Chapter 5

Intra-specific competition in prey can control chaos in a prey-predator model

5.1 Introduction

In recent times, significant research has been done on the development of the concept for Allee effect which corresponds to the positive correlation between population size/density and per-capita growth rate at low population density (8). Ecologists paid significant attention on this topic as it relates to species extinction. Furthermore, disease has been considered as one of the main cause for species disappearance, and if it is connected with the Allee effects, the interaction between them has substantial biological importance in nature (162). On the other hand, many ecological and men made activities in biology, and medicine can be better interpreted with the help of time delays; the classical books like (199; 200; 66) discussed detail topics on the relevance of time delays in practical models. As far as our knowledge, there are very few works on time delayed population dynamics in the presence of Allee effect.

*A considerable part of this chapter is under revision in the journal of Nonlinear Dynamics.*
In this chapter, the effect of strong Allee with emergent carrying capacity has been discussed.

5.2 The model

\[
\frac{dS}{dt} = S(a - bS(t - \tau))(S - \theta) - \beta SP \\
\frac{dP}{dt} = P[\alpha S - d].
\] (5.1)

All the variables and parameters are positive. The variables and parameters used in Model (5.1) are presented in the Table (5.1).

<table>
<thead>
<tr>
<th>Variables/ &amp; Parameters</th>
<th>Biological Meaning</th>
</tr>
</thead>
<tbody>
<tr>
<td>S</td>
<td>Density of prey</td>
</tr>
<tr>
<td>P</td>
<td>Density of predator</td>
</tr>
<tr>
<td>(\theta)</td>
<td>Individuals searching efficiency</td>
</tr>
<tr>
<td>(\beta)</td>
<td>Attack rate of predator</td>
</tr>
<tr>
<td>(\alpha = c\beta)</td>
<td>Net gain to P by consuming S</td>
</tr>
<tr>
<td>(\mathbf{b})</td>
<td>Intra-specific competition</td>
</tr>
<tr>
<td>(\mathbf{a})</td>
<td>Intrinsic growth rate of S</td>
</tr>
<tr>
<td>(\mathbf{d})</td>
<td>Natural death rate of P</td>
</tr>
<tr>
<td>(\tau)</td>
<td>Gestation period of P</td>
</tr>
</tbody>
</table>

Table 5.1: Variables and parameter description of different parameters for the model (5.1).

The initial conditions for the system (5.1) take the form

\[ S(\phi) = \psi_1(\phi), \quad P(\phi) = \psi_2(\phi), \quad -\tau \leq \phi \leq 0, \]
where \( \psi = (\psi_1, \psi_2)^T \in \mathcal{C}_+ \) such that \( \psi_i(\phi) \geq 0 \) \((i = 1, 2)\), \( \forall \phi \in [-\tau, 0] \), and \( \mathcal{C}_+ \) denotes the Banach space \( \mathcal{C}_+([-\tau, 0], \mathbb{R}^2_+) \) of continuous functions mapping the interval \([-\tau, 0]\) into \( \mathbb{R}^2_+ \) and denotes the norm of an element \( \psi \) in \( \mathcal{C}_+ \) by

\[
\|\psi\| = \sup_{-\tau \leq \phi \leq 0} \{|\psi_1(\phi)|, |\psi_2(\phi)|\}.
\]

For biological feasibility, we further assume that \( \psi_i(0) > 0 \), for \( i = 1, 2 \).

### 5.3 Mathematical analysis of the system (5.1) with no time delay

We first study the system (5.1) with no time lag. The system (5.1) without time delay can be written as

\[
\begin{align*}
\frac{dS}{dt} &= S(a - bS)(S - \theta) - \beta SP, \\
\frac{dP}{dt} &= P[aS - d].
\end{align*}
\]

The system (5.2) has the following boundary equilibria: \( E_0 = (0, 0) \), \( E_\theta = (\theta, 0) \) and \( E_1 = (1, 0) \). Here, \( E_0 \) is always stable, \( E_\theta \) is stable if \( \frac{a}{b} < \theta \) and \( E_1 \) is stable if \( \frac{a}{b} > \theta \).

The system has unique interior attractor \( E^* = (S^*, P^*) \), where \( S^* = \frac{d}{a} \) and \( P^* = \frac{(a-b\theta)(d-\theta)}{\beta} \). The interior equilibria exists if \( \frac{a}{b} > \frac{d}{a} > \theta \).

\( E^* \) is stable if \( S^* > \frac{a+b\theta}{2b} \).

### 5.4 Mathematical analysis of the time delay model

In this section we have studied our delay model (5.1). Here, we have studied local stability analysis of equilibria, permanence and existence of switching stability of the delay differential equation (5.1). We analyze the delay system with respect to the interior equilibria \( E^* \) only.
5.4.1 Local stability analysis

Let \( \tilde{E} = (\tilde{S}, \tilde{P}) \) be any equilibrium point of the system (5.1). The linearized system of the system (5.1) at \( \tilde{E} = (\tilde{S}, \tilde{P}) \) is

\[
\begin{align*}
\dot{x}(t) &= ((a - \tilde{S}b)(2\tilde{S} - \theta) - \beta \tilde{P})x(t) - \beta \tilde{S}y(t) - \tilde{S}b(\tilde{S} - \theta)x(t - \tau), \\
\dot{y}(t) &= x(t)\alpha \tilde{P}
\end{align*}
\]

(5.3)

Then the characteristic equation of the delayed system (5.1) around any equilibrium point \( \tilde{E} = (\tilde{S}, \tilde{P}) \) is given by

\[
\det \begin{bmatrix}
(a - \tilde{S}b)(2\tilde{S} - \theta) - \beta \tilde{P} - e^{-\lambda \tau} \tilde{S}b(\tilde{S} - \theta) - \lambda & -\beta \tilde{S} \\
\alpha \tilde{P} & -\lambda
\end{bmatrix} = 0.
\]

(5.4)

The characteristic equation at the interior equilibrium \( E^* = (S^*, P^*) \) of the dynamical system with positive delay reduces to the following transcendental equation

\[
\lambda^2 - C_1 \lambda + C_2 = [C_3 \lambda] e^{-\lambda \tau}.
\]

(5.5)

Where

\[
\begin{align*}
C_1 &= (a - \tilde{S}b)(2\tilde{S} - \theta) - \beta \tilde{P}, \\
C_2 &= \alpha \beta \tilde{S} \tilde{P}, \\
C_3 &= \tilde{S}b(\tilde{S} - \theta).
\end{align*}
\]

(5.6)

For the delay-induced system (5.1), it is known that the equilibrium point \( E^* \) will be asymptotically stable if all the roots of the corresponding characteristic equation (5.5) have negative real parts. But, the classical Routh-Hurwitz criterion cannot be used to investigate the stability of the system. Since the equation (5.5) is a transcendental equation, it has infinitely many eigenvalues. To determine the nature of the stability, we require the sign of the real parts of the roots of the characteristic
equation (5.5).

Let \( \lambda(\tau) = \zeta(\tau) + i\rho(\tau) \) be the eigenvalue of the characteristic equation (5.5), substituting this value in equation (5.5) we obtain real and imaginary parts, respectively as

\[
\zeta^2 - \rho^2 - C_1 \zeta + C_2 = [(C_3 \zeta) \cos \rho \tau + \rho C_3 \sin \rho \tau] e^{-\zeta \tau}, \tag{5.7}
\]

and

\[
2 \zeta \rho - C_1 \rho = [C_3 \rho \cos \rho \tau - (\zeta C_3) \sin \rho \tau] e^{-\zeta \tau}. \tag{5.8}
\]

A necessary condition for a stability changes of \( E^* \) is that the characteristic equation (5.5) should have purely imaginary solutions. We set \( \zeta = 0 \) in (5.7) and (5.8). Then we get,

\[
-\rho^2 + C_2 = [\rho C_3 \sin \rho \tau], \tag{5.9}
\]

\[
-C_1 \rho = [\rho C_3 \cos \rho \tau]. \tag{5.10}
\]

Eliminating \( \tau \) by squaring and adding the equations (5.9) and (5.10), we get the algebraic equation for determining \( \rho \) as

\[
\rho^4 + (C_1^2 - 2C_2 - C_3^2) \rho^2 + (C_2^2) = 0. \tag{5.11}
\]

Substituting \( \rho^2 = \theta_1 \) in equation (5.11), we obtain a quadratic equation given by

\[
k(\theta_1) = \theta_1^2 + \sigma_1 \theta_1 + \sigma_2 = 0, \tag{5.12}
\]

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where

$$\sigma_1 = C_1^2 - 2C_2 - C_3^2, \quad \sigma_2 = C_2^2.$$ 

Now $\sigma_2 > 0$ and $\sigma_1 < 0$ implies that (5.12) has at least one positive root. The following theorem gives a criterion for the switching in the stability behavior of $E^*$ in terms of the delay parameter $\tau$.

**Theorem 5.4.1.** Suppose that $E^*$ exists and locally asymptotically stable for (5.1) with $\tau = 0$. Also let $\theta_0 = \rho_0^2$ be a positive root of (5.12).

1. Then there exists $\tau = \tau^*$ such that the interior equilibrium point $E^*$ of the delay system (5.1) is asymptotically stable when $0 \leq \tau < \tau^*$ and unstable for $\tau > \tau^*$.

2. Furthermore, the system will undergo a Hopf bifurcation at $E^*$ when $\tau = \tau^*$, provided $Z(\rho)X(\rho) - Y(\rho)W(\rho) > 0$.

**Proof.** Since $\rho_0$ is a solution of the equation (5.11), that is, the characteristic equation (5.5) has the pair of purely imaginary roots $\pm i\rho_0$. From equation (5.9) and (5.10), we have $\tau_p^*$ is a function of $\rho_0$ for $p = 0, 1, 2, ...$; which is given by

$$\tau_p^* = \frac{1}{\rho_0} \arccos \left( \frac{-C_1}{C_3} \right) + \frac{2\pi p}{\rho_0}.$$ (5.13)

Now the system will be locally asymptotically stable around the interior equilibrium point $E^*$ for $\tau = 0$, if the condition $S^* > \frac{a+b\theta}{2b}$ holds. In that case by Butler’s lemma, $E^*$ will remain stable for $\tau < \tau^*$, such that $\tau^* = \min_{p \geq 0} \tau_p^*$.

Also, we can verify the following transversality condition

$$\frac{d}{d\tau} \{Re\{\lambda(\tau)\}\}_{\tau = \tau^*} > 0.$$

Differentiating equations (5.7) and (5.8), with respect to $\tau$ and then put $\zeta = 0$, we obtain

$$X(\rho) \frac{d\zeta}{d\tau} + Y(\rho) \frac{d\rho}{d\tau} = Z(\rho),$$

$$-Y(\rho) \frac{d\zeta}{d\tau} + X(\rho) \frac{d\rho}{d\tau} = W(\rho).$$ (5.14)

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Where
\[
\begin{align*}
X(\rho) &= -C_1 + \tau(\rho C_3 \sin \rho \tau) - C_3 \cos \rho \tau, \\
Y(\rho) &= -2\rho + \tau(\rho C_3 \cos \rho \tau) + C_3 \sin \rho \tau, \\
Z(\rho) &= \rho^2 C_3 \cos \rho \tau, \\
W(\rho) &= -\rho^2 C_3 \sin \rho \tau.
\end{align*}
\]
(5.15)

Solving the above system, we have
\[
\frac{d}{d\tau} \left[ \Re \{\lambda(\tau)\} \right]_{\tau = \tau^*, \rho = \rho_0} = \left[ \frac{Z(\rho) X(\rho) - Y(\rho) W(\rho)}{X^2(\rho) + Y^2(\rho)} \right]_{\tau = \tau^*, \rho = \rho_0},
\]
which shows that, \( \frac{d}{d\tau} \left[ \Re \{\lambda(\tau)\} \right]_{\tau = \tau^*, \rho = \rho_0} > 0 \) if \( Z(\rho) X(\rho) - Y(\rho) W(\rho) > 0 \). Therefore, the transversality condition is satisfied and hence Hopf bifurcation occurs at \( \tau = \tau^* \). This completes the proof of the theorem. \( \square \)

5.4.2 Uniform persistence of the system:

In this section, we present conditions for uniform persistence of the system (5.1). We denote by \( \mathbb{R}^2_+ = \{(S, P) \in \mathbb{R}^2 : S \geq 0, P \geq 0\} \) the non-negative quadrant and by \( \text{int}(\mathbb{R}^2_+) = \{(S, P) \in \mathbb{R}^2 : S > 0, P > 0\} \).

**Definition**: System (5.1) is said to be uniformly persistent if a compact region \( D \subset \text{int}(\mathbb{R}^2_+) \) exists such that every solution \( \Xi(t) = (S(t), P(t)) \) of the system (5.1) with initial conditions eventually enters and remains in the region \( D \).

**Boundedness of the solution of the delayed system (5.1)**:

One can write the first equation of the system (5.1) as
\[
\frac{dS}{S} = \left[ a(1 - bS)(S - \theta) - \beta P \right] dt.
\]
Integrating between the limits 0 and \( t \), we have
\[
S(t) = S(0) \exp \left\{ \int_0^t \left[ a(1 - bS)(S - \theta) - \beta P \right] ds \right\}.
\]
Similarly from the second equation of the system, we have
\[
P(t) = P(0) \exp \left\{ \int_0^t \left[ \alpha \frac{S(s-\tau) P(s-\tau)}{p} - d \right] ds \right\}.
\]
where $S(0) = S_0 > 0$ and $P(0) = P_0 > 0$. Therefore, $S(t) > 0$ and $P(t) > 0$.

**Lemma 5.4.2.** All the solution of the system (5.1) starting in int$(\mathbb{R}_+^2)$ are uniformly bounded with an ultimate bound $M$.

*Proof.* The proof is similar to the proof of (136). □

### 5.5 Numerical simulation

Several numerical experiments are performed on the system (5.1) to validate our theoretical findings. In the present investigation, the gestation delay ($\tau$), Allee effect ($\theta$) and intra-specific competition ($b$) are the key parameters. We investigate the system (5.1) for different values of the above parameters. This study demonstrates the feasibility of different complex dynamical behavior, including limit cycle, higher periodic oscillations and chaos.

Consider the set of parameter as $\tau = 2.5$, $\beta = 0.2$, $m = 0.4$, $a = 0.9$, $\theta = 0.1$, $\alpha = 0.8$, $b = 1$, with the initial condition $[S(0), I(0), P(0)] = [0.8, 0.1, 14]$. In the presence of strong Allee if we increase the delay parameter, the system will be more chaotic. Now keeping the other parameter values same, if we increase the emergent carrying capacity ($b$), then we observe that the system (5.1) shows 2-periodic solutions for $b = 1.04$ (see Figure (5.2)). Figure (5.3) shows the limit-cycle oscillations for $b = 1.06$. Again if we increase the value of $b$, then we observe that for $b = 1.16$ the system (5.1) converges to stable focus (see Figure (5.4)).

To make it more clear, we draw the bifurcation diagram with respect to emergent carrying capacity ($b$) for $1 < b < 1.2$. For gradual increase of $b$, the system (5.1) switches its stability from chaotic oscillation to period doubling, period doubling to limit cycle oscillation and limit cycle oscillation to stable focus. Figure (5.5) shows that for $b \in [1, 1.04)$ the interior equilibrium $E^*$ is chaotic, for $b \in [1.04, 1.06]$ it
shows period doubling, for $b \in [1.06, 1.16)$ it exhibits limit cycle oscillation and for $b \in [1.16, 1.2]$ the interior equilibrium is stable.

Figure 5.1: The figure depicts that the delayed system (5.1) is chaotic with the time delay $\tau = 2.5$ and the other parameters $\beta = 0.2, m = 0.4, a = 0.9, \theta = 0.1, \alpha = 0.8, b = 1$, with the initial condition $[S(0), I(0), P(0)] = [0.8, 0.1, 14]$. 
Figure 5.2: The figure shows that the delayed system (5.1) is two periodic with the time delay $\tau = 2.5$ and the other parameters $\beta = 0.2$, $m = 0.4$, $a = 0.9$, $\theta = 0.1$, $\alpha = 0.8$, $b = 1.04$, with the initial condition $[S(0), I(0), P(0)] = [0.8, 0.1, 14]$. 
Figure 5.3: The figure shows limit cycle oscillation of the delayed system (5.1) with the time delay $\tau = 2.5$ and the other parameters $\beta = 0.2, m = 0.4, a = 0.9, \theta = 0.1, \alpha = 0.8, b = 1.06$, with the initial condition $[S(0), I(0), P(0)] = [0.8, 0.1, 14]$. 
Figure 5.4: The delayed system (5.1) is stable with the time delay $\tau = 2.5$ and the other parameters $\beta = 0.2, m = 0.4, a = 0.9, \theta = 0.1, \alpha = 0.8, b = 1.16$, with the initial condition $[S(0), I(0), P(0)] = [0.8, 0.1, 14]$. 
5.6 Discussion

For our model, the time delay due to the gestation plays an important role. Time delay can drive a stable equilibrium to an unstable one; that is, there is a critical value $\tau^*$, such that for $\tau^* > \tau$, the positive equilibrium $E^*$ is stable, and it loses stability as $\tau$ passes through its critical magnitude from lower to higher values. It was shown that the system (5.1) experiences the Hopf-bifurcation as the delay parameter $\tau$ crosses some critical values $\tau^*$. Further increase in delay beyond the bifurcation point leads to complex dynamic behavior, including chaos. We have also shown numerically that chaos can be controlled by the emergent carrying capacity. In future we would like to study a diffusive prey–predator model with emergent carrying capacity and Allee effect to explore the possibilities of Turing instability and pattern formation.
Gateway to Chapter 6: We have observed the dynamics of predator-prey system under various ecological factors including delay. Another important ecological factors, namely hydra effect on the dynamics of eco-epidemiological scenario may provide some interesting dynamics. With this basic aim, we propose and analyze such system in the following chapter.