - \tau)^2}{2(\tau + T)^2}} \frac{\left( -t + T \right) \left( r + \frac{\sigma^2}{2} \right) + \log \left( \frac{S}{X} \right)}{2\sqrt{2\pi}\sqrt{-t + T \sigma}} \]
\[ + \frac{1}{4} r S \left( 1 + \text{Erf} \left[ \frac{(t - \tau)^2}{2(\tau + T)^2} + \log \left( \frac{S}{X} \right) \right) \right) \]
\[ \frac{1 + (-t + T) \left( r + \frac{\sigma^2}{2} \right) + \log \left( \frac{S}{X} \right)}{2\sqrt{2\pi}\sqrt{-t + T \sigma}} \]
\[ - \frac{\text{Erf} \left[ \frac{(t - \tau)^2}{2(\tau + T)^2} + \log \left( \frac{S}{X} \right) \right) \right] \]
\[ \frac{1 + (-t + T) \left( r + \frac{\sigma^2}{2} \right) + \log \left( \frac{S}{X} \right)}{2\sqrt{2\pi}\sqrt{-t + T \sigma}} \]
\[ + \frac{1}{4} r e^{-r(t - \tau)X} \frac{1 + \text{Erf} \left[ \frac{(t - \tau)^2}{2(\tau + T)^2} + \log \left( \frac{S}{X} \right) \right) \right] \]
\[ - \frac{r S}{2\sqrt{2\pi}} e^{\frac{(t - \tau)^2}{2(\tau + T)^2}} \frac{\left( -t + T \right) \left( r + \frac{\sigma^2}{2} \right) + \log \left( \frac{S}{X} \right)}{2\sqrt{2\pi}\sqrt{-t + T \sigma}} \]
\[ + \frac{1}{4} r e^{-r(t - \tau)X} \frac{1 + \text{Erf} \left[ \frac{(t - \tau)^2}{2(\tau + T)^2} + \log \left( \frac{S}{X} \right) \right) \right] \]
\[ \frac{1 + (-t + T) \left( r + \frac{\sigma^2}{2} \right) + \log \left( \frac{S}{X} \right)}{2\sqrt{2\pi}\sqrt{-t + T \sigma}} \]
\[ + \frac{1}{2} S^2 \sigma^2 e^{-\frac{(t - \tau)^2}{2(\tau + T)^2}} \frac{\left( -t + T \right) \left( r + \frac{\sigma^2}{2} \right) + \log \left( \frac{S}{X} \right)}{2\sqrt{2\pi}\sqrt{-t + T \sigma}} \]
\[ + \frac{1}{2} S^2 \sigma^2 e^{-r(t - \tau)X} \frac{1 + \text{Erf} \left[ \frac{(t - \tau)^2}{2(\tau + T)^2} + \log \left( \frac{S}{X} \right) \right) \right] \]
\[ \frac{1 + (-t + T) \left( r + \frac{\sigma^2}{2} \right) + \log \left( \frac{S}{X} \right)}{2\sqrt{2\pi}\sqrt{-t + T \sigma}} \]
Mathematica does not simplify it further.
So we conclude that the expression is too complicated to be handled by
Mathematica version that we are using. Maybe advancement in handling algebraic
expressions in future will be able to handle such expressions more efficiently and
at that time we will get the expected answer to be zero!!!
5.3 Graphical representation of Payoff functions:

In this section, we will see graphical representation based on the option pricing formulas for all the Payoff functions, which we have discussed in Chapter-4. using Mathematica. One can also compare these formulas using graphs. Here we have also given the Mathematica commands which we have used to get such graphs.

```
In[3]= Plot3D[{Logpayoff4graph[\[Sigma], X], Vanilla4graph[\[Sigma], X]},
{\[Sigma], 0, .5}, {X, 50, 150}, AxesLabel -> {"\[Sigma]", "X", "C"}, Boxed -> False,
Mesh -> False, PlotLegends -> {"Log Payoff", "Plain Vanilla Payoff"}]
```

![Figure 5.1: Graph-1](image1)

```
Plot3D[{Logpayoff4graph[\[Sigma], X], Vanilla4graph[\[Sigma], X],
AvofVanillaandLogpayoff4graph[\[Sigma], X]}, {\[Sigma], 0, .5}, {X, 50, 150},
AxesLabel -> {"\[Sigma]", "X", "C"}, Boxed -> False, Mesh -> False, PlotLegends ->
{"Log Payoff", "Plain Vanilla Payoff", "Avg. of Log and Plain Vanilla"}]
```

![Figure 5.2: Graph-2](image2)
Graph-1 is for Plain Vanilla call option and Log call option and Graph-2 is for Plain Vanilla call option, Log call option and Average of Plain Vanilla and Log call option.

In Graph-1, blue color indicates the price of call option for Plain Vanilla and orange color indicated the price of call option for Log Payoff. From the graphs, one can see that the price of call option for Log Payoff is near to zero, which is not possible in real problem. In Graph-2, blue color indicates the price of call option for Plain Vanilla and orange color indicated the price of call option for Log Payoff and green color indicates the price of call option for Average of Plain vanilla and Log Payoff. Here, for the Average we have some nonzero value of the option, it is not exactly equal to the Plain Vanilla call option but it make some sense and one can use this as a different choice of Payoff function.
Chapter 5. Use of Mathematica

Plot3D[{MLogPayoff4graph[σ, X], Vanilla4graph[σ, X]},
{σ, 0, .5}, {X, 50, 150}, AxesLabel -> {"σ", "X", "C"}, Boxed -> False,
PlotLegends -> {"MLog Payoff", "Plain Vanilla Payoff"}, Mesh -> None]

General::unfl : Underflow occurred in computation. >>
General::unfl : Underflow occurred in computation. >>
General::unfl : Underflow occurred in computation. >>
General::stop : Further output of General::unfl will be suppressed during this calculation. >>

Figure 5.3: Graph-3
Graph-3 is for Plain Vanilla call option and Modified Log call option and Graph-4 is for Plain Vanilla call option, Log call option and Average of Plain Vanilla and Modified Log call option.

In Graph-3, blue color indicates the price of call option for Plain Vanilla and orange color indicated the price of call option for Modified Log Payoff. This shows that the Modified Log call option is very close to the Plain Vanilla call option. In Graph-4, blue color indicates the price of call option for Plain Vanilla and orange color indicates the price of call option for Modified Log Payoff and green color indicates the price of call option for Average of Plain vanilla and Modified Log...
Payoff. Here, the Average is one more suitable choice for Payoff function, which is almost similar to Plain Vanilla call option.

```mathematica
Plot3D[{MLogpayoff4graph[σ, X], Logpayoff4graph[σ, X]},
   {σ, 0, .5}, {X, 50, 150}, AxesLabel -> {"σ", "X", "C"}, Boxed -> False,
   PlotLegends -> {"MLog Payoff", "Log Payoff"}, Mesh -> None]

General::unfl: Underflow occurred in computation. ➤
General::unfl: Underflow occurred in computation. ➤
General::unfl: Underflow occurred in computation. ➤
General::stop: Further output of General::unfl will be suppressed during this calculation. ➤
```

![Graph-5](image-url)
Chapter 5. Use of Mathematica

Figure 5.6: Graph-6

Graph-5 is for Modified Log call option and Log call and Graph-6 is for Modified Log call option, Log call option and Average of Log and Modified Log call option.

In Graph-5, blue color indicates the price of call option for Log Payoff and orange color indicates the price of call option for Modified Log Payoff. And in Graph-6, blue color indicates the price of call option for Log and orange color indicates the price of call option for Modified Log Payoff and green color indicates the price of call option for Average of Log and Modified Log Payoff. As we have mentioned above that Log Payoff is not practically proper choice as the value of the option is even less than the transaction cost but if we see the Average of Log and Modified Log which is more closer to the Plain Vanilla call option than the Modified Log
call option. So, Avg. of Modified Log and Log is practically better appropriate choice for the Payoff function.
Chapter 6

Conclusions and Further Scope
6.1 Introduction:

In this chapter, we have discussed the conclusions of all those results which are discussed in previous chapters and also we have discussed the scope of further research.

6.2 Conclusions and Further Scope:

In Quantitative Finance, there are many mathematical problems in several areas with direct impact to market application.

In this chapter, we draw some comments on the results, presented in this thesis. In Chapter-2 and 3 we focused our attention on different stock price models and different option pricing models. In Chapter-4 we presented pricing formulas for different payoff functions using Fourier Transform method. Fourier method in finance are used in a various situations. Many people used Fourier methods to find analytical formula for the price of derivatives.

Heston[9] used Fourier analysis to find an explicit solution of a parabolic type partial differential equation with mixed differentiation terms, which gives a model for pricing of a call option.

Carr-Madan[4] and Lewis[13] used the Fourier transform method to find the value of European call option in Fourier space and then used Fast Fourier Transformation to find the value in real space.

Fourier method have been used for variety of problems, from Physics to Mathematical Finance. From the Physics and Engineering Point of view this
method is one of the fastest method to find the heat potentials. Apart from the payoff functions which are used in Chapter-4, there are some known functions in Mathematics; namely trigonometric functions, exponential function etc... It is very difficult to find the explicit solution of BSM equation using these functions as payoff functions but can we get solution of BSM equation with small errors? In this thesis, we have derived the different formulas for European call option only. One can also do it for American call option, if possible. Here, we have derived the formulas for non-dividend paying asset and the interest rate \( r \) is constant throughout the life of the option. It might be possible to derive BSM for dividend paying asset and with \( r \) variable which varies with time.

Many authors have derived BSM formulas on multi-assets \([3]\) for some known payoff functions. In a similar way, we can also derive BSM formulas on multi-assets for averages of different payoff functions.

In Chapter-5, we have discussed basics of Mathematica. Mathematica is built to provide efficient algorithms across most of the areas.

Mathematica uses the Wolfram Notebook which allows us to do everything whatever we do to make rich documentation that include text, code, graphics etc... The language which is used in Mathematica is comparatively easy to read, write and learn. The main important thing in Mathematica for the people like us who deals with Mathematics is that, it is based on symbolic language. There are variety of functions included in the latest version of Mathematica. It is very important to implement new ideas that dramatically extent the vision and scope of the software. New versions of Mathematica are not just updates in software, each new version is a great achievement which extent the way of computations in new directions and gives important new ideas. To get new ideas in new direction is never ending process, still one can improve computational ability in upcoming versions of Mathematica so that we can handle large expressions more efficiently.
to get desired result.
List of Publications Arising From the Thesis


# List of Symbols

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\mathbb{R})</td>
<td>Set of all real numbers.</td>
</tr>
<tr>
<td>(p)</td>
<td>Probability function</td>
</tr>
<tr>
<td>(\Omega)</td>
<td>Universal set</td>
</tr>
<tr>
<td>(\mu)</td>
<td>Drift</td>
</tr>
<tr>
<td>(\sigma)</td>
<td>Volatility</td>
</tr>
<tr>
<td>(S)</td>
<td>Stock price</td>
</tr>
<tr>
<td>(S_T)</td>
<td>Stock price at time (T)</td>
</tr>
<tr>
<td>(S_0)</td>
<td>Stock price at time (t = 0)</td>
</tr>
<tr>
<td>(E[X])</td>
<td>Expected value of (X)</td>
</tr>
<tr>
<td>(W)</td>
<td>Brownian motion</td>
</tr>
<tr>
<td>(Var[X])</td>
<td>Variance of (X)</td>
</tr>
<tr>
<td>(X)</td>
<td>Strike price</td>
</tr>
<tr>
<td>(r)</td>
<td>Risk free interest rate</td>
</tr>
<tr>
<td>(C)</td>
<td>The value of call option</td>
</tr>
<tr>
<td>(\Delta)</td>
<td>Number of Shares</td>
</tr>
<tr>
<td>(\Pi)</td>
<td>Portfolio</td>
</tr>
<tr>
<td>(N(x))</td>
<td>Normal distribution i.e. (N(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-t^2/2} dt)</td>
</tr>
<tr>
<td>(B(x))</td>
<td>Binomial distribution</td>
</tr>
</tbody>
</table>
Details of the Work Presented in Conferences

1. The paper entitled as 'Black-Scholes-Merton partial differential equation and scientific computations' was presented in National conference on Emerging Trends in Engineering Technology and Management at Indus University, Ahmedabad, during Jan 31 and Feb 1, 2014.

2. The paper entitled as "Derivation of European call option for financial market using Fourier transform method" was presented in National conference on Advances in Technology and Applied Sciences at JIET, Jodhpur, during March 28-29, 2014.

3. The paper entitled as "Application of Binomial Tree in Finance" was presented in Science Excellence-2015 at Gujarat university Ahmedabad, on September 26, 2015.
Bibliography


Index

A
American Option .................. 9
Arbitrageurs ..................... 12

B
Binomial Tree Model(BTM) .... 22
Black Scholes Merton Model ... 31
BOPM ................................ 32
Brownian Motion ................. 19
BSE ................................. 6
BSM ................................. 39

C
Call option ...................... 9
CBOT ............................... 8
CME ................................. 8

E
European Option ................. 9

F
Fourier transform .............. 43
Future Contract ................ 8

G
Geometric Brownian Motion .... 20

H
Hedgers ........................... 10

L
Log payoff ........................ 47
Lognormal Model .............. 14

M
Mathematica ...................... 62
Modified Log payoff ........... 49

N
NSE ................................. 6

O
OTC ................................. 6

P
Payoff ............................. 39
Plain Vanilla Payoff .......... 45
Probability measure .......... 4
Put option ........................ 9

R
Random Variable ............... 4

S
Sample Space ..................... 3
Speculators ..................... 10