Chapter 2

Effect of Emergent Carrying Capacity in an
Eco-epidemiological System

2.1 Introduction

The dynamics of predator-prey and host-parasite interaction are two distinct and interesting subject of research. Ecological systems deal with the former one while epidemiological systems deal with the later one. But predator and parasite sometime compete together and exert simultaneous pressure on a certain target species (prey/host). This sort of interacting dynamics are under purview of eco-epidemiology (35). Such types of interaction may be beneficial for both the natural enemies. Interaction of such types may transmit the disease in prey/host population only (85; 35; 37; 36; 86), transmit the disease in the predator population only (43; 87) or both the population (88; 33). Ecological systems, epidemiological systems and even in eco-epidemiological systems, carrying capacity play a central rule. Two

\footnote{A considerable part of this chapter has been published in Mathematical Methods in the Applied Sciences, Volume 39, Pages 806–823, 2016.}
types, namely, explicit carrying capacity and implicit/emergent carrying capacity are generally used to model such systems. The later one is comparatively a new concept which has some meaningful implications in many systems. The explicit carrying capacity (fixed) is considered in most of the ecological and eco-epidemiological studies as it is easy to interpret and comparatively straightforward to estimate from real life observations. But in eco-epidemiological scenario, parasite mediated competition arises. To model such a situation, explicit carrying capacity approach may not be a better choice as it involves parasite interaction. An alternative way is to use the carrying capacity which has an emergent property compounded of an intrinsic growth rate and a susceptibility of crowding. In literature, this type of carrying capacity is known as emergent carrying capacity (1).

In most of the model based studies prey dynamics is governed by the logistic equations $\frac{dS}{dt} = rS \left(1 - \frac{S}{K}\right)$, where $r$ is the intrinsic growth rate and $K$ is the explicit carrying capacity. But, when the deaths outweigh the births, the intrinsic growth rate is negative, and as a consequence, the model prediction will be unrealistic (1; 2; 3; 4; 5; 6). Such unrealistic results arise because for the wrong assumption on explicit carrying capacity which is assumed to be independent of birth and death rates. Thus, one way to model such competition by using a carrying capacity which is the emergent property of the species related to growth rate and competition. In the explicit carrying capacity model, the carrying capacity does not depend on the intrinsic growth rate (is the maximal rate at which the population would grow under ideal conditions). The intrinsic growth rate has no effect on competitive outcomes in such models. However, in some cases where the resource is limited (such as space, rigidly fixed) and limits the population size, the explicit carrying capacity can be more appropriate, which is against the motivation behind our work.

It has already been pointed out that eco-epidemiological studies are parasite-
mediated. If the parasite infect a portion of prey population, the prey population may be sub-divided into two groups, namely, susceptible prey and infected prey population. In such a case, it is preferable to use the carrying capacity implicitly rather than explicitly. In Explicit carrying capacity, competitive abilities are same for both the susceptible prey and infected prey. But due to disease in prey, competitive abilities may differ due to infection and for that, the explicit carrying capacity modeling approach does not seem appropriate. The carrying capacity is an emergent property of the species which is given by the birth rate, death rate and different competitions among them. In Emergent carrying capacity, competitive abilities are different for both the susceptible prey and infected prey. The present paper deals with an eco-epidemiological model of a predator-prey type, where the prey population is subject to infection. We use different competition coefficients within the prey population which arises due to emergent carrying capacity. This is an extension of the initial idea of Sieber et al. (7). We observed the consequences of different intra and inter class competitions in susceptible and infected prey of the system. We show that disease-modified inter-specific competition can make the three species co-existence which is impossible in the case of explicit carrying capacity.

The rest of the paper is organized as follows: section (2.2) deals with the development of the model with explicit carrying capacity and emergent carrying capacity. Also, we have discussed their relation (explicit carrying capacity is a special case of emergent carrying capacity) and some basic dynamical features of these two models in section (2.2). Mathematical Analysis of the full model, and their submodels with emergent carrying capacity have been discussed in subsections (2.3.1) and (2.3.2) respectively. Different dynamical features like global behaviors, permanence, multi-stability are discussed in the section (2.3). Finally; in this section, we have discussed the three possible cases of competitions both analytically and numerically. The paper ends with a discussion.
2.2 Modeling framework

An eco-epidemiological model will be considered to observe the dynamics of such system under the influence of emergent carrying capacity. The beauty of such models is that the host/prey population is not only a resource for the infection, but also a prey for the predator. The host population receives two actions from parasites and predators simultaneously.

2.2.1 An eco-epidemiological model with explicit carrying capacity

An eco-epidemiological model with disease is prey population will be considered. The model consists of three population, susceptible prey(s), infected prey (I) and specialist predator (P) (the prey population is only the food source). The model is basically a combination of S-I model and Rosenzweig-MacArthur predator-prey model. It is to be noted here that such a modeling approach is well studied by considering explicit carrying capacity. But the dynamics of such system under the influence of emergent carrying capacity is not well studied and demands in-depth research (89; 90; 91; 42; 92; 35; 93; 37; 36; 94; 95; 38; 96). The following assumptions are made to formulate the model:

1. In the absence of predator and disease, susceptible prey population dynamics follows the logistic growth, which is given by \( \frac{dS}{dt} = r \left(1 - \frac{S}{K}\right) S \), where \( K \) is the environmental carrying (explicit) capacity, \( r \) is the growth rate of the susceptible prey (S).

2. In the presence of disease, total prey population is divided into two classes, namely, susceptible prey (S) and infected prey (I). We assume that the infected prey are not in a state to produce but they are competing with the susceptible one for the same resources.
3. Predator (P) can not recognize susceptible prey and infected prey. They consume both the preys at same rate, but infected prey are less beneficiary to the predator compared to the susceptible one.

4. For simplicity we assume that the predator consume both the susceptible and infected prey in a linear fashion. We also assume that the disease transmission follows mass action law. This interaction is observed when the number of contacts between individuals is proportional to the host density. Another possible mechanism was explained by Rhodes and Martin (2010) (96) in the case when the infection is caused by parasite which have a free-living stage and are released by infected host individuals. They showed the host-parasite interaction in such cases arises from the classical law of mass action. The interested readers may consult the paper of Morozov (2011) for this explanation.

With these assumptions, we can write down the following mathematical model

\[
\frac{dS}{dt} = r \left(1 - \frac{S+I}{K}\right) S - \lambda SI - \alpha SP = \left[r \left(1 - \frac{S+I}{K}\right) - \lambda I - \alpha P\right] S,
\]

\[
\frac{dI}{dt} = \lambda SI - \alpha IP - \mu I = \left[\lambda S - \alpha P - \mu\right] I, \quad (2.1)
\]

\[
\frac{dP}{dt} = \left[c_1 \alpha S + c_2 \alpha I - d\right] P = \left[\gamma_1 S + \gamma_2 I - d\right] P,
\]

The variables in the parameters description are provided in table (2.1). We like to mention here that such type of models are well studied by considering explicit carrying capacity. The infected prey are less beneficiary to the predator, compared to the susceptible prey, i.e., we assume \(c_1 \geq c_2 \Leftrightarrow \gamma_1 \geq \gamma_2\).

### 2.2.2 An eco-epidemiological model with emergent carrying capacity

In model (2.1), we assume competitive abilities are same for both the susceptible prey and infected prey. But this is not a realistic assumption as there is a disease in prey population. In the presence of disease, the infected prey population will not have the same competing ability as of susceptible one. For this reason, we modify the model
by considering different competition coefficients for two possible interactions (intra-class competition in susceptible prey \(C_S\) and inter-class competition between susceptible prey and infected prey \(C_{SI}\)). Thus, model (2.1) reduces to the following

\[
\begin{align*}
\frac{dS}{dt} &= rS - (C_SS + C_{SI} I) S - \lambda SI - \alpha SP = [r - (C_SS + C_{SI} I) - \lambda I - \alpha P] S, \\
\frac{dI}{dt} &= \lambda SI - \alpha IP - \mu I = [\lambda S - \alpha P - \mu] I, \\
\frac{dP}{dt} &= \gamma_1 SP + \gamma_2 IP - dP = [\gamma_1 S + \gamma_2 I - d] P,
\end{align*}
\]

(2.2)

All the variables and the parameters are described in the table (2.1). It is to be noted here that there is no explicit carrying capacity like logistic growth model. This type of modeling approach can be described as an upper limit of population growth which arises from reproduction, competition etc. (97; 98; 7) instead of predetermined number \(K\). The basic aim of this paper is to observe the dynamics of such system under the influence of emergent carrying capacity.

Now, we state some basic results of the model (2.2).

**Theorem 2.2.1.** *Basic dynamical features* Assume that \(-\infty < \gamma_2 < \gamma_1\), then System (2.2) is positively invariant and bounded in \(X\) with the following property

\[
\limsup_{t \to \infty} \{S(t) + I(t)\} \leq \frac{r}{C_S}.
\]

In addition, we have

\[
\text{if } \lambda S \leq \mu, \text{ then } \limsup_{t \to \infty} I(t) = 0.
\]

**Proof.** For any \(S \geq 0, I \geq 0, P \geq 0\), we have

\[
\left. \frac{dS}{dt} \right|_{S=0} = 0, \quad \left. \frac{dI}{dt} \right|_{I=0} = 0 \quad \text{and} \quad \left. \frac{dP}{dt} \right|_{P=0} = 0
\]

which implies that \(S = 0, I = 0\) and \(P = 0\) are invariant manifolds, respectively.

Due to the continuity of the system, we can easily conclude that System (2.2) is positively invariant in \(\mathbb{R}_+^3\).
Choose any point \((S, I, P) \in X\) such that \(S > \frac{r}{C_S}\), then due to the positive invariant property of (2.2), we have
\[
\left. \frac{dS}{dt} \right|_{S > \frac{r}{C_S}} = \left[ r - (C_SS + C_SI) - \lambda I - \alpha P \right] S < 0.
\]
In addition, we have \(\frac{dS}{dt} \bigg|_{S=\frac{r}{C_S}, I=0, P=0} = 0\) and \(\frac{dS}{dt} \bigg|_{S=\frac{r}{C_S}, I+P>0} < 0\), thus, we can conclude that
\[
\limsup_{t \to \infty} S(t) \leq \frac{r}{C_S}.
\]
Now we define two functions by
\[
N(t) = S(t) + I(t) \quad \text{and} \quad Z(t) = S(t) + I(t) + \frac{1}{c_1}P(t),
\]
then we have
\[
\frac{dN(t)}{dt} = rS - (C_SS + C_SI) S - \mu I - \alpha P [S + I] \leq rS \left( 1 - \frac{C_SS + C_SI(N-S)}{r} \right) - \mu I,
\]
\[
\frac{dZ(t)}{dt} = rS - (C_SS + C_SI) S - \mu I - \frac{d}{c_1} P - \alpha P \left[ 1 - \frac{c_2}{c_1} \right] I.
\]
Since \(\limsup_{t \to \infty} S(t) \leq \frac{r}{C_S}\), then for any \(\epsilon > 0\), there is a \(T\) large enough such that for any \(t > T\), we have
\[
\frac{dN(t)}{dt} \leq rS \left( 1 - \frac{C_SS + C_SI}{r} \right) + \frac{\mu}{r} - (C_SI + \mu) N \leq r \left( \frac{r}{C_S} + \epsilon \right) \left[ 1 - \frac{C_SI}{r} \left( \frac{r}{C_S} + \epsilon \right) + \frac{\mu}{r} \right] N.
\]
By applying the theory of differential inequality (99) (or Gronwall's inequality) and letting \(\epsilon \to 0\), we obtain
\[
\limsup_{t \to \infty} N(t) = \limsup_{t \to \infty} \{S(t) + I(t)\} \leq \frac{r}{C_S}.
\]
This implies that \(I(t)\) is bounded. Similarly, since \(-\infty < \gamma_2 < \gamma_1\) and \(c_2 < c_1\), then we have
\[
\frac{dZ(t)}{dt} = S \left[ r - (C_SS + C_SI) \right] - \mu I - \frac{d}{c_1} P - \alpha \left( 1 - \frac{c_2}{c_1} \right) IP \leq S \left[ r - (C_SS + C_SI) \right] - \mu I - \frac{d}{c_1} P \leq S \left[ r - (C_SS + C_SI) \right] + \min \left\{ \mu, \frac{d}{c_1} \right\} S - \min \left\{ \mu, \frac{d}{c_1} \right\} Z = L - KZ
\]
where
\[
L = \max_{0 \leq S \leq \frac{r}{C_S}, 0 \leq N \leq \frac{r}{C_S}} \left\{ S \left[ r - (C_SS + C_SI) \right] + KS \right\} \quad \text{and} \quad K = \min \left\{ \mu, \frac{d}{c_1} \right\}.
\]
This implies that \( \limsup_{t \to \infty} Z(t) = \limsup_{t \to \infty} \{S(t) + I(t) + P(t)\} \leq \frac{L}{K} \), i.e., \( P(t) \) is also bounded. Therefore, System (2.2) is positively invariant and bounded in \( X \).

\[ \square \]

In the next subsection we will see that this explicit carrying capacity model can be easily derived from the emergent carrying capacity model. Thus explicit carrying capacity is a special case of emergent carrying capacity.

### 2.2.3 Explicit carrying capacity is a special case of emergent carrying capacity

From the model (2.1) and (2.2), one can easily compute the following relation between explicit carrying capacity and emergent carrying capacity, \( C_S = C_{SI} = \frac{r}{K} \). Thus, we conclude that explicit carrying capacity is a special case of emergent carrying capacity. The following three cases arise

1. Intra-class competition in susceptible prey is less than the inter-class competition between susceptible prey with infected prey, i.e., \( C_{SI} > C_S \).

2. Infection has no rule in competitive interactions at all. In such a situation both the intra and inter class competitions are same (as explicit carrying capacity), i.e., \( C_S = C_{SI} = \frac{r}{K} \).

3. Intra-class competition in susceptible prey is greater than the inter-class competition between susceptible prey with infected prey, i.e., \( C_S > C_{SI} \). In this situation infected prey are less competitive compared to susceptible one.

**Note:** From the theorem (2.2.1), we can easily prove that (by putting \( C_S = C_{SI} = \frac{r}{K} \)) the solutions of the model (2.1) are also bounded in \( \mathbb{R}^3_+ \).

Now, we present the dynamical behavior of the system (2.2). We begin with the following submodels.
<table>
<thead>
<tr>
<th>Variables/Parameters</th>
<th>Biological meaning</th>
</tr>
</thead>
<tbody>
<tr>
<td>$S$</td>
<td>Density of susceptible prey</td>
</tr>
<tr>
<td>$I$</td>
<td>Density of infected prey</td>
</tr>
<tr>
<td>$P$</td>
<td>Density of predator</td>
</tr>
<tr>
<td>$K$</td>
<td>Explicit carrying capacity</td>
</tr>
<tr>
<td>$C_S$</td>
<td>Intra-class competition between susceptibles</td>
</tr>
<tr>
<td>$C_{SI}$</td>
<td>Inter-class impact of infected on susceptibles</td>
</tr>
<tr>
<td>$\lambda$</td>
<td>Rate of infection</td>
</tr>
<tr>
<td>$\alpha$</td>
<td>Attack rate of predator on both preys</td>
</tr>
<tr>
<td>$c_1$</td>
<td>Conversion efficiency of $P$ on $S$</td>
</tr>
<tr>
<td>$c_2$</td>
<td>Conversion efficiency of $P$ on $I$</td>
</tr>
<tr>
<td>$\gamma_1$</td>
<td>The total effect to predator by consuming susceptible prey</td>
</tr>
<tr>
<td>$\gamma_2$</td>
<td>The total effect to predator by consuming infected prey</td>
</tr>
<tr>
<td>$\mu$</td>
<td>Death rate of infected prey</td>
</tr>
<tr>
<td>$d$</td>
<td>Natural death rate of predator</td>
</tr>
</tbody>
</table>

Table 2.1: Variables and parameters used in Models (2.1) and (2.2).

### 2.3 Mathematical Analysis:

#### 2.3.1 Dynamics of submodels

In order to understand the full dynamics of (2.2), we first study the dynamics of the following two submodels

1. The prey-predator model in the absence of the disease in (2.2) is presented as

\[
\frac{dS}{dt} = rS - (C_S S)S - \alpha SP = [r - C_S S - \alpha P] S,
\]
\[
\frac{dP}{dt} = \gamma_1 S - d P = \gamma_1 \left[ S - \frac{1}{R^0} \right] P.
\]  

2. We introduce a disease free demographic reproduction number $R^0$, which gives the expected number of offspring $\gamma_1$ of an average individual predator in its...
lifetime $\frac{1}{d}$ i.e. the ratio between net gain by consuming the susceptible prey and the natural death rate of the predator. So, we have

$$R_0^p = \left[ \frac{\gamma_1}{d} \right] = \frac{\text{Total effect to predator by consuming the susceptible prey}}{\text{Predators mortality}}.$$

From the second equation of (2.4), we can easily conclude that, for the existence of predator $R_0^p$ must be greater than one, when the susceptible prey is at unit density. On the other hand, for $R_0^p \leq 1$ predator population will extinct eventually.

2. The system (2.2) in the absence of predation reduces to the following SI model.

$$\frac{dS}{dt} = [r - (C_S S + C_S I) - \lambda I] S,$$
$$\frac{dI}{dt} = [\lambda S - \mu] I = \lambda \left[ S - \frac{1}{R_0^I} \right] I. \quad (2.5)$$

We introduce the disease basic reproduction number $R_0^I$, which is the expected number of secondary cases produced by one typical infected individual in an otherwise disease free population. The average infectious period of a typical infected individual is $\frac{1}{\mu}$, and an infected individual will on average infect $\lambda$ individuals per unit of time when the population is fully susceptible. Thus, we have

$$R_0^I = \left[ \frac{\lambda}{\mu} \right] = \left[ \frac{\text{Number of secondary infections}}{\text{Inverse of the average infectious period}} \right].$$

Here also, from the second equation of (2.5), we can easily conclude that for the persistence of the disease $R_0^I$ must be greater than one, when the susceptible prey is at unit density. For the eradication of disease we need $R_0^I \leq 1$.

**Equilibria and their stability for the submodels (2.4) and (2.5)**

It is easy to check that both the submodels (2.4) and (2.5) have

$$(0,0) \text{ and } \left( \frac{r}{C_S},0 \right)$$
as their boundary equilibria. Also, both the submodels have their unique interior attractors. For convenience, we can write

\[ E^P_0 = (0, 0), \ E^P_1 = \left( \frac{r}{C_S}, 0 \right) \] and \[ E^P_i = (S^*, P^*) = \left( \frac{1}{R^P_0}, \frac{1}{\alpha} \left( r - \frac{C_s}{R^P_0} \right) \right) \]

for the submodel (2.4), while for the model (2.5), we can write

\[ E^I_0 = (0, 0), \ E^I_1 = \left( \frac{r}{C_S}, 0 \right) \] and \[ E^I_i = (S^*, I^*) = \left( \frac{1}{R^I_0}, \frac{1}{\lambda + C_{SI}} \left( r - \frac{C_s}{R^I_0} \right) \right) \].

The interior attractors for the submodels (2.4) and (2.5) exists if the conditions \( R^P_0 > \frac{C_s}{r} \) and \( R^I_0 > \frac{C_s}{r} \) hold respectively.

The stability of all equilibria for both the submodels (2.4) and (2.5) are summarized in the following proposition:

**Proposition 2.3.1.** [Local stability of equilibria for submodels (2.4) and (2.5)] The existence and local stability of boundary and interior equilibria of both submodels (2.4) and (2.5) are summarized in Table 2.2.

<table>
<thead>
<tr>
<th>Equilibria</th>
<th>Existence Condition</th>
<th>Stability Condition</th>
</tr>
</thead>
<tbody>
<tr>
<td>( E^P_0 ) and ( E^I_0 )</td>
<td>Always</td>
<td>Always saddle</td>
</tr>
<tr>
<td>( E^P_1 )</td>
<td>Always</td>
<td>LAS if ( R^P_0 &lt; \frac{C_s}{r} )</td>
</tr>
<tr>
<td>( E^I_1 )</td>
<td>Always</td>
<td>LAS if ( R^I_0 &lt; \frac{C_s}{r} )</td>
</tr>
<tr>
<td>( E^P_i )</td>
<td>( R^P_0 &gt; \frac{C_s}{r} )</td>
<td>LAS if ( R^P_0 &gt; \frac{C_s}{r} )</td>
</tr>
<tr>
<td>( E^I_i )</td>
<td>( R^I_0 &gt; \frac{C_s}{r} )</td>
<td>LAS if ( R^I_0 &gt; \frac{C_s}{r} )</td>
</tr>
</tbody>
</table>

Table 2.2: The local stability of boundary equilibria for both submodels (2.4) and (2.5). LAS \( \equiv \) Locally Asymptotically Stable.

**Proof.** The Jacobian matrix of the submodel (2.4) at its equilibrium \((S^*, P^*)\) is presented as follows

\[
J^P \bigg|_{(S^*, P^*)} = \begin{bmatrix}
    r - 2C_SS^* - \alpha P^* & -\alpha S^*\\
    \gamma_1 P^* & \gamma_1 \left( S^* - \frac{1}{R^P_0} \right)
\end{bmatrix}
\] (2.6)
while the Jacobian matrix of the submodel (2.5) at its equilibrium \((S^*, I^*)\) is presented as follows

\[
J^I\bigg|_{(S^*, I^*)} = \begin{bmatrix}
    r - (2CS^* + CSI^*) - \lambda I^* - (CSI + \lambda) S^* \\
    \lambda S^* - \frac{1}{R_0^P}
\end{bmatrix}
\]  

(2.7)

After substituting \((S^*, P^*) = E^P_u, \ u = 0, 1, i\) into (2.6), we obtain the eigenvalues for each equilibrium of the model (2.4):

1. \(E^P_0 = (0, 0)\) is always saddle, since both the eigenvalues associated with (2.6) at \(E^P_0\) are given by:
   \[
   \lambda_1 = r \ (> 0) \text{ and } \lambda_2 = -d \ (< 0)
   \]

2. \(E^P_1 = \left(\frac{r}{CS}, 0\right)\) is locally asymptotically stable if \(R_0^P < \frac{CS}{r}\) and it is a saddle if \(R_0^P > \frac{CS}{r}\) since both the eigenvalues associated with (2.6) at \(E^P_1\) are given by:
   \[
   \lambda_1 = -r \ (< 0) \text{ and } \lambda_2 = \gamma_1 \left(\frac{r}{CS} - \frac{1}{R_0^P}\right) = \begin{cases} 
   < 0 & \text{if } R_0^P < \frac{C_S}{r} \\
   > 0 & \text{if } R_0^P > \frac{C_S}{r}
   \end{cases}
   \]

3. \(E^P_i = \left(\frac{1}{R_0^P}, \frac{1}{\alpha} \left(r - \frac{CS}{R_0^P}\right)\right)\) exists and is locally asymptotically stable if \(R_0^P > \frac{C_S}{r}\).
   The characteristic equation associated with this equilibrium \(E^P_i\) is given by
   \[
   x^2 + \frac{C_S}{R_0^P} x + \frac{\gamma_1}{R_0^P} \left(r - \frac{C_S}{R_0^P}\right) = 0.
   \]
   It can be easily verified that both the roots of this quadratic equation will be either real negative or complex conjugate with negative real parts if \(R_0^P > \frac{C_S}{r}\).
   Also, the system (2.4) undergoes a supercritical Hopf-bifurcation at \(R_0^P = \frac{C_S}{r}\).

Similarly, after substituting \((S^*, I^*) = E^I_u, \ u = 0, 1, i\) into (2.7), we obtain the eigenvalues for each equilibrium:

1. \(E^I_0 = (0, 0)\) is always saddle since both the eigenvalues associated with (2.7) at \(E^I_0\) are given by:
   \[
   \lambda_1 = r \ (> 0) \text{ and } \lambda_2 = -\mu \ (< 0).
   \]
2. $E^I_1 = \left( \frac{r}{C_S}, 0 \right)$ is locally asymptotically stable if $R^I_0 < \frac{C_S}{r}$ and it is a saddle if $R^I_0 > \frac{C_S}{r}$, since both the eigenvalues associated with (2.7) at $E^I_1$ are given by:

$$\lambda_1 = -r \ (< 0) \text{ and } \lambda_2 = \frac{r}{C_S} - \frac{1}{R^I_0} = \begin{cases} < 0 & \text{if } R^I_0 < \frac{C_S}{r} \\ > 0 & \text{if } R^I_0 > \frac{C_S}{r}. \end{cases}$$

3. $E^I_i = \left( \frac{1}{R^I_0}, \frac{1}{\lambda + C_{SI}} \left( r - \frac{C_S}{R^I_0} \right) \right)$ exists and is locally asymptotically stable if $R^I_0 > \frac{C_S}{r}$. The characteristic equation associated with this equilibrium $E^I_i$ is given by

$$x^2 + \frac{C_S}{R^I_0}x + \frac{\lambda}{R^I_0} \left( r - \frac{C_S}{R^I_0} \right).$$

It can be easily verified that both the roots of this quadratic equation will be either real negative or complex conjugate with negative real parts if $R^I_0 > \frac{C_S}{r}$. Also, the system (2.5) undergoes a supercritical Hopf-bifurcation at $R^I_0 = \frac{C_S}{r}$.

\[\square\]

2.3.2 Dynamics of the full model

Finally comes to the dynamics of the full $SIP$-model with emergent carrying capacity. The full model (2.2) can be written as

\[
\begin{align*}
\frac{dS}{dt} &= rS - (C_S S + C_{SI} I) S - \lambda S I - \alpha SP = [r - (C_S S + C_{SI} I) - \lambda I - \alpha P] S, \\
\frac{dI}{dt} &= \lambda S I - \alpha I P - \mu I = \lambda \left[ S - \frac{\alpha P}{\lambda} - \frac{1}{R^I_0} \right] I, \\
\frac{dP}{dt} &= \gamma_1 S P + \gamma_2 I P - dP = \gamma_1 \left[ S + \frac{\gamma_2}{\gamma_1} I - \frac{1}{R^I_0} \right] P, \\
\end{align*}
\]

(2.8)

This system has the following boundary equilibria

$$E_0 = (0, 0, 0), \quad E_1 = \left( \frac{r}{C_S}, 0, 0 \right), \quad E_2 = E_{SP} = \left( \frac{1}{R^I_0}, 0, \frac{1}{\alpha} \left( r - \frac{C_S}{R^I_0} \right) \right) = (S_2, 0, P_2),$$

$$E_3 = E_{SI} = \left( \frac{1}{R^I_0}, \frac{1}{\lambda + C_{SI}} \left( r - \frac{C_S}{R^I_0} \right), 0 \right) = (S_3, I_3, 0)$$

**Proposition 2.3.2.** [Boundary equilibria and stability] Sufficient conditions for the existence and the local stability of boundary equilibria for system (2.1) are summarized in Table (2.3.2).
<table>
<thead>
<tr>
<th>Boundary equilibria</th>
<th>Existence Condition</th>
<th>Stability condition</th>
</tr>
</thead>
<tbody>
<tr>
<td>$E_0$</td>
<td>Always</td>
<td>Always saddle</td>
</tr>
<tr>
<td>$E_1$</td>
<td>Always</td>
<td>LAS if $R_0^I &lt; \frac{C_S}{r}$ and $R_0^P &lt; \frac{C_S}{r}$</td>
</tr>
<tr>
<td>$E_{SP}$</td>
<td>$R_0^P &gt; \frac{C_S}{r}$</td>
<td>LAS if $R_0^P &gt; \max\left{\frac{C_S}{r}, \frac{R_0^P (\lambda + C_S)}{\lambda + R_0^P}\right}$</td>
</tr>
<tr>
<td>$E_{SI}$</td>
<td>$R_0^I &gt; \frac{C_S}{r}$</td>
<td>LAS if $R_0^I &gt; \max\left{\frac{C_S}{r}, \frac{R_0^I [\gamma_1 (\lambda + C_SI) - \gamma_2 C_SI]}{\gamma_1 (\lambda + C_SI) - R_0^I r \gamma_2}\right}$</td>
</tr>
</tbody>
</table>

Table 2.3: Sufficient conditions for the existence and stability of boundary equilibria for system (2.1)

**Proof.** The Jacobian matrix of the submodel at its equilibrium $(S^*, I^*, P^*)$ is presented as follows.

$$
J_{(S^*, I^*, P^*)} = 
\begin{bmatrix}
    r - (2C_S S^* + C_SI I^*) - \lambda I^* - \alpha P^* - (C_SI + \lambda) S^* & -\alpha S^* \\
    \lambda I^* & \lambda S^* - \alpha P^* - \frac{\lambda}{R_0^P} & -\alpha I^* \\
    \gamma_1 P^* & \gamma_2 P^* & \gamma_1 S^* + \gamma_2 I^* - \frac{\gamma_1}{R_0^P}
\end{bmatrix}
$$

The local stability of equilibrium can be determined by the eigenvalues $\lambda_i$, $i = 1, 2, 3$ of the Jacobian matrix (2.3.2) evaluated at the equilibrium. After substituting $(S^*, I^*, P^*) = E_0, E_1, E_{SP}, E_{SI}$ into (2.3.2), we obtain the eigenvalues for each equilibrium:

1. $E_0 = (0, 0, 0)$ is always saddle since both the eigenvalues associated with (2.3.2) at $E_0$ are given by:

$$
\lambda_1 = r \ (> 0), \quad \lambda_2 = -\frac{\lambda}{R_0^I} \ (< 0) \quad \text{and} \quad \lambda_3 = -\frac{\gamma_1}{R_0^P} \ (< 0)
$$

2. $E_1 = \left(\frac{r}{C_S}, 0, 0\right)$ is locally asymptotically stable if both $R_0^I < \frac{C_S}{r}$ and $R_0^P < \frac{C_S}{r}$ and it is a saddle if at least one of $R_0^I$ and $R_0^P$ is greater than $\frac{C_S}{r}$, since both the eigenvalues associated with (2.3.2) at $E_1$ can be written as: $\lambda_1 = -r \ (< 0)$, $\lambda_2 = \lambda \left(\frac{r}{C_S} - \frac{1}{R_0^I}\right) \quad \left\{\begin{array}{ll}
    < 0 & \text{if} \ R_0^I < \frac{C_S}{r} \\
    > 0 & \text{if} \ R_0^I > \frac{C_S}{r}
\end{array}\right.$ and $\lambda_3 = \gamma_1 \left(\frac{r}{C_S} - \frac{1}{R_0^P}\right) \quad \left\{\begin{array}{ll}
    < 0 & \text{if} \ R_0^P < \frac{C_S}{r} \\
    > 0 & \text{if} \ R_0^P > \frac{C_S}{r}
\end{array}\right.$

Where the sign of $\lambda_i$ indicates its eigenvector pointing toward ($< 0$) or away ($> 0$) from the equilibrium in $S$-axis ($i = 1$), $I$-axis ($i = 2$) and $P$-axis ($i = 3$), respectively.
According to Proposition (2.3.1), the equilibrium \( E_{SP} \), is locally asymptotically stable only if it is locally asymptotically stable in the submodel (2.4) and

\[
\left. \frac{dI}{dt} \right|_{E_{SP}} = \lambda S - \alpha P - \frac{\lambda}{R_0^P} I_{E_{SP}} < 0 \Leftrightarrow R_0^P > \frac{R_0^P (\lambda + C_S)}{\lambda + r R_0^I}.
\]

Similarly, the equilibrium \( E_{SI} \) is locally asymptotically stable only if it is locally asymptotically stable in the submodel (2.5) and

\[
\left. \frac{dP}{dt} \right|_{E_{SI}} = \gamma_1 S + \gamma_2 I - \frac{\gamma_1}{R_0^P} P_{E_{SI}} < 0 \Leftrightarrow R_0^P > \frac{R_0^P [\gamma_1 (\lambda + C_{SI}) - \gamma_2 C_S]}{\gamma_1 (\lambda + C_{SI}) - R_0^P \gamma_2 r}.
\]

\[\square\]

### 2.3.3 Permanence:

In order to prove permanence of the system (2.2), we present the uniform persistence theory for infinite dimensional systems following the theory of Hale (100). Let \( X \) be a complete metric space. Suppose that \( X^0 \) is open and dense in \( X \) and \( X^0 \cap X_0 = \Phi \). Assume that \( T(t) \) is a \( C^0 \) semigroup on \( X \) satisfying

\[
T(t) : X^0 \to X^0, \ T(t) : X_0 \to X_0.
\]

Let \( T_b(t) = T(t)|_{X_0} \) and let \( A_b \) be the global attractor for \( T_b(t) \).

To investigate the permanence of the system (2.2) the following lemmas are useful.

**Lemma 2.3.1.** [Hale (100)] If \( T(t) \) satisfies (2.9) and we have the following:

1. There is a \( t_0 \geq 0 \) such that \( T(t) \) is compact for \( t \geq t_0 \);
2. \( T(t) \) is a point dissipative in \( X \);
3. \( \hat{A}_b = \cup_{x \in A_b} \omega(x) \) is isolated and thus has an acyclic covering \( \hat{M} \), where \( \hat{M} = \{ M_1, M_2, \ldots, M_n \} \);
4. $W^s(M_i) \cap X^0 = \emptyset$ for $i = 1, 2, ..., n$. Then $X_0$ is an uniform repeller with respect to $X^0$, i.e. there is an $\epsilon > 0$ such that for any $x \in X^0$, $\liminf_{t \to \infty} d(T(t)x, X_0) \geq \epsilon$, where $d$ is the distance of $T(t)x$ from $X_0$.

**Theorem 2.3.2.** System (2.2) is permanent provided

1. $\frac{r-a(P_2+\epsilon_1)-C_4(\mu+a(P_2+\epsilon_1))}{\lambda-C_{SI}} > 0$, where $\epsilon_1$, is sufficiently small.

2. $\gamma_2(I_3-\epsilon_2) > d$, where $\epsilon_2$, is sufficiently small.

**Proof.** In this section, we shall show that the boundary planes of $R^3_+$ repel the positive solutions of system (2.2) uniformly. Let us define

$$C_1 = \{(S,I,P) \in R^3_+ : S(0) \neq 0, I(0) = 0\},$$

$$C_2 = \{(S,I,P) \in R^3_+ : P(0) = 0, S(0)I(0) \neq 0\},$$

$$C_3 = \{(S,I,P) \in R^3_+ : I(0) = 0, S(0)P(0) \neq 0\}.$$

If $C_0 = C_1 \cup C_2 \cup C_3$ and $C^0 = \text{int } R^3_+$, it suffices to show that there exists an $\epsilon_0 > 0$ such that for any solution $u_t$ of system (2.2) initiating from $C_0^0$, $\liminf_{t \to \infty} d(u_t, C_0) \geq \epsilon_0$.

Now, we verify below that the conditions of Lemma (2.3.1) are satisfied. By definition of $C_0$ and $C^0$ and system (2.2), it is easy to see that $C_0$ and $C^0$ are positively invariant. Moreover, it is clear that conditions 1 and 2 of Lemma (2.3.1) are satisfied. Thus we only need to verify conditions 3 and 4. There are three constant solutions $\tilde{E}_1$, $\tilde{E}_2$ and $\tilde{E}_3$ in $C_0$ corresponding to $(S(t) = \frac{r}{C_1}, I(t) = 0, P(t) = 0)$, $(S(t) = S_2, I(t) = 0, P(t) = P_2)$ and $(S(t) = S_3, I(t) = I_3, P(t) = 0)$ respectively, i.e. if $(S(t), I(t), P(t))$ is any solution of system (2.2) initiating from $C_1$ with $S(0) > 0$, it follows that $S(t) \to \frac{r}{C_1}$, $I(t) \to 0$ and $P(t) \to 0$ as $t \to \infty$. If $(S(t), I(t), P(t))$ is a solution of system (2.2) initiating from $C_2$ with $S(0) > 0$, $P(0) > 0$, it follows that $S(t) \to S_2$, $I(t) \to 0$, $P(t) \to P_2$ as $t \to \infty$. If $(S(t), I(t), P(t))$ is a solution of
system (2.2) initiating from \(C_3\) with \(S(0) > 0, I(0) > 0\), it follows that \(S(t) \rightarrow S_3, I(t) \rightarrow I_3, P(t) \rightarrow 0\) as \(t \rightarrow \infty\).

This shows that invariant sets \(\tilde{E}_1, \tilde{E}_2\) and \(\tilde{E}_3\), are isolated invariant, then \(\{\tilde{E}_1, \tilde{E}_2, \tilde{E}_3\}\) is an isolated as well as an acyclic covering, satisfying condition 3 of Lemma (2.3.1).

We now show that \(W^s(\tilde{E}_1) \cap C_0 = \emptyset, W^s(\tilde{E}_2) \cap C_0 = \emptyset\) and \(W^s(\tilde{E}_3) \cap C_0 = \emptyset\).

The proof for the first is simple, so we ignore it. For the proof of the second equation, we assume contradiction \(W^s(\tilde{E}_2) \cap C_0 \neq \emptyset\), then there exists a positive solution \((S(t), I(t), P(t))\) of system (2.2) such that \((S(t), I(t), P(t)) \rightarrow (S_2, 0, P_2)\) as \(t \rightarrow +\infty\). Let us choose \(\epsilon_1 > 0\) small enough such that

\[
\frac{r - \alpha (P_2 + \epsilon_1) - C_s \frac{(\mu + \alpha(P_2 + \epsilon_1))}{\lambda} - C_{SI}}{\lambda - C_{SI}} > 0
\]

and

\[
P_2 - \epsilon_1 < P(t) < P_2 + \epsilon_1
\]

for some large \(t > t_1\), where \(t_1\) be sufficiently large. Then from third equation of the system (2.2) we have, for \(t > t_1\)

\[
\frac{dS(t)}{dt} \geq S \left[ r - (C_S S + C_{SI} I) - \lambda I - \alpha (P_2 + \epsilon_1) \right] \tag{2.10}
\]

\[
\frac{dI(t)}{dt} \geq I \left[ \lambda S - \mu - \alpha (P_2 + \epsilon_1) \right].
\]

Now let us consider

\[
\frac{dx_1(t)}{dt} \geq x_1 \left[ r - (C_S x_1 + C_{SI} x_2) - \lambda x_2 - \alpha (P_2 + \epsilon_1) \right] \tag{2.11}
\]

\[
\frac{dx_2(t)}{dt} \geq x_2 \left[ \lambda x_1 - \mu - \alpha (P_2 + \epsilon_1) \right].
\]

Let \(v = (v_1, v_2)\) and \(\zeta > 0\) be small enough such that \(\zeta v_1 < S(t_1), \zeta v_2 < I(t_1)\). If \((x_1(t), x_2(t))\) is a solution of system (2.2) satisfying \(x_i(t_1) = \zeta v_i, i = 1, 2\). We know from comparison theorem \(S(t) > x_1(t), I(t) > x_2(t)\) for all \(t > t_1\). \(T(t)\) is easy to see that system has a unique positive equilibrium

\[
(x_1^*, x_2^*) = \left( \frac{\mu + \alpha (P_2 + \epsilon_1)}{\lambda}, \frac{r - \alpha (P_2 + \epsilon_1) - C_s \frac{(\mu + \alpha(P_2 + \epsilon_1))}{\lambda} - C_{SI}}{\lambda - C_{SI}} \right).
\]
which is globally asymptotically stable, from ... Now $S(t) > x_1(t), I(t) > x_2(t)$ for all $t > t_1$ and $\lim_{t \to \infty} x_2(t) = x_2^*$. This is a contradiction. Hence $W^*(\bar{E}_2) \cap C^0 = \Phi$.

Let $W^*(\bar{E}_3) \cap C^0 \neq \Phi$. Then there exists a positive solution $(S(t), I(t), P(t))$ of system such that $(S(t), I(t), P(t)) \to (S_3, 0, P_3)$ as $t \to \infty$. Let us choose $\epsilon_2 > 0$ small enough such that $I_3 - \epsilon_2 < I(t) < I_3 + \epsilon_2$ for $t > t_2$ and $\gamma_2(I_3 - \epsilon_2) > d$.

Then from third equation of the system (2.2) we have, for $t > t_2$

$$\frac{dP(t)}{dt} \geq P(\gamma_2(I_3 - \epsilon_2) - d) \quad (2.12)$$

Now let us consider

$$\frac{dz(t)}{dt} \geq z(\gamma_2(I_3 - \epsilon_2) - d) \quad (2.13)$$

Let $u_1$ and $w > 0$ be small enough such that $wu_1 < P(t_2)$. If $z_1$ is a solution of system (2.13) satisfying $z_1(t_1) = wu_1$, we know from comparison theorem (see (37)), $P(t) \geq y_1(t)$ for all $t > t_2$. From condition 1 of Lemma (2.3.1), we observe that the solution $z_1$ of Equation (2.13), $\lim_{t \to \infty} z_1(t) \to \infty$, when the conditions 1 - 2 of this theorem is satisfied.

Since $P(t) \geq z_1(t)$ for all $t > t_2$ and so $\lim_{t \to \infty} P(t) \to 0$. This is a contradiction. Hence $W^*(\bar{E}_3) \cap C^0 = \Phi$.

Then we conclude from Lemma (2.3.1) that $C_0$ repels the positive solutions of (2.2) uniformly. Hence the system (2.2) is permanent. This proves the theorem. □

### 2.3.4 Global behavior

In this subsection we look at the disease-free equilibrium point from the global perspective.
Theorem 2.3.3. The disease-free equilibrium $E_{SP}$ is globally asymptotically stable if, the following conditions are satisfied

1. $R_0^p < \frac{\lambda + C_S}{\mu + r}$,

2. $\alpha$ is small and

3. $r > \mu + \frac{(C_S - \lambda)\gamma_1}{\gamma_2}$

Proof. We rewrite system (2.2) as

$$
\frac{dS}{dt} = Sf_1(S, I, P),
\frac{dI}{dt} = If_2(S, I, P),
\frac{dP}{dt} = Pf_3(S, I, P),
$$

(2.14)

where $f_1(S, I, P) = r - (C_S S + C_S I) - \lambda I - \alpha P$, $f_2(S, I, P) = \lambda S - \alpha P - \mu$ and $f_3(S, I, P) = \gamma_1 S + \gamma_2 I - d$. The system (2.2) has a unique disease-free planner equilibrium $E_{SP} = (S_1, 0, P_1)$, where $S_1 = \frac{1}{R_0^p}$ and $P_1 = \frac{1}{\alpha} \left( r - \frac{C_S}{R_0^p} \right)$. Let us define

$$
F_1(x) = f_2(x, 0, P_1),
F_2(y) = -f_1(S_1, y, P_1),
F_3(z) = -[f_2(S_1, 0, z) + f_1(S_1, 0, z)].
$$

As $\frac{\partial f_2}{\partial S} > 0$, $\frac{\partial f_1}{\partial I} < 0$, $\frac{\partial f_2}{\partial P} < 0$ and $\frac{\partial f_1}{\partial P} < 0$ the functions $F_1$, $F_2$, $F_3$ are strictly increasing in $x$, $y$, $z$, respectively. We next consider the function

$$
V = \int_{S_1}^{S} \frac{F_1(x)}{x} dx + \int_{0}^{P} \frac{F_2(y)}{y} dy + \int_{P_1}^{P} \frac{F_3(z)}{z} dz.
$$

From the construction of $V$, it is seen that $V$ is positive definite in the region

$$
\Omega = \left\{ (S, I, P) : S \geq S_1; I \geq 0; P \geq P_1 \right\};
$$
also, \( V(S_1, 0, P_1) = 0 \). Now,

\[
\frac{dV}{dt} = F_1(S)f_1(S, I, P) + F_2(I)f_2(S, I, P) + F_3(P)f_3(S, I, P)
\]

\[
= F_1(S) \left[ f_1(S, I, P) - f_1(S_1, I, P_1) \right] + F_2(I) \left[ f_2(S, I, P) - f_2(S, 0, P_1) \right]
\]

\[
+ F_3(P) \left[ f_3(S, I, P) - f_3(S_1, 0, P) \right] + F_1(S)f_1(S_1, I, P_1) + F_2(I)f_2(S, 0, P_1)
\]

\[
+ F_3(P)f_3(S_1, 0, P)
\]

\[
= F_1(S) \left[ (S - S_1) \frac{\partial f_1}{\partial S}(\tilde{S}, I, P) + (P - P_1) \frac{\partial f_1}{\partial P}(S, I, \tilde{P}) \right]
\]

\[
+ F_2(I) \left[ I \frac{\partial f_2}{\partial I}(S, \tilde{I}, P) + (P - P_1) \frac{\partial f_2}{\partial P}(S, I, \tilde{P}) \right]
\]

\[
+ F_3(P) \left[ (S - S_1) \frac{\partial f_3}{\partial S}(\tilde{S}, I, P) + I \frac{\partial f_3}{\partial I}(S, \tilde{I}, P) \right].
\]

Now, from the definition of \( f_i \)s, \( \frac{\partial f_1}{\partial S} < 0 \), \( \frac{\partial f_2}{\partial P} < 0 \), \( \frac{\partial f_3}{\partial S} > 0 \), and \( \frac{\partial f_3}{\partial I} > 0 \). Again, \( F_3(P) < 0 \) will hold provided \( \alpha \) is small and \( r > \mu + \frac{(C_S - \lambda)d}{\gamma_1} \), also \( f_2(S_1, 0, P_1) > 0 \) provided \( R_0^P < \frac{\lambda + C_S}{\mu + r} \). The above condition also implies that \( F_1(S) > 0 \). Moreover, \( F_2(I) \geq 0 \) is always true.

Therefore, \( \frac{dV}{dt} \leq 0 \) when \( \frac{\partial f_1}{\partial S} < 0 \), \( \frac{\partial f_2}{\partial P} < 0 \), where \( S_1 < \tilde{S} < S \), \( 0 < \tilde{I} < I \), \( P_1 < \tilde{P} < P \). Now,

\[
\frac{\partial f_1}{\partial S} = -C_S < 0,
\]

\[
\frac{\partial f_2}{\partial P} = 0,
\]

Therefore, \( \frac{dV}{dt} \leq 0 \) when \( R_0^P < \frac{\lambda + C_S}{\mu + r} \), \( \alpha \) is small and \( r > \mu + \frac{(C_S - \lambda)d}{\gamma_1} \), and in that case, by LaSalle’s theorem (101), \( E_{SP} \) is a global attractor for the system. \( \square \)

### 2.3.5 Interior attractor:

In this subsection, we explore the sufficient conditions for the existence of the interior equilibrium of system (2.2) and its stability. It can be easily calculated that the system (2.2) has a unique interior equilibria namely

\[
E^* = (S^*, I^*, P^*) = \left( \frac{\gamma_1 R_0^P (\lambda + C_{SI}) - \gamma_2 R_0^P (\lambda + r R_0^I)}{R_0^P R_0^I \left[ \frac{\gamma_1 (\lambda + C_{SI}) - \gamma_2 (\lambda + C_S)}{\gamma_2} \right]}, \frac{\gamma_1}{\gamma_2} \left( \frac{1}{R_0^P} - S^* \right), \frac{\lambda}{\alpha} \left( S^* - \frac{1}{R_0^I} \right) \right).
\]

Here, the \( S^* \) is positive if either both numerator and denominator are positive or negative. Thus two possible cases arises
1. Both numerator and denominator positive gives
\[(\lambda + C_{SI}) > \frac{2\gamma_1}{\gamma_1} \max \left\{ \frac{R_0^c(\lambda + R_0^c)}{R_0^c}, (\lambda + C_S) \right\}. \]

2. Both numerator and denominator negative gives
\[(\lambda + C_{SI}) < \frac{2\gamma_1}{\gamma_1} \min \left\{ \frac{R_0^c(\lambda + R_0^c)}{R_0^c}, (\lambda + C_S) \right\}. \]

Again \( I^* \) is positive if \( S^* < \frac{1}{R_0^c} \) and \( P^* \) is positive if \( S^* > \frac{1}{R_0^c} \), thus \( I^* \) and \( P^* \) both exist if \( \frac{1}{R_0^c} < S^* < \frac{1}{R_0^c} \).

Thus the unique interior attractor exists if either of the following two conditions holds
\[
(\lambda + C_{SI}) > \frac{2\gamma_1}{\gamma_1} \max \left\{ \frac{R_0^c(\lambda + R_0^c)}{R_0^c}, (\lambda + C_S) \right\} \& R_0^p < \frac{\gamma_1 R_0^c(\lambda + C_{SI}) - \gamma_2 R_0^c(\lambda + R_0^c)}{\gamma_1(\lambda + C_{SI}) - \gamma_2(\lambda + C_S)} < R_0^c(2.16)
\]
\[
(\lambda + C_{SI}) < \frac{2\gamma_1}{\gamma_1} \min \left\{ \frac{R_0^c(\lambda + R_0^c)}{R_0^c}, (\lambda + C_S) \right\} \& R_0^p < \frac{\gamma_1 R_0^c(\lambda + C_{SI}) - \gamma_2 R_0^c(\lambda + R_0^c)}{\gamma_1(\lambda + C_{SI}) - \gamma_2(\lambda + C_S)} < R_0^c(2.17)
\]

The Jacobian matrix at the interior equilibrium \( E^* \) is given by
\[
J \bigg|_{E^*=(S^*,I^*,P^*)} = \begin{bmatrix}
-C_SS^* & -(C_{SI} + \lambda)S^* & -\alpha S^* \\
\lambda I^* & 0 & -\alpha I^* \\
\gamma_1 P^* & \gamma_2 P^* & 0
\end{bmatrix}
\]
(2.18)

The stability of the unique interior attractor is determined by the eigenvalues evaluated at \( E^* \), whose characteristic equation is given by
\[
\lambda^3 + [C_SS^*] \lambda^2 + [\gamma_2 \alpha I^* P^* + (C_{SI} + \lambda)\lambda S^* I^* + \alpha \gamma_1 S^* P^*] \lambda \\
+ \alpha [C_S \gamma_2 - (C_{SI} + \lambda)\gamma_1 + \lambda \gamma_2] S^* I^* P^* = 0
\]
(2.19)
\[
\Leftrightarrow (\lambda_1 - \lambda) (\lambda_2 - \lambda) (\lambda_3 - \lambda) = 0
\]
with \( \lambda_i; \ i = 1; 2; 3 \) being roots of (2.19). If all real parts of \( \lambda_i; \ i = 1; 2; 3 \) are negative, then we have
\[
\Sigma_{i=1}^3 \lambda_i = [C_SS^*] > 0
\]
\[
\Sigma_{i,j=1, i\neq j}^3 \lambda_i \lambda_j = [\gamma_2 \alpha I^* P^* + (C_{SI} + \lambda)\lambda S^* I^* + \alpha \gamma_1 S^* P^*] > 0
\]
\[
\Pi_{i=1}^3 \lambda_i = \alpha [C_S \gamma_2 - (C_{SI} + \lambda)\gamma_1 + \lambda \gamma_2] S^* I^* P^* > 0
\]
\[
(\Sigma_{i=1}^3 \lambda_i) (\Sigma_{i,j=1, i\neq j}^3 \lambda_i \lambda_j) - \Pi_{i=1}^3 \lambda_i = (C_{SI} + \lambda) I^* (\lambda C_SS^* + \alpha \gamma_1 P^*) \\
+ \alpha P^* (\gamma_1 C_SS^* - \lambda \gamma_2 I^*) > 0
\]
From the above inequalities, the unique interior attractor \( E^* \) is stable if

\[
\frac{\alpha P^*[\gamma_1 I^* - \gamma_1 C_S S^*]}{I^* [\lambda C_S S^* + \alpha \gamma_1 P^*]} < (\lambda + C_{SI}) < \frac{\gamma_1 (C_S + \lambda)}{\gamma_1}
\]  

(2.20)

**Note: Existence and stability of interior equilibrium:** From existence conditions (2.16) and (2.17) along with stability condition (2.20), it can be easily conclude that the interior equilibrium exists and is locally asymptotically stable if

\[
\gamma_1 (\lambda + C_{SI}) < \gamma_2 \min \left\{ \frac{R_0^P (\lambda + r I^0)}{R_0^I}, (\lambda + C_S) \right\}, R_0^P \left\{ \frac{\gamma_1 R_0^I (\lambda + C_{SI}) - \gamma_2 R_0^P (\lambda + r I^0)}{\gamma_1 (\lambda + C_{SI}) - \gamma_2 (\lambda + C_S)} \right\} < R_0^I
\]

and

\[
\frac{\alpha P^*[\gamma_2 I^* - C_S \gamma_1 S^*]}{I^* [\lambda C_S S^* + \alpha \gamma_1 P^*]} < (\lambda + C_{SI})
\]

and unstable otherwise.

**Theorem 2.3.4.** The interior equilibrium \( E^* \) is globally asymptotically stable if, the following conditions are satisfied

\[
4 \mu C_S > (C_{SI} + \lambda)^2 + \frac{\mu \alpha^2}{d}
\]

\[
(\gamma_1 + \gamma_2) \frac{C}{C_S} M = dP^*
\]

\[
\lambda \left( \frac{C}{C_S} \right)^2 = \mu I^*
\]

(2.21)

**Proof.** Define the function \( L(S, I, P) = L_1(S, I, P) + L_2(S, I, P) + L_3(S, I, P) \),

\[
L_1 = (S - S^* - S^* \ln \frac{S}{S^*}), \quad L_2 = \frac{1}{2} (I - I^*)^2, \quad L_3 = \frac{1}{2} (P - P^*)^2
\]

we will show that the function \( L \) is a Lyapunov function. Note that \( L(S, I, P) \) vanishes at \( E^* \) and it is positive for all \( S, I, P > 0 \). Hence \( E^* \) represents its global
minimum.

\[
\begin{align*}
\frac{dt_1}{dt} &= (S - S^*) [r - C_SS - C_SI - \lambda I - \alpha P] \\
&= (S - S^*) [C_S (S^* - S) + (C_SI + \lambda) (I^* - I) + \alpha (P^* - P)] \\
&= -C_S (S - S^*)^2 - (S - S^*) (I - I^*) (C_SI + \lambda) - \alpha (S - S^*) (P - P^*)
\end{align*}
\]

\[
\frac{dt_2}{dt} = (I - I^*) [\lambda SI - \alpha PI - \mu I] \\
= (I - I^*) [\lambda SI - \alpha PI - \mu (I - I^*) - \mu I^*] \\
= -\mu (I - I^*)^2 + (I - I^*) [\lambda SI - \alpha PI - \mu I^*] \tag{2.22}
\]

Similarly,

\[
\begin{align*}
\frac{dt_3}{dt} &= (P - P^*) [\gamma_1 SP + \gamma_2 IP - dP] \\
&= (P - P^*) [\gamma_1 SP + \gamma_2 IP - d(P - P^*) - dP^*] \\
&= -d (P - P^*)^2 + (P - P^*) [\gamma_1 SP + \gamma_2 PI - dP^*]
\end{align*}
\]

we find

\[
\begin{align*}
\frac{dt}{dt} &= -C_S (S - S^*)^2 - \mu (I - I^*)^2 - d (P - P^*)^2 - (S - S^*) (I - I^*) (C_SI + \lambda) \\
&\quad - \alpha (S - S^*) (P - P^*) + (I - I^*) (\lambda SI - \beta PI - \mu I^*) \\
&\quad + (P - P^*) [\gamma_1 SP + \gamma_2 PI - dP^*] \\
&\leq -C_S (S - S^*)^2 - \mu (I - I^*)^2 - d (P - P^*)^2 - (S - S^*) (I - I^*) (C_SI + \lambda) \\
&\quad - \alpha (S - S^*) (P - P^*) + (I - I^*) (\lambda SI - \mu I^*) \tag{2.23} \\
&\quad + (P - P^*) [\gamma_1 SP + \gamma_2 PI - dP^*] \\
&\leq -C_S (S - S^*)^2 - \mu (I - I^*)^2 - d (P - P^*)^2 - (S - S^*) (I - I^*) (C_SI + \lambda) \\
&\quad - \alpha (S - S^*) (P - P^*) + (I - I^*) \left( \lambda \left( \frac{r}{c_s} \right)^2 - \mu I^* \right) \\
&\quad + (P - P^*) \left[ \left( \gamma_1 + \gamma_2 \right) \frac{r}{c_s} M - dP^* \right]
\end{align*}
\]

Assuming \((\gamma_1 + \gamma_2) \frac{r}{c_s} M = dP^*\) and \(\lambda (\frac{r}{c_s})^2 = \mu I^*\),
we get \( \frac{dL}{dt} \leq -C_S (S - S*)^2 - \mu (I - I*)^2 - d (P - P*)^2 - (S - S*) (I - I*) (C_{SI} + \lambda) - \alpha (S - S*) (P - P*) \).

The right hand side of the above expression can be written as \( -v^T Q v \), where
\[
v = ((S - S*), (I - I*), (P - P*))^T\]
and the symmetric matrix \( Q \) is given by
\[
Q = \begin{pmatrix}
C_S & \frac{C_{SI} + \lambda}{2} & \frac{\alpha}{2} \\
\frac{C_{SI} + \lambda}{2} & \mu & 0 \\
\frac{\alpha}{2} & 0 & d
\end{pmatrix}.
\]
Thus, if the matrix \( Q \) is positive definite, then \( \frac{dL}{dt} < 0 \). We need all of the principal minors of \( Q \), namely \( P_1, P_2, P_3 \), to be positive, i.e.,

\[
\begin{align*}
P_1 &= C_S > 0, \\
P_2 &= \mu C_S - \frac{(C_{SI} + \lambda)^2}{4} > 0 \quad (2.24) \\
P_3 &= d \left[ \mu C_S - \frac{(C_{SI} + \lambda)^2}{4} \right] - \frac{\mu(\alpha)^2}{4} > 0
\end{align*}
\]

Now, \( \frac{dL}{dt} \leq 0 \) if the symmetric matrix \( Q \) is positive definite. \( Q \) is positive definite if
\[
4\mu C_S > (C_{SI} + \lambda)^2 + \frac{\mu(\alpha)^2}{d}.
\]
Thus, the function \( L \) is a Lyapunov function and global stability for \( E^* \) follows. \( \square \)

### 2.3.6 Multi-Stability:

In this subsection, we investigate the possibility of multistability of our model system. We will see that our model may be bistable with or without the coexistence equilibrium. Two different forms of bistable scenarios are possible: (a) our system may be bi-stable between two planner equilibria where susceptible prey coexists with infected prey or predator; (b) our system may be bi-stable between the coexistence equilibrium and the planner equilibrium where both preys coexists in a predator free environment.

The equilibrium point \( E_{SI} \) exists and is locally asymptotically stable if
\[
R_0^I > \max \left\{ \frac{C_S}{r}, \frac{R_0^P}{\gamma_2} \left[ \frac{\gamma_1 (\lambda + C_{SI}) - \gamma_2 C_S}{\gamma_2 (\lambda + C_{SI}) - R_0^P \gamma_2 r} \right] \right\}.
\]

36
The equilibrium point \( E_{SP} \) exists and is locally asymptotically stable if

\[
R_0^P > \max \left\{ \frac{C_S}{r}, \frac{R_0^l(\lambda + C_S)}{\lambda + rR_0^l} \right\}.
\]

The unique interior attractor \( E_* \) exists and is locally asymptotically stable if

\[
\gamma_1(\lambda+C_{SI}) < \gamma_2 \min \left\{ \frac{R_0^l(\lambda + rR_0^l)}{R_0^l}, (\lambda+C_S) \right\}, \quad R_0^P < \frac{\gamma_1R_0^l(\lambda + C_{SI}) - \gamma_2R_0^p(\lambda + rR_0^l)}{\gamma_1(\lambda + C_{SI}) - \gamma_2(\lambda + C_S)} < R_0^l
\]

and

\[
\frac{\alpha P^* \left[ \lambda \gamma_2 I^* - C_S \gamma_1 S^* \right]}{I^* \left[ \lambda C_S S^* + \alpha \gamma_1 P^* \right]} < (\lambda + C_{SI}).
\]

Where, \( S^* \), \( I^* \) and \( P^* \) are given in the subsection (2.3.5).

Now, both the planner equilibria \( E_{SI} \) and \( E_{SP} \) are locally asymptotically stable if (i.e., bi-stability occurs between the equilibrium points \( E_{SI} \) and \( E_{SP} \))

\[
R_0^I > \max \left\{ \frac{C_S}{r}, \frac{R_0^P (\gamma_1 (\lambda + C_{SI}) - \gamma_2 C_S)}{\gamma_1 (\lambda + C_SI) - R_0^l \gamma_2 r} \right\} \quad \text{and} \quad R_0^P > \max \left\{ \frac{C_S}{r}, \frac{R_0^l (\lambda + C_S)}{\lambda + rR_0^l} \right\}.
\]

Again, the equilibria \( E_{SI} \) and \( E_* \) are locally asymptotically stable if (i.e., bi-stability occurs between the equilibria \( E_{SI} \) and \( E_* \))

\[
R_0^I > \max \left\{ \frac{C_S}{r}, \frac{R_0^P (\gamma_1 (\lambda + C_{SI}) - \gamma_2 C_S)}{\gamma_1 (\lambda + C_SI) - R_0^l \gamma_2 r} \right\}, \quad \frac{\alpha P^* \left[ \lambda \gamma_2 I^* - C_S \gamma_1 S^* \right]}{I^* \left[ \lambda C_S S^* + \alpha \gamma_1 P^* \right]} < (\lambda + C_{SI}),
\]

\[
\gamma_1(\lambda+C_{SI}) < \gamma_2 \min \left\{ \frac{R_0^l (\lambda + rR_0^l)}{R_0^l}, (\lambda+C_S) \right\}, \quad R_0^P < \frac{\gamma_1R_0^l(\lambda + C_{SI}) - \gamma_2R_0^p(\lambda + rR_0^l)}{\gamma_1(\lambda + C_{SI}) - \gamma_2(\lambda + C_S)} < R_0^l.
\]

**Note:** For existence of unique interior equilibrium \( I^* = \frac{\gamma_1}{\gamma_2} \left( \frac{1}{R_0^l} - S^* \right) \), we have

\[
\frac{1}{R_0^l} - S^* > 0 \quad \Rightarrow \quad \frac{\gamma_2 \left[ R_0^p \left( \lambda + rR_0^l \right) - R_0^l (\lambda + C_S) \right]}{R_0^l R_0^p \left[ \gamma_1 (\lambda + C_SI) - \gamma_2 (\lambda + C_S) \right]} > 0 \quad \text{(2.25)}
\]

From the stability condition of interior equilibrium, we have

\[
\gamma_1 (\lambda + C_{SI}) < \gamma_2 (\lambda + C_S) \quad \Rightarrow \quad \gamma_1 (\lambda + C_{SI}) - \gamma_2 (\lambda + C_S) < 0
\]
Therefore, (2.25) holds

\[
\begin{align*}
&\text{if } R^P_0 (\lambda + r R^I_0) - R^I_0 (\lambda + C_S) < 0 \\
&\iff R^P_0 < \frac{R^I_0 (\lambda + C_S)}{\lambda + r R^I_0}
\end{align*}
\]

But for the stability of \( E_{SP} \), \( R^P_0 \) must be greater than \( \frac{R^I_0 (\lambda + C_S)}{\lambda + r R^I_0} \). Therefore, the equilibrium points \( E_{SP} \) and \( E^* \) can not be stable together. Thus, bi-stability between the equilibria \( E_{SP} \) and \( E_\ast \) can not be possible. Also, ti-stability among the equilibria \( E_{SI}, E_{SP} \) and \( E_\ast \) can not be possible for the model (2.2).

### 2.3.7 Different Strength of Competitions:

Let us consider possible three cases in more details.

**Strong inter-class competition (Inter-class competition is greater than intra-class competition):**

First, we consider the case when inter-class competition is greater than intra-class competition, that is \( C_{SI} > C_S \). From the second inequality of (2.20) we will get for the stability of the interior attractor as

\[
\gamma_1 (\lambda + C_{SI}) < \gamma_2 (C_S + \lambda).
\]

But in this case \( \gamma_1 > \gamma_2 \) and \( C_{SI} > C_S \). Thus the above inequality can not be true in this case. Therefore, three species coexistence can not be possible when inter-class competition is greater than the intra-class competition.

Here the unique interior attractor exists, which is always unstable (see figure (2.1(a))). In this case bi-stability occurs between the equilibrium points \( E_{SI} \) and \( E_{SP} \) (see figures yellow region in (2.1(b)) and (2.2(a)))
Explicit carrying capacity (Inter-class competition is equal to intra-class competition):

Second, we consider the case when both the intra and inter class competitions are equal, i.e., $C_{SI} = C_S$. From the second inequality of (2.20) we will get for the stability of the interior attractor as

$$\gamma_1 (\lambda + C_{SI}) < \gamma_2 (C_S + \lambda) \Leftrightarrow \gamma_1 < \gamma_2,$$

which is a contradiction. Thus the above inequality can not be true in this case. Therefore, three species coexistence can not be possible when inter-class competition is equal to the intra-class competition.

We have already discussed in the sub-section (2.2.3) that when both intra-class competition ($C_S$) and inter-class competition ($C_{SI}$) are equal and the relation $C_S = C_{SI} = \frac{r}{K}$ holds, then the models (2.1) and (2.2) are identical. Thus the co-existence of all the three populations are not possible if we consider the explicit carrying capacity.

In this case also the interior attractor exists, which is unstable (see figure (2.3(a))). Here also bi-stability occurs between the planner equilibrium points $E_{SI}$ and $E_{SP}$ (see figures, yellow region in (2.3(b)) and (2.2(b))).

Strong intra-class competition (Inter-class competition is less than intra-class competition):

Finally, we have considered the case when intra-class competition is greater than the inter-class competition, i.e., $C_S > C_{SI}$. In this case the unique interior attractor is locally asymptotically stable if it exists and the condition (2.20) is true. Thus, three species coexistence is possible in this case.
In this case interior attractor is stable. The figure (2.4(a)) shows the existence and stability of the unique interior attractor. In this case bi-stable occurs in two cases, first, bi-stability between the equilibria $E_{SI}$ and $E_{SP}$ (see figures, yellow region in (2.4(b)) and (2.5(a))), second, bi-stability occurs between the equilibria $E_{SI}$ and $E_*$ (see figures, cyan region in (2.4(b)) and (2.5(b))).
(a) Existence and stability for the unique interior equilibrium for the model (2.2) for strong inter-class competition

(b) Stability region of all the equilibrium points for the model (2.2) for strong inter-class competition

Figure 2.1: We fix parameter values as $r = 2.82; \alpha = 0.6; d = 0.15; \lambda = 0.03; \mu = 0.14; \gamma_1 = 0.006; \gamma_2 = 0.002; C_S = 0.01; C_{SI} = 0.05$ and vary the parameter $R_I^0$ from 0 to 0.2 and $R_P^0$ from 0 to 0.07. For the model (2.2), $E_0$ is always unstable. For figure (2.1(a)) Cyan-region is for the existence of unique interior attractor, which is always unstable. For figure (2.1(b)) (a) Blue-region is the stability region for the equilibrium point $E_1$, (b) Green-region is the stability region for the equilibrium point $E_{SP}$, (c) Black-region is the stability region for the equilibrium point $E_{SI}$, (d) Yellow-region is the stability region for both the equilibrium points $E_{SP}$ and $E_{SI}$ (bi-stability between these two equilibria).
(a) Bi-stability between the equilibrium points $E_{SI}$ and $E_{SP}$ for the model (2.2) for strong inter-class competition.

(b) Bi-stability between the equilibrium points $E_{SI}$ and $E_{SP}$ for the model (2.2) for explicit carrying capacity.

Figure 2.2: Multi-stability phase-plot for the model (2.2). (2.2(a))- Bi-stability between the planner equilibrium points $E_{SI}$ (green lines) and $E_{SP}$ (blue lines) for strong inter class competition, when $R_I^0 = 0.1$, $R_P^0 = 0.02$ and $C_S = 0.01$ and $C_{SI} = 0.05$. (2.2(b))- Bi-stability between the planner equilibrium points $E_{SI}$ (green lines) and $E_{SP}$ (green lines) for equal intra and inter class competition, when $R_I^0 = 0.15$, $R_P^0 = 0.03$ and $C_S = r/K = C_{SI}$, $K = 100$. We fix the other parameter values as $r = 2.82$; $\alpha = 0.6$; $\lambda = 0.03$; $\gamma_1 = 0.006$; $\gamma_2 = 0.002$.
(a) Existence and stability for the unique interior equilibrium for the model (2.2) for explicit carrying capacity

(b) Stability region of all the equilibrium points for the model (2.2) for explicit carrying capacity

Figure 2.3: We fix parameter values as $r = 2.82; \ K = 100; \ \alpha = 0.6; \ \delta = 0.15; \ \lambda = 0.03; \ \mu = 0.14; \ \gamma_1 = 0.006; \ \gamma_2 = 0.002; \ C_S = \frac{r}{K}; \ C_{SI} = \frac{r}{K}$ and vary the parameter $R^I_0$ from 0 to 0.2 and $R^P_0$ from 0 to 0.07. For the model (2.2), $E_0$ is always unstable. For figure (2.3(a)) Cyan-region is for the existence of unique interior attractor, which is always unstable. For figure (2.3(b)) (a) Blue-region is the stability region for the equilibrium point $E_1$, (b) Green-region is the stability region for the equilibrium point $E_{SP}$, (c) Black-region is the stability region for the equilibrium point $E_{SI}$, (d) Yellow-region is the stability region for both the equilibrium points $E_{SP}$ and $E_{SI}$ (bi-stability between these two equilibria).
(a) Existence and stability for the unique interior equilibrium for the model (2.2) for strong intra-class competition

(b) Stability region of all the equilibrium points for the model (2.2) for strong intra-class competition

Figure 2.4: We fix parameter values as $r = 2.82; K = 100; \alpha = 0.6; d = 0.15; \lambda = 0.03; \mu = 0.14; \gamma_1 = 0.006; \gamma_2 = 0.002; C_S = \frac{4}{45}; C_{SI} = \frac{2}{315}$ and vary the parameter $R_0^I$ from 0 to 0.2 and $R_0^P$ from 0 to 0.07. For the model (2.2), $E_0$ is always unstable. For figure (2.4(a)) (a) Red-region is for the stability of the unique interior attractor and (b) Red-region union Cyan-region is for the existence of unique interior attractor. For figure (2.4(b)) (a) Blue-region is the stability region for the equilibrium point $E_1$, (b) Green-region is the stability region for the equilibrium point $E_{SP}$, (c) Black-region is the stability region for the equilibrium point $E_{SI}$, (d) Red-region is the stability region for the equilibrium point $E_*$, (e) Yellow-region is the stability region for both the equilibrium points $E_{SP}$ and $E_{SI}$ (bi-stability between these two equilibria), (f) Cyan-region is the stability region for both the equilibrium points $E_{SI}$ and $E_*$ (bi-stability between these two equilibria).
Figure 2.5: Multi-stability phase-plot for the model (2.2). (2.5(a))- Bi-stability between the planner equilibrium points $E_{SI}$ (blue lines) and $E_{SP}$ (black lines) for strong intra-class competition when $R_{0}^I = 0.3$ and $R_{0}^P = 0.05$. (2.5(b))- Bi-stability between the equilibrium points $E_{SI}$ (black lines) and $E_*$ (blue lines) for strong intra-class competition when $R_{0}^I = 0.45$ and $R_{0}^P = 0.039$. We fix parameter values $r = 2.82; \alpha = 0.6; \lambda = 0.03; \gamma_1 = 0.006; \gamma_2 = 0.002; C_S = \frac{4}{45}; C_{SI} = \frac{2}{315}$

2.4 Discussion

In this paper, we have considered a prey-predator model with disease in prey population. We assume that infected population does not reproduce, but compete for the same resources with susceptible population. Most of the previous studies it is considered that both the susceptible and infected are equally competing for resource (in case of explicit carrying capacity), and the upper limit of the population growth is fixed by a pre-assigned number $K$ (89; 90; 91; 42; 92; 35; 36; 94; 95; 38; 102; 103). But competitive ability may differ due to infection in such competition models (in case of emergent carrying capacity) and upper limit of the population growth cannot
be fixed, it is an emergent property based on reproduction, competition, actual life-history traits, etc. (7). Also, we have shown that by considering equal competition on susceptible and infected prey, our model can be reduced to the fixed carrying capacity model, thus it can be easily applied for the case where such modeling approaches more applicable. Therefore, it will be better to consider the emergent carrying capacity modeling approach rather than the explicit carrying capacity one.

Sieber et al. (7) considered an eco-epidemiological model with disease in prey population, that allows different competition among and between susceptible and infected prey individuals. They showed that disease-induced competition can facilitate the co-existence of all the three populations and tremendously alter the stability nature of the system. In their model, they considered infected prey can reproduce (in that case offspring are infected by birth), and disease can spread horizontally and vertically. It is well known that a wide range of pathogens are spread by the mixture of horizontal and vertical transmission; among these are microsporidians (104; 105), helminths (106), bacteria (107) and viruses of plants and animals (108), including some important human pathogens as HIV, several hepatitis viruses. But there are several diseases like cholera, tuberculosis, etc., which can be transmitted only horizontally. Thus in our model, we have considered that disease transmitted only horizontally and infected prey cannot reproduce (infected prey dies out before reproduction), thus we have not included intra and inter specific competition in the subpopulation of infected prey. We have analyzed our full eco-epidemiological model and associated submodels in term’s of disease basic reproduction number and disease free demographic reproduction number. We have shown the positivity and boundedness of solutions and permanence of our full model. Also by constructing Lyapunov functions, we proved the global stability of the planner equilibrium where susceptible prey coexists with predator in a disease-free environment and the co-existence equilibrium where all the three species coexists. We also analyzed the multi-stability of our model. Our system is bi-stable in two different scenarios: (a)
bi-stable between the planner equilibria where susceptible prey co-exists with infected prey or predator, and (b) bi-stable between coexistence equilibrium and the planner equilibrium where susceptible prey coexists with infected prey.

From the previous eco-epidemiological models with explicit carrying capacities, it has been proved that three species coexistence is possible only when a predator is more benefited by consuming infected prey compared to the susceptible one (102; 103). Also with fixed carrying capacities, the three species coexistence is possible when predators feed only on infected prey by discriminating them from susceptible one (93). Also, some other researchers have proven that the coexistence of all the populations is possible with fixed carrying capacity, under certain conditions like the predators are not too dominant and system is cyclic, non-stationary (38; 109). Whereas Morozov (110) showed that with fixed carrying capacity coexistence of susceptible-infected-predator is possible when the disease transmission depends on the predator population. Without the above additional conditions, the coexistence is impossible and competition exclusion occurs. Thus the system is to be either disease-dominated or predator-dominated (either predator extinct or disease extinction depends on the superior competitor among them)(111).

It is well known that predator-prey models with emergent carrying capacity can show more biologically relevant dynamics in compare to the explicit carrying capacity models (97; 98). For detailed discussion, we have considered three different strength of competitions (a) strong inter-class competition, (b) equal intra and inter class competition (case for the fixed carrying capacity), and (c) strong intra-class competition. We have analyzed multi stability in three different cases. Like previous study by Sieber et al. (7) one of the most interesting findings is that the coexistence of all the three populations is possible due to different competitive abilities between susceptible and infected prey (coexistence is possible only when intra-class
competition on susceptible prey is greater than the inter-class competition between susceptible and infected prey), which is impossible for the case of fixed carrying capacity. From the bi-stable scenarios, it can be easily concluded that the coexistence state is fragile for the case of strong intra-class competition. Sustainability of predator depends on the initial infected prey density. If a large number of preys are infected in the early stage, due to less beneficiary effect of infected prey on predator in compare to susceptible prey, predator population becomes extinct and the system goes to the predator free susceptible-infected prey equilibrium. On the other hand, when the initial density of infected prey is not too large then coexistence is possible for the above case. But our system does not support the disease induced extinction like Sieber et al. (7). The above bi-stable scenario can be applied as external perturbations in the form of adding infected preys to eradicate predator (where predators are harmful) or save the natural predator by culling infected preys (where predators are beneficial) for the same environmental conditions (since other parameters are fixed). For the other two cases (strong inter-class competition and fixed carrying capacity) when coexistence equilibrium is unstable, same external perturbations can be applied to control the competition exclusion, whether disease wins or predator wins in the forms of stable planner equilibria, whether susceptible prey coexists with infected prey or predator.

**Gateway to Chapter 3:** In chapter 2, we basically interested in an eco-epidemiological model of a predator-prey type, where the prey population is subject to infection. We observed the dynamics of such system under the influence of emergent carrying capacity. The effect Allee parameter is of considerable interest in ecological perspective. We like to introduce weak Allee effect in eco-epidemiological model and to observe the dynamics under the influence of emergent carrying capacity.