1.1 Ever since the publication of Cauchy's Analyse algébrique (1821) which laid to the rigorous foundation of the theory of infinite series, the attention of some of the foremost men of mathematics has been focused upon the problems concerning divergent series.

Given an infinite series of real or complex terms \( \sum_{n=0}^{\infty} a_n \), let \( \{s_n\} \) denote the sequence of its 'partial sums', i.e., let

\[
S_n = \sum_{k=0}^{n} a_k \quad (n=0, 1, 2, \ldots)
\]

Suppose that there is a number \( s \) such that \( \{s_n\} \) converges to \( s \), i.e., given any \( \epsilon > 0 \), \( |s_n - s| < \epsilon \) for all sufficiently large \( n \). Then Cauchy defines the 'sum' of \( \sum_{n=0}^{\infty} a_n \) as the number \( s \). Since there cannot exist two such numbers \( s \), this definition of the 'sum' of an infinite series is unique. Any infinite series which has a sum is Cauchy sense is said to be convergent. A series which is not convergent is called divergent.

Towards the end of the last century various methods of associating 'sums' with series which are not convergent in the classical sense of Cauchy, nor properly divergent, in other words oscillatory, were developed as generalizations of the classical concept of convergence.*

* For the concept of convergence of an infinite series see Hobson(1).
These methods called summability methods have been found to be increasingly useful in the study of divergent series, about which, prior to this, hardly any thing fruitful could be thought of, so much so that, even Abél wrote, "Divergent series are the invention of the devil and it is shameful to base on them any demonstrations what-soever".

Just as the concept of convergence had led to the development of its extensions under general title of summability methods, so also, by analogy, the concept of absolute convergence led to the process of absolute summability. Thus, series which may not be absolutely convergent in the classical sense, may still be found to be absolutely summable in some suitable sense.

We shall give a precise formulation of several specific forms of this now-rather-well-known concept of absolute summability.

A particularly interesting aspect of the recent researches on divergent series is the one that concerns the general problem of determining suitably sequences of factors \( \{c_n\} \) such that the series \( \sum_{n=1}^{\infty} c_n a_n \) is summable or absolutely summable in a prescribed sense, while in general, \( \sum_{n=1}^{\infty} a_n \) itself is not so summable.

+ for an account of summability methods, reference may be made, e.g., to Hobson(1) and Szàz (1).

** The earliest work in absolute summability, appears to be contained in Fekete (1). The theory of absolute Cesàro summability was extensively developed by Kogbetliantz (1). Other sources of the various kinds of absolute summability are mentioned at appropriate places in the sequel.
Such factors are termed as summability factors. If the summability in question is absolute, then the factors are naturally called absolute summability factors.

We begin by giving a resume of the results obtained neither to, on summability and absolute summability, in the context of which the problems dealt with in the subsequent chapters suggest themselves.

1.2 The process of summability of divergent series that are in common use are either T-processes based upon the formation of the \( \{t_n\} \) auxiliary means defined by the sequence to sequence transformation

\[
t_n = \sum_{m=0}^{\infty} c_{m,n} s_n \quad (1.2.1)
\]

\[\{s_n\}\] being the sequence of partial sums of the infinite series \( \sum a_n \) in question, or \( \Phi \)-processes based upon the formation of a functional transformation \( t(x) \), defined either by the sequence to function transformation, defined either by the sequence to function transformation

\[
t(x) = \sum \phi_n(x) s_n \quad (1.2.2)
\]

or more generally, the transformation

\[
t(x) = \int \phi(x, y) s(y) dy \quad (1.2.3)
\]

where \( x \) is a continuous parameter, and the function

\[
\phi_n(x), \sum \phi(x, y)
\]

is defined over an appropriate interval of \( x \).

* Hardy (1) p.43, Theorem 2. These conditions are necessary and sufficient for the truth of the assertion that the sequence \( \{t_n\} \) has a limit whenever the sequence \( \{s_n\} \) has a limit and that, the two limits are same.

By \( \sum \) we mean \( \sum_{n} \), Also we sometimes write this as \( \sum_{n} \) only.
The series \( \sum a_n \) or the sequence \( \{s_n\} \) is said to be summable by a \( T \)-process to the finite sum \( s \), if the corresponding oscillatory sequence as defined in (1.2.1) tends to a limit \( S \) as \( n \to \infty \).

If any two processes, namely \( P \) and \( Q \) are inclusive of each other, ordinarily, or absolutely, they are said to be equivalent or absolutely equivalent, symbolically,

\[
P \equiv Q \quad \text{or} \quad |P| \equiv |Q|.
\]

The sequence to sequence transformation (1.2.1) is said to be conservative or absolutely conservative if the convergence (or absolute convergence) of the sequence \( \{s_n\} \) implies that of the sequence \( \{t_n\} \) in each case, and is said to be regular (or absolutely regular) if the convergence (or absolute convergence) of the sequence \( \{s_n\} \) and \( s_n \to s \quad \text{as} \quad n \to \infty \), imply the convergence (or absolute convergence) of the sequence \( \{t_n\} \) and \( t_n \to s \quad \text{as} \quad n \to \infty \).

It should be noted that an absolutely conservative transformation is not necessarily conservative (see Morley(1)).

1.3 Absolute Cesàro summability

The earlier definition of any special method of absolute summability is that of Cesàro summability, introduced by Fekete (1) and (2) in 1911.

Fekete has defined \([C, \lambda]\) - summability for integral values of \( \lambda \). It was Kogbetliantz (1) and (2), who gave the general definition of absolute Cesàro summability, while
laying the foundation of its fundamental theories like the 'consistency theorem', theorems on the multiplication of absolutely summable series, and the absolute Cesaro summability factor theorems.

We write,

\[ S_n^d = \frac{1}{A_n^{d-1}} \sum_{n=0}^{\infty} A_n^{d-1} \alpha_n , \quad (d>1) \]

where \( \{ A_n^d \} \) is defined by the power series expression.

\[ (1-x)^{-d-1} = \sum_{n=0}^{\infty} A_n^d x^n , \quad \left( \left| x \right| < 1 \right) \]

The series \( \sum a_n \) or the sequence \( \{ S_n \} \) is said to be \((C,\alpha)\) summable to the value \( s \) if

\[ S_n^d \to s \quad \text{as} \quad n \to \infty. \]

The series \( \sum a_n \) is said to be absolutely summable \((C,\alpha)\) or summable \((C,\alpha)\), \((d>1)\), if the sequence \( \{ S_n^d \} \) is of bounded variation, that is to say,

\[ \sum_n \left| S_n^d - S_{n-1}^d \right| < \infty. \]

1.4 Abel Summability (see Zygmund (1)).

A series \( \sum a_n \) is said to be summable by Abel's method, or summable \( A \) to the sum \( s \) (finite), if the series

\[ f(x) = a_0 + a_1 x + a_2 x^2 + \ldots \]

is convergent for \( |x| < 1 \) and

\[ \left. \frac{d}{dx} f(x) \right|_{x=1} = \sum_{n=0}^{\infty} a_n. \]

If \( f(x) \) is of bounded variation in \( \mathbb{R}^+ \), then the series is said to be summable \( A \) to the sum \( s \).
Whittaker (1) proved in the case of infinite series, that, absolute convergence implies absolute Abel summability.

Fekete (3) established the inclusion relation

\[ |C, A| \subset |A| \]

for integral values of \( \alpha > 1 \).

A combination of Fekete's result with consistency theorems for absolute Cesàro summability leads to the conclusion that

\[ |C, A| \subset |A| \]

for every positive \( \alpha \) however large.

1.5 Summability \((R, \lambda, 1)\) (see Hardy (1))

Let \( 0 < \lambda_1 < \lambda_2 < \cdots < \lambda_n \to \infty \)

as \( n \to \infty \),

\[ \lambda_n = \lambda_n - \lambda_{n-1}, \quad (n > 1) \]

and

\[ \lambda_0 = \lambda_0 = 0 \]

Let the series \( \sum a_n \) be the series with the sequence \( \sum_{n=1}^{\infty} S_n \) for its \( n^{th} \) partial sum.

The \((R, \lambda, 1)\) mean of the sequence \( \sum_{n=1}^{\infty} S_n \) is given by

\[ t_n = \frac{1}{\lambda_{n+1}} \sum_{k=0}^{n} (\lambda_{n+1} - \lambda_k) \]

The series \( \sum a_n \) is said to be summable by Riesz mean of the type \( \lambda \) and order 1, or summable, if

The series is absolutely summable by Riesz means of type \( \lambda \) and order 1, or summable, if
where $K$ is an absolute positive constant, not necessarily the same at each occurrence.

When,

$$M_n = \frac{1}{n+1}$$

the $(R,\lambda,1)$ mean reduces to the familiar Riesz logarithmic mean of order 1, written as $(R,\log n,1)$.

**1.6 $(R,K)$ Summability**

Let $W$ be a continuous parameter and let

$$R^k(\omega) = \sum_{n=0}^{\infty} \left(1 - \frac{n}{\omega}\right)^k \lambda_n, \quad (k \geq 0).$$

If

$$R^k(\omega) \to s, \quad \text{when } \omega \to \infty,$$

then we say that $\sum \lambda_n$ is summable $(R,\lambda,k)$ (or briefly $(R,K)$) to the sum $s$. This is a case when $\lambda_n = 1$ in the $(R,\lambda,1)$ summability.

We find that (see Hardy (1) p.113) summability $(R,\lambda,k)$ is equivalent to summability $(L,k)$.

**1.7 Summability (L)**

The infinite sequence $\sum \lambda_n$ is said to be summable (L) or summable in the logarithmic means to the sum $s$ (finite), if the expression

$$\lambda_n \to s$$

tends to a finite limit $s$ as $n \to \infty$ in the open interval

and we write shortly as

For this definition see Hardy (1) p.81 and Borwein (1).
1.8 Strong Summability (Hardy (1))

A series \( \sum a_n \) with partial sum \( S_n \) is said to be strongly summable with index \( k, (\kappa > 0) \), if
\[
\lim_{n \to \infty} \frac{1}{n+1} \sum_{j=0}^{n} |S_{j+1} - S_j|^k = 0.
\]

1.9 \((N,p,q)\) Summability

This was defined by Borwein (1), and is called as the generalized Norlund transform. And the properties of this summability was investigated by Das (1) and (2).

The \((N,p,q)\) transform of \( S_n = \sum_{j=0}^{n} a_j \) is defined by
\[
\phi_n = \sum_{j=0}^{n} a_j p_{n-j} q^{n-j}
\]
where,
\[
V_n = \sum_{j=0}^{n} p_{n-j} q^{n-j}.
\]
\[
(\phi_n = V_n = \sum a_j \\
\neq 0, \text{ for } n \geq 0).
\]

The series \( \sum a_n \) or the sequence \( S_n \) is said to be summable \((N,p,q)\) to the sum \( s \) (finite), if
\[
\phi_n \to s \text{ as } n \to \infty;
\]
and is said to be absolutely summable \((N,p,q)\) or summable \(N,p,q\), if \( \phi_n \) is of bounded variation, that is,
\[
\sum_{j=0}^{n} |\phi_{j+1} - \phi_j| < \infty
\]
and when this happens, we write
\[
\sum a_n = \phi.
\]

The method \((N,p,q)\) reduces to the method \((N,p)\) when (see Hardy (1) p.57); to the Euler-Knopp method when,
(see Hardy (1) p.178); to the method, \((\alpha, \lambda; \beta)\) (see Borwein(1))
when
\[
\rho_n = \left(\frac{n+\alpha-1}{\lambda}\right), \quad \nu_n = \left(\frac{n+\beta}{\lambda}\right).
\]

We note that,
\[
\lambda_n = \rho_n - \rho_{n-1},
\]
and,
\[
Q_n = \nu_0 + \nu_1 + \cdots + \nu_n,
\]
and,
\[
\frac{1}{\lambda_{n-1}} \sum_{n=0}^{\infty} \rho_{n-1} \nu_n s_n = \frac{1}{\lambda_{n-1}} \sum_{n=0}^{\infty} (\rho_{n-1} - \rho_{n-2}) \sum_{m=0}^{n} \nu_0 s_m
\]
\[
= \frac{1}{\lambda_{n-1}} \sum_{n=0}^{\infty} \lambda_{n-1} t_n Q_0,
\]
where,
\[
t_n = \frac{1}{Q_0} \sum_{m=0}^{\infty} \nu_0 s_m = \frac{1}{Q_0} \sum_{m=0}^{\infty} (Q_0 - Q_{m-1}) a_m.
\]

Here \(t_n\) is the \(\left(\alpha, \beta, \lambda_0, \nu_0\right)\) mean (see Hardy (1)),
which is equivalent to \(\left(\alpha, \beta, \lambda_0, \nu_0\right)\) mean of \(\lambda\) (see Hardy (1) p.113).

**1.10 \((J, q, k)\) Summability**

This was studied for the first time by Das (1).

The \(\lambda\) mean of \(\nu\) is given by,
\[
\lambda \nu = \sum_{n=0}^{\infty} \rho_n \nu_n s_n
\]

where,
\[
\lambda \nu
\]
and

are convergent for
If
\[ J_k(x) \to s, \quad \text{as } x \to \infty \quad \text{(through positive values)}, \]
then \( \{S_n\} \) is said to be summable \((J,q,k)\) to the sum \(s\).

Absolute summability \(|J,q,k|\) of \(\{S_n\}\) is defined, when
\[ \int_0^\infty |J_k(x)| \, dx < \infty. \]

When \( |z| \leq 1 \) and \( k = 1 \), it is known that (see Das (2))
\((J,q,k)\) is the same as \((J,q)\) (see Hardy (1) p.79). If \( k = 0 \)
and \( Q_n - Q_{n+1} \neq 0 \), then \( J_k(x) \) mean is the Abel mean and
in the case \( Q_n - Q_{n+1} = 0 \), \((Q_n - Q_{n+1})^k\) may be defined as in 1.9
so that the \( J_k(x) \) mean is defined.

1.11 Fourier Series

Poisson defines the sum of the trigonometrical series
\[ \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos n\theta + b_n \sin n\theta) \quad (1.11.1) \]
as the limit when \( \rho \to 1 \) of the associated power series
\[ \frac{a_0}{2} + \sum_{n=1}^{\infty} \left( a_n \cos n\theta + b_n \sin n\theta \right) \rho^n. \]

By 'Fourier's theorem' we mean here, the theorem that,
if \( f(x) \) belongs to an appropriate class of functions and
is representable by a trigonometrical series \((1.11.1)\), in the
sense that the series \((1.11.1)\) converges to \( f(x) \) in the
open interval \((-\pi, \pi)\), then
\[ a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx \, dx. \]
Thus the theorem asserts that if a trigonometrical series converges to \( f(x) \) for \(-\pi < x < \pi\), then it necessarily is the Fourier Series of \( f(x) \).

The formulae (1.11.2) and (1.11.3) are older than Fourier. Burkhardt, in his article in the Enzyklopadie, traces the formula for \( a_n \) back to Clairaut (1757). They were familiar to Euler who gave the ordinary deduction of them by term by term integration in 1777.

1.12 Let the function \( f(t) \) be integrable in the sense of Lebesgue in the interval \((-\pi, \pi)\) and periodic with a period \( 2\pi \). Let the Fourier Series of \( f(t) \) be represented by the trigonometric series (1.11.1).

We may without any loss of generality assume that the constant term in the expression (1.11.1) vanishes and then the Fourier Series of \( f(t) \) may be represented in the form,

\[
\sum_{n=1}^{\infty} [a_n \cos nt + b_n \sin nt] = \sum_{n=1}^{\infty} a_n \cos nt + \sum_{n=1}^{\infty} b_n \sin nt \quad (1.12.1)
\]

The series

\[
\sum_{n=1}^{\infty} b_n \cos nt = \sum_{n=1}^{\infty} a_n \sin nt = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \sin nt \, dt \quad (1.12.2)
\]

is known as the conjugate series of the Fourier Series.

The \( r \)-th derivative (for \( r = 1, 2, \ldots \)) of the Fourier Series at \( t = \pi \), may be written as

\[
\sum_{n=1}^{\infty} \frac{(-1)^n}{n^r} a_n \cos nt + \sum_{n=1}^{\infty} \frac{(-1)^n}{n^r} b_n \sin nt \to \frac{d^r}{dt^r} f(t) \quad (1.12.3)
\]

We write,
-12-

\[ g_\gamma(t) = \frac{\gamma}{t^\gamma} \int_0^t (t-u)^{-\gamma} g(u) \, du, (x > 0) \]
\[ g_0(t) = g(t) \]
\[ u(t) = g_\gamma(t) - p(t) \]

Where, \( p(t) \) is a polynomial given by
\[ p(t) = \sum_{p > 1} C_p \frac{t^p}{p!} \]

The \( \gamma \)th generalized derivative of the Fourier Series at \( t = x \) may be defined as
\[ u^{(\gamma)}(t) = \lim_{\delta \to 0} \frac{u(t+\delta) - u(t)}{\delta} \]  
(see Zygmund (1))

(The function \( g_\gamma(t) \) is the Riemann–Liouville integral of order \( \gamma \) of \( g(t) \)).

1.13 Summability of Fourier Series

Here we mention different results on summability of Fourier Series and factored Fourier series which naturally suggests our work included in this thesis.

Theorem A (Whittaker (1))

The series
\[ \sum_{\gamma} \int_0^t (t-u)^{-\gamma} g(u) \, du \]
is summable \( A \) almost every where.

Extending this result, Prasad (1) proved the following theorem:

Theorem B

The series
is summable \(| A |\) almost everywhere.

Since summability \(| C,1 |\) \Rightarrow\ summability \(| A |\)
(Fekete (3)), theorem A was improved by Hsiang (1),
in the following form:

**Theorem C**

If

\[ \int_0^t |\varphi(u)| \, du = o(t), \quad t \to +\infty, \]

then the series

\[ \sum_{n=1}^{\infty} A_n(x_0) / n^d \]

is summable \(| C,1 |\) (for \( d > 0 \)), where,

\[ \varphi(t) = \frac{1}{2} \left[ f(x+t) + f(x-t) - 2f(x) \right] \]

**Theorem D**

If \( \{ \lambda_n \} \) be any one of the sequences

\[ \left\{ \frac{1}{n \log n} \right\}^+ \]

\[ \left\{ \frac{1}{n \log n \log \log n} \right\}^+ \]

\[ \left\{ \frac{1}{n \log n \cdots \log \log \cdots \log n} \right\}^+ \]

(\( d > 0 \)), and

\[ \int_0^t |\varphi(u)| \, du = o(t) \]

as \( t \to +\infty \), then the series \( \sum_{n=1}^{\infty} A_n(x_0) \) is summable:

\( | A | \) for almost all values of \( t \).

An extension of Theorem D was obtained by Izumi and
Kawata (1), in the form of the following theorem.
Theorem E

If \( \{\lambda_n\} \) is a convex sequence such that the series \( \sum \frac{\lambda_n}{n} \) is convergent, then the series \( \sum \lambda_n A_n(t) \) is summable \( |A| \) for almost all values of \( t \).

We observe that while Prasad (Theorem D) starts with the condition

\[
\int_0^t |g(u)| \, du = o(t) \quad (1.13.1)
\]
as \( t \to 0 \), which holds in the Lebesgue set, and therefore holds almost everywhere, (see Zygmund(1)p.703 and Titchmarsh (1) p. 11.6), Chow (1) bases the proof of the following Theorem F, upon J. Marcinkiewicz's (1) theorem concerning strong summability of Fourier series, assume that

\[
\sum_{n=1}^{\infty} \left( S_n(t) - f(t) \right)^2 = o(n) \quad (1.13.2)
\]
as \( n \to \infty \), for almost all values of \( t \).

It is known (Zygmund (1), Hardy & Littlewood (1)) that the condition (1.13.1) does not imply the condition (1.13.2) for \( t \to x \), when \( f(t) \) is integrable (1).

Chow (1) established the following theorem:

Theorem F

If \( \sum \lambda_n \) is a convex sequence such that the series

+ for properties of convex sequence, see, e.g., Zygmund (1) p.58.
$\sum \lambda_n$ is convergent, then the series $\sum \lambda_n A_n(t)$ is summable $[c, 1]$ for almost all values of $t$, when (1.13.2) holds.

Now the question naturally arises whether it is possible to prove the summability $[c, 1]$ of the series $\sum \lambda_n A_n(t)$ at $t=x$, when (1.13.1) holds.

An attempt in this direction has been made by Pati(2) in the form of the following theorem:

**Theorem G**

If $\{\lambda_n\}$ be a convex sequence such that

then a necessary and sufficient condition that is summable $[c, 1]$, when ever (1.11.1) holds, is that

$$\sum \lambda_n S_n(x) \sim f(x) < \infty.$$  

Cheng (1) has however obtained the following result:

**Theorem H**

If $\{\lambda_n\}$ is any one of the sequences:

$$\log \log (\log \log n)^{1+\delta}$$

$$(\delta > 0), \quad \text{when} \quad (1.13.1) \quad \text{holds, then} \quad \sum \lambda_n \sim \lambda_n(t)$$

is summable $[c, 1], \quad [c, \infty]$ at $t=x$.

Theorem H has been extended by Pati (1) to the case in which $\{\lambda_n\}$ is a convex sequence so that $\sum \lambda_n$ is convergent, and he proves the following theorem:

**Theorem I.** Pati (1)
Theorem 1 Pati (1)

If \( \{A_n\} \) is a convex sequence such that the series \( \sum \lambda_n \) is convergent, then the series \( \sum \lambda_n A_n(t) \) at \( t=x \) is summable \( \langle \xi, \lambda \rangle \), for every \( \lambda \triangleright 1 \), provided that (1.13.1) holds.

On the other hand, Cheng (2) proved the following theorem:

Theorem 2

If the Fourier series \( \sum A_n(t) \) is multiplied by one of the following factors
\[
\left\{ \frac{1}{(\log n)^{1+\varepsilon}} \right\}, \left\{ \frac{1}{(\log n)^{1+\varepsilon} (\log \log n)^{1+\varepsilon}} \right\}, \ldots
\]
when (1.13.1) holds, then the resulting series is summable \( \langle \xi, 1 \rangle \) at the point \( x \).

The second theorem of Hsing (1) is the following:

Theorem 3

If
\[
\int_0^t |\phi(u)| \, du = o \left( \frac{t}{\log t} \right),
\]
as \( t \to \infty \), then the series
\[
\sum_{n=1}^{\infty} \frac{A_n(\omega)}{\log n} e^{-\lambda n} \nu_n
\]
is summable \( \langle \xi, 1 \rangle \) for every \( \lambda \triangleright 1 \).

Now we come over to results on \( \langle \xi, 1 \rangle \) summability factors. Mohapatra (1) proved the following theorem:
Theorem L Mohapatra (1)

Let

\[ \int_0^t |g(u)| \, du = o \left( \frac{t}{(\log \frac{t}{e})^x} \right) \quad 0 < x < 1, \]

Then \( \sum A_n(x) E_n \) is summable \( \{R_{\log n}, 1\} \), where\( \{E_n\} \) satisfies the following conditions:

(i) \( E_n = o (\log n) \)

(ii) \( \sum_{n=1}^{\infty} \frac{n \log (n+1)}{\log \log (n+1)^{\frac{1}{2}}} \left| \Delta^2 E_n \right| < \infty, 0 < \lambda < 1 \)

(iii) \( \sum_{n=1}^{\infty} \frac{\log (n+1)}{\log \log (n+1)^{\frac{1}{2}}} \left| \Delta^2 E_n \right| < \infty, \lambda = 1 \)

where, \( E_n' = E_n / \log (n+1) \).

Next coming over to results on the summability\( \{L\} \) of the Fourier series, Nanda (1) & (2) established the following theorems:

Theorem M Nanda (1)

If as \( t \to 0 \)

\[ \int \frac{g(u)}{u} \, \omega(u) = o \left( \frac{1}{\log \frac{1}{e}} \right) \]

then the series \( \frac{1}{2 \pi} \sum f(n) \omega(n) \) is summable to the sum \( s \).

Theorem N Nanda (2)

Let
where $C$ is a function of $x$.

If as $t \to 0$,
\[
\int_{t}^{+\infty} \frac{g(u)}{u} \, du = o \left( \log \frac{1}{t} \right)
\]
then the series \( \frac{1}{2} a_0 + \sum_{n=1}^{\infty} A_n(t) \) is summable (L) to $s$.

Theorem M was extended by Nanda & Das (1) in the form of the following theorem:

**Theorem M Nanda and Das (1)**

Let $\lambda > 0$

If as $t \to +\infty$,
\[
\int_{t}^{+\infty} \frac{g(u)}{u} \, du = o \left( \log \frac{1}{t} \right)
\]
then the series \( \frac{1}{2} a_0 + \sum_{n=1}^{\infty} A_n(t) \) is summable (L).

Next, we state some of the results of Das (1) and (2) on $(N,p,q)$ and $(J,q,k)$ summability.

**Theorem P Das (2)**

In order that \( (K^p, \rho_{ul}, \varphi) \) should imply $(J,q,k)$, it is necessary and sufficient that the condition
\[
\sum_{n=0}^{\infty} \rho_{ul}^{\lambda + n} \varphi^{-n} \to 0 \quad \text{as} \quad \lambda \to \infty
\]
should hold.

**Theorem Q Das (2)**

If \( \sum_{n=0}^{\infty} \rho_{ul}^{\lambda + n} \varphi^{-n} \to 0 \), then it is summable \( (J,q,k) \) to the same sum.

**Theorem R Das (1)**

Let the power series...
have radii of convergence and respectively, such that

\[ S_\phi = S_\psi , \]

and

\[ p(z) = \sum_{n=0}^{\infty} A_n z^n = 0, \quad (|z| < 1) \]

Also let any one of the three sets of the following conditions be fulfilled:

(i) \[ a_n > b, \quad a_n > b, \quad q_n > 0, \quad q_n > 0, \]

\[ \text{and } \chi(x) = \frac{a}{n \pi} x^m = \infty \quad (x \to \phi) \]

if \( S_\phi \) is finite

(ii) \[ p_0 = 0, \quad q_0 > 0, \quad q_n > 0, \quad S_\phi = 1, \]

\[ \chi^m (\chi_n = 0, \chi_n = 0) \text{ is real and such that } \]

\[ |\chi_n/\chi_n| > 0, \quad \chi_n^m \to \infty \quad (n \to \infty) \]

(iii) \[ p_0 = 0, \quad q_0 > 0, \quad q_n > 0, \quad S_\phi = 1, \]

\[ \chi_n (\chi_n = 0, \chi_n \to 0) \text{ is complex and such that } \]

\[ |\chi_n| > 0, \quad |\chi_n^m| \to \infty, \]

where \( \chi \) is non zero and finite;

then, if \( \sum a_n \) is summable \((N,p,q)\) to \( s \), it is summable \((J,q)\) to the same sum.

The following results obtained by Das(2) as the corollaries of theorem R and other theorems.

Theorem 5. Das (2)

If \( \sum a_n \) is summable \((N, \frac{L}{\log x}, \frac{L}{\log y})\) to the sum \( s \), then it is summable \((L)\) to the same sum.

Theorem T. Das (2)

If \( \sum a_n \) is summable \((N, \frac{L}{x}, \frac{L}{x})\) to the sum \( s \), then
it is summable ($N, \frac{1}{n^k}$) to the same sum and this implies, it is summable ($L$) to the same sum.

The following are the two theorems of F.T. Wang (2) on $(R,k)$ summability. We give only the case $k=1$ of his theorems.

**Theorem U**

Let $$f(t) = q(t)$$
and $$\gamma_n(t) = \sum_{n=1}^{\infty} \frac{q(n)}{n} \, dt.$$

Suppose that $$\int_0^t |q(w)| \, dw = O[t(\log t)^{\frac{1}{2}}].$$

The necessary and sufficient condition that the Fourier series \( \sum_{n=1}^{\infty} A_n(t) \) should be summable $(R,1)$ for $t=x$, to the sum $s$, is that

$$\gamma(t) = O(\log t)$$

and

$$\int_0^t \gamma_n(t) \, dt = O[t(\log t)^{\frac{1}{2}}].$$

when $t \to 0$.

**Theorem V**

Suppose that $$\int_0^t |q(w)| \, dw = O[t^{\frac{1}{2}} \log(t)]$$

Then the necessary and sufficient condition that the Fourier series \( \sum_{n=1}^{\infty} A_n(t) \) should be summable $(R,1)$ to sum $s$ is that

as $t \to 0$. 

1.14 Proposed Work

In the context of the unsolved problem arising out of Theorems A to k, as discussed in 1.13, it is now worth trying for the same problem with the condition
\[ \int_{t}^{1} \frac{f'(u)}{u} \, du = O(\log \frac{1}{t}), \quad t \to 0 \]
instead of the condition
\[ \int_{0}^{t} \left| f'(u) \right| \, du = o \left\{ t (\log \frac{1}{t}) \right\}, \quad t \to 0, \]
as has been tried by many authors previously.

We state below one of the Theorems in this context, we have established in Chapter 2.

**Theorem 1.**

If
\[ \int_{t}^{1} \frac{f'(u)}{u} \, du = O(\log \frac{1}{t}) \]
then the series \( \sum a_n A_n(x) \) is summable \( \text{r.c.} \) at \( t = x \), provided that the sequence \( \{ a_n \} \) satisfies the following conditions:
\[ \sum_{n=1}^{\infty} \frac{1}{n} |a_n| (\log n)^{\alpha} < \infty \]
\[ \sum_{n=1}^{\infty} |a_n| r^n (\log n)^{\alpha} < \infty \] for some \( r < 1 \).

Again looking at Theorem 1 above, we may avoid the modulus sign inside the integral, and one of our Theorems established in Chapter 3 is the following:

**Theorem 2.**

If,
Theorem 2

If

$$\int \frac{\varphi(u)}{u} \, du = O \left( \log \frac{1}{t} \right)$$

then \( \sum \lambda_n A_n(x) \) is summable \(|c,1|\) at \( t = x \), provided that the sequence \( \{ \lambda_n \} \) satisfies the following conditions:

$$\sum_{n=1}^{\infty} \left| \lambda_n \right|^2 (\log n)^2 < \infty$$

In Chapter 4 the case \( \alpha > 0 \) of Theorem 2 stated above, has been established as follows:

Theorem 3

If \( \alpha > 0 \), and

$$\int \frac{\varphi(u)}{u} \, du = O \left( \log \frac{1}{t} \right),$$

then the series \( \sum \lambda_n A_n(x) \) is summable \(|c,1|\) at \( t = x \), provided that the sequence \( \{ \lambda_n \} \) satisfies the following conditions:

$$\sum_{n=1}^{\infty} \left| \lambda_n \right|^2 (\log n)^2 < \infty$$

where as Chapters 2, 3, & 4 deal with every such sequence \( \{ \lambda_n \} \) in general, we may consider a particular value of \( \alpha \) to be a convex sequence of the type \( \cdots \) and the following Theorem in this context, has been established in Chapter 5.
**Theorem 4**

If \[ \int_{t}^{1} \frac{f(n)}{n} du = O(1) \]
then the series \[ \sum (\log n)^{-1 - \varepsilon} A_n(x) \] is summable \( |C, 1| \)
at \( t = x \).

Looking again at Theorem 1 of Nanchapatra as given in 1.13, the integral condition on \( f(t) \) seems much heavier, and this may be relaxed to get suitable results. One of our Theorems in this connection, established in Chapter 6 is the following:

**Theorem 5**

If
\[
\int_{t}^{1} \left| \frac{f(n)}{n} \right| du = O(\log \frac{1}{t})^{1-\varepsilon}, \quad (0 \leq \varepsilon \leq 1)
\]
then the series \[ \sum A_n(x) E_n \] is summable \( |D, \log n| \)
where the sequence \( E_n \) satisfies the following conditions:
\[
\sum_{n=2}^{\infty} \frac{1}{n} \left| E_n \right| (\log n)^{1-\varepsilon} < \infty
\]

Now coming over to Theorem N of Nanda & Das (1) as given in 1.13, a generalisation of this Theorem may be derived, and in this context, for the derived Fourier series, we have established the following Theorem in Chapter 7:
**Theorem 6**

Let \( p \) be a positive integer and

\[
G_p(t) = t^{-p} \mathcal{Y}(t)
\]

where,

\[
\mathcal{Y}(t) = \int_0^t k_p(u) \, du, \quad \text{(Eq. 1.12)}
\]

If for a suitably chosen polynomial \( p(t) \),

\[
\int_0^t G_p(u) u^{-p} \, du = t \left( \log \frac{1}{t} \right), \quad (t \to 0^+)
\]

then the series \( \sum_{n=1}^{\infty} \frac{d^n}{dt^n} (a_n \cos nx + b_n \sin nx) \)

is summable \((L)\) to \( C_r \).

In Chapter 8, we have generalized the \((N,p,q)\) method of summability and have established some results generalizing the result of Das (2) as given in Theorems \( P, Q \) and \( R \) in 1.13.

In Chapter 9, we have applied the \( N, p, q \) of summability to Fourier series and established the following Theorem:

**Theorem 7**

Let \( g(t) = q(t) / \sin bt \).

If,

\[
(i) \quad \int_0^t \mathcal{F} \, du = \int_0^t \mathcal{G} \, du
\]

and

\[
(ii) \quad \int_0^t \mathcal{K} \, du = \int_0^t \mathcal{L} \, du
\]

then the series \( \sum_{n=1}^{\infty} \) is summable to the sum \( s \).

This completes in brief, our proposed work established in different Chapters in this Thesis.