Chapter 3

Locally Dually Flat Finsler Space with Two Special Metrics

3.1 Introduction

In 2000, S. I. Amari and H. Nagaoka [163] introduced the concept of dually flat metrics during the study of information geometry on a Riemannian space. The notion of dually flatness was extended to Riemannian-Finsler space by Z. Shen [206].

In this chapter, we deal with the necessary and sufficient conditions for a Finsler metric $F^2 = \alpha^2 + \beta^2 + 3\alpha\beta$ to be locally dually flat as well as locally projectively flat and also characterize the locally dually flat Finsler metric $F = \frac{\alpha^2}{\alpha + \beta}$.

A Finsler metric $F = F(x, y)$ on the manifold $M$ is called locally dually flat if there exists a coordinate system $(x^i)$ such that the spray coefficients $G^i$ satisfy

$$G^i = -\frac{1}{2}g^{ij}\dot{\partial}_j H,$$

where $H = H(x, y)$, is a local scalar function. Such coordinate system is termed as adopted coordinate system. These Finsler metrics were
studied in Finsler information geometry with application to information geometry for the first time [206].

The Finsler metrics given by

\[ F = \sqrt{(1 - |x|^2)|y|^2 - <x, y>^2} \pm \frac{<x, y>}{1 - |x|^2} \]

are examples of a locally dually flat metrics.

In [206], it was shown that a Finsler metric \( F = F(x, y) \) is dually flat if and only if it satisfies

\[ y^k \dot{\partial}_l \partial_k F^2 = 2 \partial_l F^2. \]  

(3.1.1)

It has been proved that a locally Minkowskian metric is locally dually flat but the converse is not necessarily true.

Now, we define a locally projectively flat metric.

A Finsler metric is called locally projectively flat if there exists a local coordinate system in which each geodesic is a straight line as a point set. Equivalently, the spray coefficients \( G^i \) satisfy

\[ G^i = Py^i, \]

where \( P = P(x, y) \) is positively homogeneous of degree one in \( y^i \), which is equivalent to

\[ y^k \dot{\partial}_l \partial_k F = 2 \partial_l F. \]

The necessary and sufficient conditions for a Finsler metric \( F \) to be locally dually flat as well as locally projectively flat are given by

(3.1.2) \[ \partial_k F = CF\dot{\partial}_k F, \]

where \( C \) is a constant[198].

**Lemma 3.1.1:** [198] Suppose \( F = F(x, y) \) be a Finsler metric on any open subset \( V \) of \( \mathbb{R}^n \). Then \( F \) is locally flat and projectively flat on \( V \) if and only if it satisfies the condition (3.1.2).
A Finsler metric $F$ on an $n$-dimensional manifold $M$ is called of *isotropic $S$-curvature* if it is characterized by the equation

\[
S = (n + 1)c(x)F,
\]

where $c$ is a scalar function on $M$.

Suppose $F = \alpha \phi(s)$ be an $(\alpha, \beta)$ metric, where $s = \frac{\beta}{\alpha}$, $\alpha^2 = a_{ij}y^i y^j$ is the Riemannian metric, $\beta = b_i y^i$ is differential 1-form and $\phi = \phi(s)$ is a $C^\infty$-function on the interval $(-b_0, b_0)$ with certain regularity.

Let $\Gamma^i_{jk}$ be the Levi-Civita connection for the Riemannian metric tensor $a_{ij}$. Then

\[
b_{i|j} = \partial_j b_i - b_r \Gamma^r_{ji},
\]

which implies

\[
b_{i|j} dx^j = db_i - b_r \Gamma^r_{ji} dx^j.
\]

Let us write $dx^l = \theta^l$ and $\Gamma^r_{ji} dx^j = \theta^r_i$. Then

\[
b_{i|j} \theta^j = db_i - b_r \theta^r_i.
\]

Suppose that

\[
r_{ij} = \frac{1}{2}(b_{i|j} + b_{j|i})
\]

and

\[
s_{ij} = \frac{1}{2}(b_{i|j} - b_{j|i}).
\]

From the above two equations, it is clear that $\beta$ is closed if and only if $s_{ij} = 0$. In the case $r_{ij} = s_{ij} = 0$, the $(\alpha, \beta)$ metric is called trivial.
With the help of relations

\[
\begin{align*}
(a) \quad r_{i0} &= r_{ij} y^j, \\
(b) \quad r_{00} &= r_{ij} y^j y^i, \\
(c) \quad r_j &= b^i r_{ij}, \\
(d) \quad s_{i0} &= s_{ij} y^j, \\
(e) \quad s_j &= b^i s_{ij}, \\
(f) \quad r_0 &= r_j y^j, \\
(g) \quad s_0 &= s_j y^j,
\end{align*}
\]

(3.1.4)

we can obtain following formula for mean Cartan torsion tensor of an \((\alpha, \beta)\) metric on direct computation as

\[
I_i = -\frac{\Phi(\phi - s\phi')}{2\Delta \phi \alpha^2},
\]

where

\[
\Phi = -(Q - sQ')(n\Delta + 1 + sQ) - (b^2 - s^2)(1 + sQ)Q''
\]

and

\[
\Delta = 1 + sQ + (b^2 - s^2)Q'.
\]

Clearly, an \((\alpha, \beta)\) metric \(F = \alpha \phi(s), s = \frac{\beta}{\alpha}\), will be Riemannian metric if and only if \(\Phi = 0\). Hence, we suppose \(\Phi \neq 0\) for non-Riemannian case.

**Theorem 3.1.1:** [145] Suppose \(F = \alpha \phi(s), s = \frac{\beta}{\alpha}\), be an \((\alpha, \beta)\) metric on an \(n\)-dimensional manifold \(M\) \((n > 2)\), where \(\alpha^2 = a_{ij} y^i y^j\) is the Riemannian metric and \(\beta = b_i(x) y^i\) is a differential 1-form on \(M\). If \(F\) is not Riemannian and \(\phi'(s) \neq 0\), then \(F\) is locally dually flat on \(M\) if and only if \(\alpha, \beta\) and \(\phi = \phi(s)\) satisfy the following

\[
\begin{align*}
(1) \quad r_{00} &= \frac{2}{3} \theta \beta + [\tau + \frac{2}{3}(b^2 \tau - \theta b^i)] \alpha^2 + \frac{1}{3}(3k_3 - 2 - 3k_3b^2)\tau \beta^2, \\
(2) \quad s_{k0} &= \frac{1}{3}(\beta \theta_k - \theta b_k), \\
(3) \quad G^m_{\alpha} &= \frac{1}{3}[2\theta + (3k_1 - 2)\tau \beta] y^m + \frac{1}{3}(\theta^m - \tau b^m) \alpha^2 + \frac{1}{2} k_3 \tau \beta^2 b^m,
\end{align*}
\]

and

\[
\tau [s(k_2 - k_3 s^2)(\phi \phi' - s \phi^2 - s \phi \phi'') - (\phi'^2 + \phi \phi'') + k_1 \phi(\phi - s \phi')] = 0,
\]

where \(\tau = \tau(x)\) is a scalar function, \(\theta = \theta_i(x) y^i\) is 1-form on \(M\) with
\[ \theta^l = a^{lm} \theta_m \text{ and} \]
\[ k_1 = \Pi(0), \quad k_2 = \frac{\Pi'(0)}{Q(0)}, \]
\[ k_3 = \frac{1}{6Q^2(0)}[3Q''(0)\Pi'(0) - 6\Pi^2(0) - Q(0)\Pi''(0)], \]
\[ Q = \frac{\phi'}{\phi - s\phi'}, \quad \Pi = \frac{\phi'^2 + \phi''}{\phi^2 - s\phi\phi'}. \]

**Theorem 3.1.2:** [197] Let \( F = \alpha\phi(s), \ s = \frac{s}{\alpha} \) be an \((\alpha, \beta)\) metric on a manifold \( M \) and \( b = ||x||_\alpha \). Suppose that \( \phi \neq t_1\sqrt{1 + t_2s^2} + t_3s \) for the constants \( t_1, t_2 \) and \( t_3 \), where \( t_1 > 0 \). Then \( F \) is of isotropic \( S \)-curvature (3.1.3) if and only if any of the following conditions holds:

(i) If \( \beta \) satisfies

\[ (3.1.5) \quad r_{ij} = \in (b^2a_{ij} - b_ib_j), \quad s_j = 0, \]

where \( \in = \in (x) \) is a scalar function, and \( \phi = \phi(s) \) satisfies

\[ (3.1.6) \quad \Phi = -2(n + 1)k\frac{\phi\Delta^2}{b^2 - s^2}, \]

where \( k \) is a real constant. In this case, \( S = (n + 1)cF \), with \( c = k\epsilon \).

(ii) If \( \beta \) satisfies

\[ (3.1.7) \quad r_{ij} = 0, \quad s_j = 0. \]

In this case, \( S = 0 \), regardless of choice of particular \( \phi \).

### 3.2 Characterization of Locally Dually Flat Finsler Metric \( F^2 = \alpha^2 + \beta^2 + 3\alpha\beta \)

In this section, we consider a locally dually flat Finsler metric \( F \) satisfying \( F^2 = \alpha^2 + \beta^2 + 3\alpha\beta \), where \( \alpha^2 = a_{ij}(x)y^iy^j \) is the Riemannian metric and \( \beta = b_iy^i \) is a differential 1-form. For such metric, we propose the following:

**Theorem 3.2.1:** The necessary and sufficient conditions for a Finsler metric \( F \) satisfying \( F^2 = \alpha^2 + \beta^2 + 3\alpha\beta \) to be locally dually flat are given by
\[
\begin{align*}
(1) \quad r_{00} &= \frac{2}{3} \theta \beta - \frac{5}{3} \tau \beta^2 + [\tau + \frac{2}{3}(\tau b^2 - b_m \theta^m)] \alpha^2 \\
(2) \quad s_{k0} &= \frac{1}{3}(\beta \theta_k - \theta b_k) \\
(3) \quad G^m_\alpha &= \frac{1}{3}(2 \theta + \tau \beta) y^m - \frac{1}{3}(\tau b^m - \theta^m) \alpha^2.
\end{align*}
\]

Proof. Let us consider the \((\alpha, \beta)\) metric satisfying \(F^2 = \alpha^2 + \beta^2 + 3\alpha \beta\).

Since, sufficient condition can be easily verified, we have to prove only necessary condition. Suppose \((\alpha, \beta)\) metric

\[(3.2.1) \quad F^2 = \alpha^2 + \beta^2 + 3\alpha \beta = (\alpha + \beta)^2 + \alpha \beta.\]

be locally dually flat.

Differentiating equation \((3.2.1)\) partially with respect to \(x^k\), we get

\[
\partial_k F^2 = 2\alpha \partial_k \alpha + 2\beta \partial_k \beta + 3\beta \partial_k \alpha + 3\alpha \partial_k \beta.
\]

\[
= (2\alpha + 3\beta) \partial_k \alpha + (2\beta + 3\alpha) \partial_k \beta.
\]

Since, we have the following identities:

\[
\partial_k \alpha = \frac{y^m}{\alpha} \dot{\partial}_k G^m_\alpha, \quad \partial_k \beta = b_{m|k} y^m + b_m \dot{\partial}_k G^m_\alpha, \quad \dot{\partial}_k s = \frac{\alpha b_k - s y_k}{\alpha^2},
\]

where \(s = \frac{\beta}{\alpha}\) and \(y_k = a_{jk} y^j\) then, the above equation can be written as

\[
\partial_k F^2 = (2\alpha + 3\beta) \partial_k \alpha + \partial_k \beta (3\alpha + 2\beta)
\]

\[
= (2\alpha + 3\beta) \frac{y^m}{\alpha} \dot{\partial}_k G^m_\alpha + (3\alpha + 2\beta) [b_{m|k} y^m + b_m \dot{\partial}_k G^m_\alpha]
\]

\[
= (2 + \frac{3\beta}{\alpha}) y^m \dot{\partial}_k G^m_\alpha + (3\alpha + 2\beta) b_{m|k} y^m + (3\alpha + 2\beta) b_m \dot{\partial}_k G^m_\alpha.
\]

\[(3.2.2) \quad \partial_k F^2 = [2y^m + \frac{3\beta}{\alpha} y^m + 2\beta b_m + 3\alpha b_m] \dot{\partial}_k G^m_\alpha + (3\alpha + 2\beta) b_{m|k} y^m.
\]

Similarly,

\[(3.2.3) \quad \partial_l F^2 = [2y^m + 3sy^m + 2\beta b_m + 3\alpha b_m] \dot{\partial}_l G^m_\alpha + (3\alpha + 2\beta) b_{m|l} y^m.
\]

Differentiating equation \((3.2.3)\) partially with respect to \(y^k\), we obtain

\[
\dot{\partial}_k \partial_l F^2 = [2a_{mk} + 2b_m b_k + \frac{3a_{kj} y^j}{\alpha} b_m + 3s a_{mk} + 3y_m \frac{(\alpha^2 b_k - \beta y_k)}{\alpha^3}] \dot{\partial}_l G^m_\alpha
\]

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Transvecting the above equation with $y^l$, we get

\[(3.2.4)\]

\[y^l \partial_k \partial_l F^2 = 2[2a_{mk} + 2bm_bk + \frac{3b_my_k}{\alpha} + 3\beta a_{mk} + 3y_m \frac{(\alpha^2b_k - \beta y_k)}{\alpha^3}]G^m_{\alpha} \]

\[\quad + [2y_m + 3\beta y_m + 2\beta b_m + 3\alpha b_m] \hat{\partial}_k G^m_{\alpha} + (3\alpha + 2\beta) b_k y^l \]

\[\quad + (2b_k + 3\frac{y_k}{\alpha}) b_m y \cdot y^l.
\]

From equations (3.2.2) and (3.2.4), equation (3.1.1) becomes

\[(3.2.5)\]

\[2[2a_{mk} + 2bm_bk + \frac{3b_my_k}{\alpha} + 3\beta a_{mk} + 3y_m \frac{(\alpha^2b_k - \beta y_k)}{\alpha^3}]G^m_{\alpha} \]

\[\quad - [2y_m + 3\beta y_m + 2\beta b_m + 3\alpha b_m] \hat{\partial}_k G^m_{\alpha} + (3\alpha + 2\beta) y^l [b_{k|l} - 2b_{l|k}] \]

\[\quad + (2b_k + 3\frac{y_k}{\alpha}) b_m y \cdot y^l = 0.
\]

On simplifying equation (3.2.5), we have

\[(3.2.6)\]

\[2[2a_{mk}\alpha^3 + 2bm_bk\alpha^3 + 3b_my_k\alpha^2 + 3\beta\alpha^2 a_{mk} + 3y_m (\alpha^2b_k - \beta y_k)]G^m_{\alpha} \]

\[\quad - [2y_m\alpha^3 + 3\beta\alpha^2 y_m + 2\beta b_m\alpha^3 + 3\alpha^4 b_m] \hat{\partial}_k G^m_{\alpha} + (3\alpha^4 + 2\beta\alpha^3) \]

\[\quad [b_{k|0} - 2b_{0|k}] + (2b_k\alpha^3 + 3y_k\alpha^2)r_{00} = 0.
\]

Let us suppose

\[A_1 = 2a_{mk}\alpha^3 + 2bm_bk\alpha^3 + 3b_my_k\alpha^2 + 3\beta\alpha^2 a_{mk} + 3y_m (\alpha^2b_k - \beta y_k),\]

\[B_1 = [2y_m + 3\beta y_m + 2\beta b_m + 3\alpha b_m],\]

\[C_1 = (2b_k + 3\frac{y_k}{\alpha})\]

and

\[D_1 = (3\alpha + 2\beta).
\]

Then equation (3.2.6) can be reduced as

\[(3.2.7)\]

\[\frac{2A_1}{\alpha^3} G^m_{\alpha} - B_1 \hat{\partial}_k G^m_{\alpha} + C_1 r_{00} + D_1 (b_{k|0} - 2b_{0|k}) = 0.
\]
Equation (3.2.6) may also be written in the form

\[ \alpha^4 [-3b_m \dot{\hat{G}}_m^m + 3(b_{k|l} - 2b_{l|k})y^l] + \alpha^3 [(4a_{mk} + 4b_mb_k)G^m_m + 2b_k r_{00} + 2\beta (b_{k|l} - 2b_{l|k})y^l - (2y_m + 2\beta b_m) \dot{\hat{G}}_m^m] + \alpha^2 \\
[(6b_my_k + 6\beta a_{mk} + 6y_mb_k)G^m_m + 3y_k r_{00} - 3\beta y_m \dot{\hat{G}}_m^m ] - 6\beta y_my_k G^m_m = 0. \]

If, we put

\[ A_2 = \frac{3}{2} [(3s_{k0} - r_{k0}) - b_m \dot{\hat{G}}_m^m] , \]
\[ B_2 = \frac{1}{2} [(4a_{mk} + 4b_mb_k)G^m_m + 2\beta (3s_{k0} - r_{k0}) + 2b_k r_{00} - (2y_m + 2\beta b_m) \dot{\hat{G}}_m^m] , \]
\[ C_2 = \frac{1}{2} [(6b_my_k + 6\beta a_{mk} + 6y_mb_k)G^m_m + 3y_k r_{00} - 3\beta y_m \dot{\hat{G}}_m^m ] \]

and

\[ D_2 = 3\beta y_my_k G^m_m . \]

where \( b_{k|l} y^l = s_{k0} \) and \( b_{l|k} y^l = r_{k0} \).

Then, the above equation becomes

\[ 2A_2 \alpha^4 + 2B_2 \alpha^3 + 2C_2 \alpha^2 - 2D_2 = 0. \]

This is nothing but

(3.2.8) \[ A_2 \alpha^4 + B_2 \alpha^3 + C_2 \alpha^2 - D_2 = 0, \]

which is a polynomial in \( \alpha \). Here, the coefficients of \( \alpha \) are vanish, therefore the coefficients of \( \alpha^3 \) will also be vanish, i.e. \( B_2 = 0 \). Hence, we have

(3.2.9) \[ (2a_{mk} + 2b_mb_k)G^m_m + \beta (3s_{k0} - r_{k0}) + b_k r_{00} - (y_m + \beta b_m) \dot{\hat{G}}_m^m = 0. \]

So, (3.2.8) becomes

\[ \frac{3}{2} [(3s_{k0} - r_{k0}) - b_m \dot{\hat{G}}_m^m] \alpha^4 + \frac{1}{2} [(6b_my_k + 6\beta a_{mk} + \]
\[ 6 y_m b_k G^m_{\alpha} + 3 y_k r_{00} - 3 \beta y_m \dot{G}^m_{\alpha}] \alpha^2 - 3 \beta y_m y_k G^m_{\alpha} = 0. \]

Multiplying above equation by \( \frac{2}{3} \), we get

\[
(3.2.10) \quad [(3s_{k0} - r_{k0}) - b_m \dot{\dot{G}}^m_{\alpha}] \alpha^4 + [(2b_m y_k + 2 \beta a_{mk} + 2 y_m b_k) G^m_{\alpha} + y_k r_{00} - \beta y_m \dot{G}^m_{\alpha}] \alpha^2 - 2 \beta y_m y_k G^m_{\alpha} = 0.
\]

Transvecting equation (3.2.10) with \( b^k \) and applying \( b^k a_{mk} = b_m \) and \( s_{k0} b^k = s_0 \), we get

\[
(3.2.11) \quad (2b_m + 2b_m b^2) G^m_{\alpha} + \beta (3s_0 - r_0)
\]

\[ + b^2 r_{00} - b^k (y_m + \beta b_m) \dot{\dot{G}}^m_{\alpha} = 0. \]

We have

\[
(3.2.12) \quad \dot{\dot{G}}^m_{\alpha} = \dot{\dot{G}}^m_{\alpha} - a_{mk} G^m_{\alpha}
\]

and

\[
(3.2.13) \quad \dot{\dot{G}}^m_{\alpha} = b_m \dot{\dot{G}}^m_{\alpha}.
\]

With the help of (3.2.12) and (3.2.13), (3.2.11) may be written as

\[
(3.2.14) \quad 2b_m (1 + b^2) G^m_{\alpha} + \beta (3s_0 - r_0) + b^2 r_{00} - \beta b^k
\]

\[ \dot{\dot{G}}^m_{\alpha} - b^k [\dot{\dot{G}}^m_{\alpha} - a_{mk} G^m_{\alpha}] = 0. \]

or

\[
(3.2.15) \quad b^k \dot{\dot{G}}^m_{\alpha} + \beta b^k \dot{\dot{G}}^m_{\alpha} = b_m (3 + 2b^2) G^m_{\alpha} + b^2 r_{00} + \beta (3s_0 - r_0).
\]

Transvecting equation (3.2.10) with \( b^k \), we get

\[
[(3s_0 - r_0) - b_m b^k \dot{\dot{G}}^m_{\alpha}] \alpha^4 + [(2b_m \beta + 2 \beta b_m + 2 y_m \dot{G}^m_{\alpha} + y_k r_{00} - \beta y_m \dot{G}^m_{\alpha}] \alpha^2 - 2 \beta^2 y_m G^m_{\alpha} = 0.
\]
or
\[
[(3s_0 - r_0) - b^k \dot{k}(b_m G^m_{\alpha})] \alpha^4 + [(2\beta b_m + 2\beta b_m + 2y_m b^2)G^m_{\alpha} + \beta r_{00}] \alpha^2 - \beta b^k \dot{k}(y_m G^m_{\alpha}) \alpha^2 - 2\beta^2 y_m G^m_{\alpha} = 0.
\]

This implies
\[
(3s_0 - r_0) \alpha^4 - b^k \alpha^4 \dot{k}(b_m G^m_{\alpha}) + 4\beta b_m G^m_{\alpha} \alpha^2 + 2\alpha^2 y_m b^2 G^m_{\alpha} - \beta b^k \dot{k}(y_m G^m_{\alpha}) + 4\beta r_{00} \alpha^2 - 2\beta^2 y_m G^m_{\alpha} = 0.
\]

On simplifying above equation, we get
\[
\frac{\beta \alpha^2 \dot{k}(y_m G^m_{\alpha}) b^k + \alpha^4 (\dot{k} b_m G^m_{\alpha}) b^k}{(3s_0 - r_0) \alpha^4 + (2b^2 y_m G^m_{\alpha} + 5\beta b_m G^m_{\alpha} + \beta r_{00}) \alpha^2 - 2\beta^2 y_m G^m_{\alpha}} = \frac{\beta \alpha^2 \dot{k}(y_m G^m_{\alpha}) b^k + \alpha^4 (\dot{k} b_m G^m_{\alpha}) b^k}{(3s_0 - r_0) \alpha^4 + (2b^2 y_m G^m_{\alpha} + 5\beta b_m G^m_{\alpha} + \beta r_{00}) \alpha^2 - 2\beta^2 y_m G^m_{\alpha}}.
\]

After multiplying (3.2.15) by \(\alpha^4\) and (3.2.16) by \(\beta\), subtracting them, we have
\[
\alpha^2 (\alpha^2 - \beta^2) \dot{k}(y_m G^m_{\alpha}) b^k - (3\alpha^4 - 5\beta^2 \alpha^2) b_m G^m_{\alpha} = 2b^2 \alpha^4 b_m G^m_{\alpha} + b^2 \alpha^4 r_{00} - 2\beta b^2 \alpha^2 y_m G^m_{\alpha} - \beta^2 b^2 \alpha^2 r_{00} + 2\beta^3 y_m G^m_{\alpha}.
\]

This implies
\[
\alpha^2 (\alpha^2 - \beta^2) \dot{k}(y_m G^m_{\alpha}) b^k - 3\alpha^4 b_m G^m_{\alpha} + 3b_m \beta^2 \alpha^2 G^m_{\alpha} = 2b^2 \alpha^4 b_m G^m_{\alpha} - 2b_m \beta^2 \alpha^2 G^m_{\alpha} + b^2 \alpha^4 r_{00} - 2\beta b^2 \alpha^2 y_m G^m_{\alpha} - \beta^2 b^2 \alpha^2 r_{00} + 2\beta^3 y_m G^m_{\alpha} = r_{00} \alpha^2 (\alpha^2 b^2 - \beta^2) - 2\beta y_m G^m_{\alpha} (b^2 \alpha^2 - \beta^2)
\]
\[
+ 2\alpha^2 b_m G^m_{\alpha} (b^2 \alpha^2 - \beta^2).
\]

(3.2.17) \(\alpha^2 (\alpha^2 - \beta^2) [\dot{k}(y_m G^m_{\alpha}) b^k - 3b_m G^m_{\alpha}]\)
\[
\alpha^2 b^2 - \beta^2 \\right) \alpha^2 m \Gamma \alpha + r_{00} \alpha^2 - 2 \beta y m G_m \alpha^2.
\]

Since \( \alpha^2 \), \( \alpha^2 - \beta^2 \) and \( \alpha^2 b^2 - \beta^2 \) are relatively prime in \( y^i \), then there exists a function \( \tau = \tau(x) \) on \( M \) such that

\[
\text{(3.2.18)} \quad \dot{\partial}_k (G_m^m y^m) b^k - 3b_m G_m^m = \tau(\alpha^2 b^2 - \beta^2).
\]

\[
\text{(3.2.19)} \quad 2\alpha^2 b_m G_m^m + r_{00} \alpha^2 - 2\beta y_m G_m^m = \tau(\alpha^2 - \beta^2) \alpha^2.
\]

\[
\text{(3.2.20)} \quad 2\beta y_m G_m^m = (2b_m G_m^m + r_{00} - \tau \alpha^2 + \tau \beta^2) \alpha^2.
\]

Suppose

\[
y_m G_m^m = \theta \alpha^2.
\]

Differentiating it with respect to \( y^k \), we get

\[
\text{(3.2.21)} \quad \dot{\partial}_k (y_m G_m^m) = \alpha^2 \theta_k + 2\theta y_k,
\]

where \( \theta = \theta_k y^k \).

Putting the value of \( y_m G_m^m = \theta \alpha^2 \) in equation (3.2.20), we get

\[
\text{(3.2.22)} \quad 2\beta \theta \alpha^2 = (2b_m G_m^m + r_{00} - \tau \alpha^2 + \tau \beta^2) \alpha^2,
\]

then equation (3.2.22) implies

\[
\text{(3.2.23)} \quad b_m G_m^m = \beta \theta - \frac{1}{2} r_{00} + \frac{1}{2} \tau \alpha^2 + \frac{1}{2} \tau \beta^2.
\]

Differentiating equation (3.2.23) with respect to \( y^k \), we have

\[
\text{(3.2.24)} \quad \dot{\partial}_k (b_m G_m^m) = \beta \theta_k + \theta b_k - r_{k0} + \tau y_k - \tau b_k.
\]

Using equations (3.2.21), (3.2.23) and (3.2.24), the equations (3.2.10) and (3.2.11) become

\[
\text{(3.2.25)} \quad \beta (3s_{k0} + \theta b_k - \beta \theta_k) + (\tau b_k - \theta_k) \alpha^2 + 3a_{mk} G_m^m - (2\theta + \tau \beta) y_k = 0
\]
and

\begin{equation}
(3.2.26) \quad [(3s_{k0} + \theta b_k - \beta \theta_k) + (\tau b_k - \theta_k) \beta] \alpha^2 - (2\theta + \tau \beta) y_k \beta + 3\beta a_{mk} G^m_{\alpha} = 0,
\end{equation}

respectively.

Multiplying equation (3.2.25) with $\beta$ and subtracting (3.2.26) with it, we get

\begin{align*}
(3s_{k0} + \theta b_k - \beta \theta_k) \beta^2 &+ (\tau b_k - \theta_k) \beta \alpha^2 + 3\beta a_{mk} G^m_{\alpha} - (2\theta + \tau \beta) y_k \beta - (3s_{k0} + \theta b_k - \beta \theta_k) \alpha^2 - \tau b_k \beta \alpha^2 \\
&+ \theta_k \beta \alpha^2 + (2\theta + \tau \beta) y_k \beta - 3\beta a_{mk} G^m_{\alpha} = 0.
\end{align*}

On simplification, above equation becomes

\begin{equation}
(3.2.27) \quad (\beta^2 - \alpha^2)(3s_{k0} + \theta b_k - \beta \theta_k) = 0.
\end{equation}

Above equation implies

\begin{equation}
(3.2.28) \quad (3s_{k0} + \theta b_k - \beta \theta_k) = 0.
\end{equation}

From (3.2.28), we obtain

\begin{equation}
(3.2.29) \quad s_{k0} = \frac{1}{3}(\beta \theta_k - \theta b_k).
\end{equation}

Transvecting equation (3.2.28) with $a^{kl}$, we get

\begin{equation}
(3.2.30) \quad 3s^l_0 - \beta \theta^l + \theta b^l = 0.
\end{equation}

Also transvecting equation (3.2.25) with $a^{kl}$, we get

\begin{equation}
(3.2.31) \quad (3s^l_0 + \beta \theta^l - \theta b^l) \beta + (\tau b^l - \theta^l) \alpha^2 + 3G^l_{\alpha} - (2\theta + \tau \beta) y^l = 0.
\end{equation}

From equation (3.2.30) and (3.2.31), we obtain the result

\begin{equation}
G^m_{\alpha} = \frac{1}{3}(2\theta + \tau \beta) y^m - \frac{1}{3}(\tau b^m - \theta^m) \alpha^2.
\end{equation}
Substituting the values of $G^m_\alpha$ in equation (3.2.23), we get

$$(3.2.32)\ b_m(\frac{1}{3}(2\theta + \tau \beta)y^m - \frac{1}{3}(\tau b^m - \theta^m)\alpha^2) = \beta \theta - \frac{1}{2}r_{00} + \frac{\tau \alpha^2}{2} - \frac{\tau \beta^2}{2}.$$ 

After simplification of equation (3.2.32), we get another result

$$(3.2.33)\ r_{00} = \frac{2}{3}\theta \beta - \frac{5}{3}\tau \beta^2 + [\tau + \frac{2}{3}(\tau b^2 - b_m\theta^m)]\alpha^2.$$ 

This complete the proof. \qed

3.3 Characterization of Locally Dually Flat Finsler Metric $F = \frac{\alpha^2}{\alpha + \beta}$

**Theorem 3.3.1:** Suppose $F = \frac{\alpha^2}{\alpha + \beta}$ be a metric on the manifold $M$ of dimension $n > 2$. Necessary and sufficient conditions for metric $F$ to be locally dually flat are given by

1. $r_{00} = \frac{2}{3}\theta \beta + [\tau + \frac{2}{3}(b^2\tau - \theta b^2)]\alpha^2 + \frac{1}{3}(25 + 54b^2)\tau \beta^2$,
2. $s_{k0} = \frac{1}{3}(\beta \theta_k - \theta \theta_k)$
3. $G^m_\alpha = \frac{1}{3}[2\theta + 7\tau \beta]y^l + \frac{1}{3}(\theta^l \tau b^l)\alpha^2 - 9\tau \beta^2 b^l$, 
   \[\tau[s(9 + 18s^2)(\phi \phi' - s \phi^2 - s \phi \phi'') - (\phi'^2 + \phi \phi'') + 3\phi(\phi - s \phi')] = 0.\]

**Proof.** Comparing the $(\alpha, \beta)$ metric $F = \frac{\alpha^2}{\alpha + \beta}$ with $F = \alpha \phi(s)$, we get

\[\phi(s) = \frac{1}{1 + s}.\]

Differentiating this equation with respect to $s$, we get

\[\phi' = -\frac{1}{(1 + s)^2}, \quad \phi''(s) = \frac{2}{(1 + s)^3}.\]

Putting the values of $\phi(s)$ and $\phi'(s)$ in

\[Q = \frac{\phi'(s)}{\phi(s) - s \phi'(s)},\]

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we get

\[ Q = -\frac{1}{(1 + 2s)}. \]

Differentiating this with respect to \( s \), we obtain

\[ Q' = \frac{2}{(1 + 2s)^2}, \]
\[ Q'' = -\frac{8}{(1 + 2s)^3} \]

and

\[ Q''' = \frac{48}{(1 + 2s)^4}. \]

Putting the values of \( \phi(s) \), \( \phi'(s) \) and \( \phi''(s) \) in

\[ \pi = \frac{\phi^2 + \phi\phi''}{\phi^2 - s\phi\phi'}, \]

we get

\[ \Pi = \frac{3}{1 + 3s + 2s^2}. \]

Differentiating this equation with respect to \( s \), we have

\[ \Pi' = -\frac{3(4s + 3)}{(1 + 3s + 2s^2)^2}; \]
\[ \Pi'' = \frac{6(12s^2 + 18s + 7)}{(1 + 3s + 2s^2)^3} \]

and

\[ \Pi''' = -\frac{18(32s^3 + 72s^2 + 56s + 15)}{(1 + 3s + 2s^2)^4}. \]

Using above, we obtain

\[ Q = -1, \quad Q' = 2, \quad Q'' = -8, \quad Q''' = 48, \]
\[ \Pi(0) = 3, \quad \Pi'(0) = -9, \quad \Pi''(0) = 42, \quad \Pi'''(0) = -270, \]
\[ k_1 = 3, \quad k_2 = 9, \quad k_3 = -18. \]

With the help of above values and Theorem 3.1.1, we get

\[ [s(9 + 18s^2)(\phi\phi' - s\phi^2 - s\phi\phi'')] \]
\[-(\phi'^2 + \phi\phi''') + 3\phi(\phi - s\phi')\] = 0

and

\[\tau = 0.\]

Combining the above equations, we get the result

\[\tau[s(9 + 18s^2)(\phi\phi' - s\phi'^2 - s\phi''')]
\[-(\phi'^2 + \phi\phi''') + 3\phi(\phi - s\phi')] = 0.\]

This completes the proof.

Suppose \(\phi = \phi(s)\) be a \(C^\infty\) function on the \((-b_0, b_0)\) for a number \(b \in [0, b_0]\) such that

\[(3.3.1) \Phi = -(Q - sQ')(n\Delta + 1 + sQ) - (b^2 - s^2)(1 + sQ)Q'',\]

where \(\Delta = 1 + sQ + (b^2 - s^2)Q'.\)

Putting the values of \(Q\) and \(Q'\), \(\Delta\) becomes

\[(3.3.2) \Delta = \frac{1}{(1 + 2s)^2}(1 + 3s + 2b^2)\]

By using \(Q\), the equation (3.3.1) can be written as:

\[(3.3.3) \Phi = -(Q - sQ')(n + 1)\Delta + (b^2 - s^2)[(Q - sQ')Q' - (1 + sQ)Q'']\]

**Theorem 3.3.2:** Let \(F = \frac{\alpha^2}{\alpha + \beta}\) be a locally dually flat \((\alpha, \beta)\)-metric on the manifold of dimension \(n > 3\). Suppose that \(F\) is of isotropic \(S\)-curvature \(S = (n+1)cF\), where \(c = c(x)\) is a scalar function on \(M\). Then \(F\) is locally projectively flat in adopted coordinate system and \(G^i = 0\).

**Proof.** Suppose \(G^i = G^i(x, y)\) and \(= \bar{G}^i_\alpha(x, y)\) be the coefficients of \(F\) and \(\alpha\) respectively, in the same coordinate system. Thus, we have

\[(3.3.4) G^i = \bar{G}^i_\alpha + Py^i + Q^i,\]
where

\[(3.3.5)\quad P = \alpha^{-1}\Theta - 2Q\alpha s_0 + r_{00}.\]

Thus

\[(3.3.6)\quad Q^i = \alpha Q_s^i + \Psi - 2Q\alpha s_0 + r_{00}b^i.\]

\[(3.3.7)\quad \Theta = \frac{\phi\phi' - s(\phi\phi'' + \phi^2)}{2\phi[(\phi - s\phi') + (b^2 - s^2)\phi'']}.\]

\[(3.3.8)\quad \Psi = \frac{1}{2} \frac{\phi''}{(\phi - s\phi') + (b^2 - s^2)\phi''},\]

and

\[(3.3.9)\quad \Psi = \frac{1}{1 + 2b^2 + 3s}.\]

Now, we suppose that the case (i) of the Theorem 3.1.2 holds, then for a
metric \(F = \frac{\alpha^2}{\alpha + \beta}\)

\[(3.3.10)\quad \Delta = \frac{1 + 2b^2 + 3s}{(1 + 2s)^2}.\]

It follows that the \((1 + 2s)^2\) is a polynomial in \(s\) of degree two.

\[(3.3.11)\quad \phi\Delta^2 = \frac{\phi(1 + 2b^2 - 3s)^2}{(1 - 2s)^4}.\]

Hence, if case (ii) of Theorem 3.1.2 holds, then we get

\[(3.3.12)\quad \Phi = -2(n + 1)k\phi \frac{(1 + 2b^2 + 3s)^2}{(b^2 - s^2)(1 + 2s)^4}\]

\[(3.3.13)\quad (b^2 - s^2)(1 + 2s)^4\Phi = -2(n + 1)k\phi(1 + 2b^2 + 3s)^2.\]

This shows that \((b^2 - s^2)(1 + 2s)^4\Phi\), is not a polynomial in \(s\) (if \(k = 0\), by taking the Cartan torsion equation, we have a contradiction).

Now, we put

\[\phi\Delta^2 = \frac{\bar{\Delta}}{(1 - 2s)^4},\]
where \( \bar{\Delta} = \phi(1 + 2b^2 + 3s)^2 \).

But our assumption \( F \) is not Randers type metric. Thus, \( \bar{\Delta} \) is not a polynomial in \( s \). So, the expression \( (b^2 - s^2)(1 + 2s)^4\Phi \) is also not a polynomial in \( s \).

Next, we consider another formula for

\[
\Phi = -(Q - sQ')(n + 1)\Delta + (b^2 - s^2)
[\[(Q - sQ')Q' - (1 + sQ)Q'']
\]

where

\[
(Q - sQ') = -\frac{1 + 4s}{(1 + 2s)^2},
\]

\[
(1 + sQ) = \frac{1 + s}{(1 + 2s)}
\]

\[
\Phi = \frac{(1 + 4s)}{(1 + 2s)^2}(n + 1)\frac{(1 + 2b^2 + 3s)}{(1 + 2s)^2} + \frac{6(b^2 - s^2)}{(1 + 2s)^4}.
\]

This implies

\[
\Phi = \frac{(n + 1)(1 + 4s)(1 + 2b^2 + 3s) + 6(b^2 - s^2)}{(1 + 2s)^4}.
\]

From equations (3.3.13) and (3.3.17), we see that \( (b^2 - s^2)(1 + 2s)^4\Phi \) is a polynomial of degree 4 in \( s \) and in \( b \) both, as the coefficient of \( s^4 \) is 6, which is not equal to zero. Hence, it is impossible that \( \Phi = 0 \). Therefore, equation (3.1.3) of Theorem 3.1.2 does not hold. Thus, case(i) of Theorem 3.1.2 holds. So, we have

\[
r_{00} = 0.
\]

\[
s_j = 0.
\]

Substituting \( r_{00} = 0 \) in Theorem 3.3.1 (1), we get

\[
\tau + \frac{2}{3}(b^2\tau - \theta\theta)\alpha^2 = -\beta\left[\frac{2}{3}\theta - \frac{\pi\beta}{3}(25 + 54b^2)\right].
\]
Since $\alpha^2$ is irreducible polynomial of $y_i$, equation (3.3.20) reduces to

(3.3.21) \[ \tau + \frac{2}{3}(b^2\tau - \theta_i b_i) = 0. \]

(3.3.22) \[ \frac{2}{3}\theta + \frac{\tau\beta}{3}(25 + 54b^2) = 0. \]

From (3.3.22), we get

(3.3.23) \[ \theta = \frac{1}{2}(25 + 54b^2)\tau\beta. \]

Also the Theorem 3.3.1 (2) provides

(3.3.24) \[ \theta b^2 - \beta b_m\theta^m = 0. \]

From equation (3.3.21) and (3.3.23), we have

\[ \tau = 0. \]

Putting the value of $\tau = 0$ in (3.3.23), we get

\[ \theta = 0. \]

Thus, from the Theorem 3.3.1, we get

\[ r_{00} = 0, \quad s_{ij} = 0 \quad \text{and} \quad G^i_\alpha = 0. \]

Since, $r_{00} = 0$, $s_{ij} = 0$.

Hence, from (3.3.5) and (3.3.6), we obtain

(3.3.25) \[ P = 0, \quad Q^i = 0. \]

Using (3.3.25) in (3.3.4), we obtain $G^i = 0$, which completes the proof. \qed

**Theorem 3.3.3:** Suppose $F = \frac{\alpha^2}{\alpha + \beta}$ be a non-Riemannian metric on $n$-dimensional ($n > 3$) manifold $M$. Then $F$ is locally dually flat with isotropic $S$-curvature. Moreover, $S = (n + 1)cF$ if and only if the structure is locally Minkowskian.
Proof. Since, in adapted coordinate system the metric $F = \frac{\alpha^2}{\alpha + \beta}$ is dually flat and projectively flat with isotropic $S$-curvature. From Lemma 3.1.1, we have

$$\partial_k F = CF \dot{\partial}_k F.$$ 

Then, the spray coefficients $G^i = Py^i$ will be

$$P = \frac{1}{2} CF.$$ 

Since, $G^i = 0$, $P = 0$ and also $C = 0$. This implies that $\partial_k F = 0$, and then $F$ is a locally Minkowskian metric in adapted coordinate system. $\square$