CHAPTER V

5.1 In Chapter II, III and IV, we have considered in general, every such sequence \( \{ \lambda_n \} \) and found out different results in each case. Now let us consider in particular, \( \{ \lambda_n \} \) to be a convex sequence and we thus have the following theorem for

Theorem

If

\[
\int_0^\infty \frac{|g(u)|}{u} \, du = O(1),
\]

then

\[
\sum (\log n)^{-\frac{1}{2}} A_n(z) \text{ is summable for } t \in [0, 1] \text{ at } t=\infty.
\]

This result may be compared with the theorem of Cheng (1), which we state below.

Theorem A

If

\[
\int_0^t |g(u)| \, du = o(t),
\]

as \( t \to \infty \), then

\[
\sum (\log n)^{-\frac{1}{2}} A_n(z) \text{ is summable for } t \in [0, 1], (x \in \mathbb{R}).
\]

The case \( \{\varepsilon_1 \} \) of Theorem A is still open. Hence, to consider this case, i.e. \( \{\varepsilon_1 \} \) summability factors instead of \( \{\varepsilon_1 \} \) summability factors, we have slightly changed the integral (5.1.2) to the form (5.1.1) and we
have been able to establish the above theorem. We have also avoided the \( o' \) condition in Theorem A, to a \( O' \) condition in our theorem.

Before going to the proof of the theorem, we now have the following lemmas.

**Lemma 1** Cheng (2)

Let

\[
H(n, t) = \frac{1}{n+1} \sum_{j=2}^{n} \frac{(n+2)^{1+c}}{\log ((n+2)^{1+c})}.
\]

Then following is the order of \( H(n, t) \):

\[
H(n, t) = \begin{cases} 
0 \left( \frac{n}{(\log n)^{1+\epsilon}} \right), & n < 1, \\
0 \left( \frac{1}{t (\log n)^{1+\epsilon}} \right), & n \geq 1. 
\end{cases}
\]

**Lemma 2**

\[
\int_{t}^{\pi} \frac{Q(u)}{u} du = O(1),
\]

Therefore,

\[
\int_{0}^{t} |Q(u)| du = O(t).
\]

**Proof:**

Let

\[
\Phi(t) = \int_{t}^{\pi} |Q(u)| du.
\]

Then we have almost everywhere,

\[
\Phi(t).
\]

Therefore,
Proof of the Theorems

Let \( S_n(x) \) denote the \( n \)th Cesaro mean of order one of the sequence \( \left\{ \frac{n A_n(x)}{\log n + \varepsilon} \right\} \).

As in Theorem 1 (2.1), we have to prove the convergence of

\[
\sum n^{-1} |S_n(x)|.
\]

we have,

\[
|S_n(x)| = \left| \frac{2}{\pi} \int_0^T G(t) \left| \frac{1}{n+1} \sum_{\nu} \frac{e^{i(\nu+1)T}(\log(\nu+2))^{1+\varepsilon}}{[\log(\nu+2)]^{1+\varepsilon}} \right| dt \right|
\]
By lemma 1, and lemma 2,

\[
J_1 = \int_0^\infty \left| g(t) \right| \frac{t^{\frac{1}{n+1}}}{\sum_{j=0}^{n} \log (v+2) \left| \frac{t^{j+\epsilon}}{t} \right|} \, dt
\]

\[
= 0 \left( \frac{\eta}{(\log n)^{1+\epsilon}} \right) \int_0^\infty \frac{\left| g(t) \right|}{t} \, dt
\]

\[
= 0 \left( \frac{\eta}{(\log n)^{1+\epsilon}} \right) \cdot 0 \left( \frac{1}{n} \right)
\]

\[
= 0 \left( \frac{1}{(\log n)^{1+\epsilon}} \right)
\]

and

\[
J_2 = \int_{1/n}^{\infty} \left| g(t) \right| \frac{t^{\frac{1}{n+1}}}{\sum_{j=0}^{n} \log (v+2) \left| \frac{t^{j+\epsilon}}{t} \right|} \, dt
\]

\[
= 0 \left( \frac{1}{(\log n)^{1+\epsilon}} \right) \int_{1/n}^{\infty} \frac{\left| g(t) \right|}{t} \, dt
\]

\[
= 0 \left( \frac{1}{(\log n)^{1+\epsilon}} \right) \cdot 0 \left( 1 \right)
\]

\[
= 0 \left( \frac{1}{(\log n)^{1+\epsilon}} \right)
\]

Hence,

\[
\sum_{n=1}^{\infty} \left| \frac{1}{n^{1+\epsilon}} \right| \left| J_n \right| \
\leq \sum_{n=1}^{\infty} \frac{1}{n^{1+\epsilon}} \left| J_1 \right| + \sum_{n=1}^{\infty} \frac{1}{n^{1+\epsilon}} \left| J_2 \right|
\]

\[
= \sum_{n=1}^{\infty} \frac{1}{n^{1+\epsilon}} \left| J_n \right|
\]

This completes the proof of the theorem.