CHAPTER 4

STRONGLY b*- CONTINUOUS FUNCTIONS

4.1 INTRODUCTION

Strong forms of continuous maps have been introduced and investigated by several mathematicians. Strongly continuous maps, perfectly continuous maps, completely continuous maps, clopen continuous maps were introduced by Levine (1960), Noiri (1984b), Munshi & Bassan (1982) and Reilly & Vamanamurthy (1983) respectively. Ekici (2004) introduced and studied a new class of functions called almost contra pre-continuous functions. Several authors have introduced and studied stronger and weaker forms of irresolute maps and homeomorphism.

In this Chapter, we introduce the notion of strongly b*-continuous function, sb* - irresolute maps and Homeomorphisms and also we study some of their basic properties.

4.2 STRONGLY b*-CONTINUOUS FUNCTIONS

In this section, we introduce and investigate strongly b*- continuous functions and their relationships with other continuous maps. Also we study sb*- open and closed maps in topological spaces.

**Definition 4.2.1:** Let X and Y be topological spaces. A map \( f : X \rightarrow Y \) is called strongly b* - continuous (briefly, sb*- continuous) if the inverse image of every open set in Y is sb* - open in X.
Theorem 4.2.2: If a map \( f : X \rightarrow Y \) is continuous, then it is sb* - continuous.

Proof: Let \( f : X \rightarrow Y \) be continuous. Let \( F \) be any open set in \( Y \). The inverse image of \( F \) is open in \( X \). Since every open set is an sb*-open set, inverse image of \( F \) is an sb*-open set in \( X \). Therefore \( f \) is sb* - continuous.

Remark 4.2.3: The converse of the above Theorem need not be true as seen from the following example.

Example 4.2.4: Consider \( X = \{d, e, h\} \) and \( \tau = \{\emptyset, \{d, h\}, X\} \), \( Y = \{a, b, c\} \) and \( \sigma = \{\emptyset, \{b\}, \{a, c\}, Y\} \). Let \( f : (X, \tau) \rightarrow (Y, \sigma) \) be defined by \( f(d) = a, f(h) = b, f(e) = c \). Then \( f \) is sb*-continuous. But \( f \) is not continuous since for the open set \( U = \{a, c\} \) in \( Y \), \( f^{-1}(U) = \{d, e\} \) is not open in \( X \).

Theorem 4.2.5: Let \( f : (X, \tau) \rightarrow (Y, \sigma) \) be a map from a topological space \( (X, \tau) \) into a topological space \( (Y, \sigma) \). The statement (a) \( f \) is sb* - continuous is equivalent to the statement (b) the inverse image of each open set in \( Y \) is sb*-open in \( X \).

Proof: Assume that \( f : X \rightarrow Y \) is sb*-continuous. Let \( G \) be open in \( Y \). Then \( G^c \) is closed in \( Y \). Since \( f \) is sb*-continuous, \( f^{-1}(G^c) \) is sb* - closed in \( X \). But \( f^{-1}(G^c) = X - f^{-1}(G) \). Thus \( X - f^{-1}(G) \) is sb*-closed in \( X \) and so \( f^{-1}(G) \) is sb*-open in \( X \). Therefore \((a) \Rightarrow (b)\).

Conversely assume that the inverse image of each open set in \( Y \) is sb*- open in \( X \). Let \( F \) be any closed set in \( Y \). Then \( f^{-1}(F^c) \) is sb* - open in \( X \). But \( f^{-1}(F^c) = X - f^{-1}(F) \). Thus \( X - f^{-1}(F) \) is sb* - open in \( X \) and so \( f^{-1}(F) \) is sb*-closed in \( X \). Therefore \( f \) is sb*-continuous. Hence \((b) \Rightarrow (a)\). Thus \((a) \) and \((b) \) are equivalent.
Theorem 4.2.6: Let \( f : X \to Y \) be an sb* - continuous map from a topological space \( X \) in to a topological space \( Y \) and let \( H \) be a closed subset of \( X \). Then the restriction \( f/H : H \to Y \) is sb* - continuous where \( H \) is endowed with the relative topology.

Proof: Let \( F \) be any closed subset in \( Y \). Since \( f \) is sb* - continuous, \( f^{-1}(F) \) is sb* - closed in \( X \). Intersection of a closed set and a sb*-closed sets is an sb*-closed set. Thus if \( f^{-1}(F) \cap H = H_1 \), then \( H_1 \) is an sb*-closed set in \( X \). Since \( (f/H)^{-1}(F) = H_1 \), it is sufficient to show that \( H_1 \) is an sb*-closed set in \( H \).

Let \( G_1 \) be any open set of \( H \) such that \( H_1 \subseteq G_1 \). Let \( G_1 = G \cap H \) where \( G \) is open in \( X \). Now \( H_1 \subseteq G \cap H \subseteq G \). Since \( H_1 \) is sb* - closed in \( X \), \( \overline{H}_1 \subseteq G \). Now \( cl_H(H_1) = \overline{H}_1 \cap H = G \cap H = G \), where \( cl_H(A) \) is the closure of a subset \( A \subseteq H \) in a subspace \( H \) of \( X \). Therefore \( f/H \) is sb* - continuous.

Remark 4.2.7: In the above Theorem, the assumption of closedness of \( H \) cannot be removed as seen from the following example.

Example 4.2.8: Let \( X = \{a, b, c\} \), \( \tau = \{\emptyset, \{b\}, X\} \), \( Y = \{p, q\} \) and \( \sigma = \{\emptyset, \{p\}, Y\} \). Let \( f : (X, \tau) \to (Y, \sigma) \) be defined by \( f(a) = f(c) = q, f(b) = p \). Now \( H = \{a, b\} \) is not closed in \( X \). Then \( f \) is sb* - continuous but the restriction \( f/H \) is not sb*-continuous. Since for the closed set \( F = \{q\} \) in \( Y \), \( f^{-1}(F) = \{a, c\} \) and \( f^{-1}(F) \cap H = \{a\} \) is not sb*-closed in \( H \).

Theorem 4.2.9: A map \( f : X \to Y \) is sb* - continuous if and only if the inverse image of every closed set in \( Y \) is sb* - closed in \( X \).

Proof: Let \( F \) be a closed set in \( Y \). Then \( F^c \) is open in \( Y \). Since \( f \) is sb* -continuous, \( f^{-1}(F^c) \) is sb* - open in \( X \). But \( f^{-1}(F^c) = X - f^{-1}(F) \) and so \( f^{-1}(F) \) is sb* - closed in \( X \).
Conversely let the inverse image of every closed set in \( Y \) be an \( \text{sb}^* \)-closed set in \( X \). Let \( V \) be an open set in \( Y \), then \( V^c \) is closed in \( Y \). Now, by the assumption, \( f^{-1}(V^c) = X - f^{-1}(V) \) is an \( \text{sb}^* \)-closed set in \( Y \). Therefore \( f^{-1}(V) \) is \( \text{sb}^* \)-open in \( X \). Then \( f \) is \( \text{sb}^* \)-continuous.

**Theorem 4.2.10:** If a function \( f : X \to Y \) is \( \text{sb}^* \)-continuous, then it is \( b \)-continuous.

**Proof:** Assume that a map \( f : X \to Y \) is \( \text{sb}^* \)-continuous. Let \( V \) be an open set in \( Y \). Since \( f \) is \( \text{sb}^* \)-continuous, \( f^{-1}(V) \) is \( \text{sb}^* \)-open and hence \( b \)-open in \( X \). Therefore \( f \) is \( b \)-continuous.

**Remark 4.2.11:** The converse of the above Theorem need not be true as seen from the following example.

**Example 4.2.12:** Let \( X = Y = \{a, b, c\} \), with \( \tau = \{\emptyset, \{a\}, \{c\}, \{a,c\}, X\} \), \( \sigma = \{\emptyset, \{b\}, \{c\}, \{b,c\}, Y\} \) and \( f = \{(a,b),(b,b),(c,c)\} \). Then \( f \) is \( b \)-continuous but not \( \text{sb}^* \)-continuous. Since the inverse image of the open set \( \{b\} \) in \( Y \) is \( \{a,b\} \) in \( X \), it is not \( \text{sb}^* \)-open.

**Theorem 4.2.13:** If a map \( f : X \to Y \) is \( \alpha \)-continuous, then it is \( \text{sb}^* \)-continuous.

**Proof:** Assume that \( f \) is \( \alpha \)-continuous. Let \( V \) be an open set in \( Y \). Since \( f \) is \( \alpha \)-continuous, \( f^{-1}(V) \) is \( \alpha \)-open and hence it is \( \text{sb}^* \)-open in \( X \). Then \( f \) is \( \text{sb}^* \)-continuous.

**Remark 4.2.14:** The converse of the above Theorem need not be true as seen from the following example.
Example 4.2.15: Let $X = Y = \{a, b, c\}$ with $\tau = \{\phi, \{b\}, \{a, c\}\}$ and $\sigma = \{\phi, \{a, c\}, Y\}$. Consider $f : X \to Y$ defined by $f(a) = f(b) = b, f(c) = c$. This function $f$ is sb*-continuous but not $\alpha$-continuous, since the preimage of the open set $\{a, c\}$ in $Y$ is $\{c\}$ in $X$ which is not $\alpha$-open.

Theorem 4.2.16: If a map $f : X \to Y$ is sb*-continuous, then it is wg-continuous.

Proof: Assume that a map $f : X \to Y$ is sb*-continuous. Let $V$ be an open set in $Y$. Since $f$ is sb*-continuous, $f^{-1}(V)$ is sb*-open and hence it is wg-open in $X$. Then $f$ is wg-continuous.

Remark 4.2.17: The converse of the above Theorem need not be true as seen from the following example.

Example 4.2.18: Let $X = Y = \{a, b, c\}$ with $\tau = \{\phi, \{b\}, \{a, c\}\}$ and $\sigma = \{\phi, \{a, \{b, a\}\}, Y\}$ and $f$ be the identity map. Then $f$ is wg-continuous but not sb*-continuous, as the inverse image of the open set $\{a\}$ in $Y$ is $\{a\}$ in $X$ which is not sb*-open.

Theorem 4.2.19: If a map $f : X \to Y$ is w-continuous, then it is sb*-continuous.

Proof: Let $f : X \to Y$ be w-continuous and $V$ be an open set in $Y$. Then $f^{-1}(V)$ is w-open and hence sb*-open in $X$. Then $f$ is sb*-continuous.

Remark 4.2.20: The converse of the above Theorem need not be true as seen from the following example.

Example 4.2.21: Let $X = Y = \{a, b, c\}$ with $\tau = \{\phi, \{b\}, \{a, c\}\}$ and $\sigma = \{\phi, \{b, c\}, Y\}$ and $f$ be the identity map. Then $f$ is sb*-continuous but not w-continuous, as the inverse image of the open set $\{b, c\}$ in $Y$ is $\{b, c\}$ in $X$ which is not w-open.
Theorem 4.2.2: If a map \( f : X \rightarrow Y \) is \( \text{sb}^* \)-continuous, then it is semi pre -continuous.

Proof: Let \( f : X \rightarrow Y \) be \( \text{sb}^* \)-continuous and \( V \) be an open set in \( Y \). Then \( f^{-1}(V) \) is an \( \text{sb}^* \)-open set and hence an semi pre -open set in \( X \). Then \( f \) is semi pre-continuous.

Remark 4.2.23: The converse of the above Theorem need not be true as seen from the following example.

Example 4.2.24: Let \( X = Y = \{a, b, c\} \) with \( \tau = \{\phi, \{a\}, \{c\}, \{a, c\}, X\} \) and \( \sigma = \{\phi, \{b, c\}, Y\} \) and \( f \) be the identity map. Then \( f \) is semi pre -continuous but not \( \text{sb}^* \)-continuous, since the inverse image of the open set \( \{b, c\} \) in \( Y \) is \( \{b, c\} \) in \( X \) which is not \( \text{sb}^* \)-open.

Remark 4.2.25: The following example shows that the \( g \)-continuous function and \( \text{sb}^* \)-continuous function are independent.

Example 4.2.26: Consider \( X = Y = \{a, b, c\} \) with \( \tau = \{\phi, \{b\}, X\} \) and \( \sigma = \{\phi, \{a\}, \{b, c\}, Y\} \). Let the function \( f : (X, \tau) \rightarrow (Y, \sigma) \) be defined by \( f(a) = b, f(b) = c, f(c) = a \). This function \( f \) is \( g \)-continuous but not \( \text{sb}^* \)-continuous since the inverse image of the open set \( \{a\} \) in \( Y \) is \( \{c\} \) in \( X \) which is not \( \text{sb}^* \)-open.

Example 4.2.27: Consider \( X = Y = \{a, b, c\} \) with \( \tau = \{\phi, \{a\}, \{a, b\}, X\} \) and \( \sigma = \{\phi, \{a, b\}, Y\} \). Let the function \( f : (X, \tau) \rightarrow (Y, \sigma) \) be defined by \( f(a) = f(c) = b, f(b) = c \). Here the inverse image of the open set \( \{a, b\} \) in \( Y \) is \( \{a, c\} \) in \( X \) which is \( \text{sb}^* \)-open but not \( g \)-open. Therefore this function is \( \text{sb}^* \)-continuous but not \( g \)-continuous.
Remark 4.2.28: The following example shows that the $\alpha g$-continuous function and $\text{sb*}$-continuous function are independent.

Example 4.2.29: Consider $X = Y = \{a, b, c\}$ with $\tau = \{\phi, \{b\}, X\}$ and $\sigma = \{\phi, \{c\}, Y\}$. Let the function $f : (X, \tau) \rightarrow (Y, \sigma)$ be defined by $f(a) = a, f(b) = b, f(c) = c$. Here the inverse image of the open set $\{c\}$ in $Y$ is $\{c\}$ in $X$ which is an $\alpha g$-open set but not $\text{sb*}$-open. Therefore the defined function is $\alpha g$-continuous but not $\text{sb*}$-continuous.

Example 4.2.30: Consider $X = Y = \{a, b, c\}$ with $\tau = \{\phi, \{a, b\}, X\}$ and $\sigma = \{\phi, \{a, c\}, Y\}$. Let the function $f : (X, \tau) \rightarrow (Y, \sigma)$ be defined by $f(a) = c, f(b) = b$ and $f(c) = a$. Here the inverse image of the open set $\{a, c\}$ in $Y$ is $\{a, c\}$ in $X$ which is $\text{sb*}$-open but not $\alpha g$-open. Therefore the defined function is $\text{sb*}$-continuous but not $\alpha g$-continuous.

Remark 4.2.31: The following example shows that the $\text{sb*}$-continuous function and $\text{sg}$-continuous function are independent.

Example 4.2.32: Consider $X = Y = \{a, b, c\}$ with $\tau = \{\phi, \{a\}, \{c\}, \{a, c\}, X\}$ and $\sigma = \{\phi, \{a, c\}, Y\}$. Let the function $f : (X, \tau) \rightarrow (Y, \sigma)$ be defined by $f(a) = b, f(b) = a, f(c) = c$. Here the inverse image of the open set $\{a, c\}$ in $Y$ is $\{b, c\}$ in $X$ which is an $\text{sg}$-open set but not $\text{sb*}$-open. Therefore the defined function is $\text{sg}$-continuous but not $\text{sb*}$-continuous.

Example 4.2.33: Consider $X = Y = \{a, b, c\}$ with $\tau = \{\phi, \{a, c\}, X\}$ and $\sigma = \{\phi, \{a, b\}, Y\}$. Let the function $f : (X, \tau) \rightarrow (Y, \sigma)$ be defined by $f(a) = c, f(b) = b$ and $f(c) = a$. Here the inverse image of the open set $\{a, b\}$ in $Y$ is $\{b, c\}$ in $X$ which is $\text{sb*}$-open but not $\text{sg}$-open. Therefore the defined function is $\text{sg}$-continuous but not $\text{sb*}$-continuous.
Remark 4.2.34: The following example shows that the sb*-continuous function and semi-continuous function are independent.

Example 4.2.35: Consider \( X = Y = \{a, b, c\} \) with \( \tau = \{\phi, \{a, c\}, X\} \) and \( \sigma = \{\phi, \{a\}, \{b\}, \{a, b\}, Y\} \). Let the function \( f : (X, \tau) \rightarrow (Y, \sigma) \) be defined by \( f(a) = a, f(b) = c, f(c) = b \). Here the inverse image of the open set \( \{a\} \) in Y is \( \{a\} \) in X which is not semi-open but it is sb*-open. Therefore the defined function is sb*-continuous but not semi-continuous.

Example 4.2.36: Consider \( X = Y = \{a, b, c\} \) with \( \tau = \{\phi, \{a\}, \{c\}, \{a, c\}, X\} \) and \( \sigma = \{\phi, \{b, c\}, Y\} \). Let the function \( f : (X, \tau) \rightarrow (Y, \sigma) \) be defined by \( f(a) = a, f(b) = c, f(c) = b \). Here the inverse image of the open set \( \{b, c\} \) in Y is \( \{b, c\} \) in X which is semi-open but not sb*-open. Therefore the defined function is semi-continuous but not sb*-continuous.

Remark 4.2.37: From above remarks and known results, the following diagram is obtained. None of the implications is reversible.

![Diagram showing relationships of sb*-continuous maps](image)

**Figure 4.1 Relationships of sb*-continuous maps**

In this figure \( A \leftrightarrow B \) means A and B are independent of each other.
Definition 4.2.38: Let $X$ and $Y$ be topological spaces. A map $f: X \to Y$ is called a strongly $b^*$-closed (briefly, $sb^*$ - closed) map if the image of every closed set in $X$ is an $sb^*$ - closed set in $Y$.

Theorem 4.2.39: Every closed map is $sb^*$-closed.

Proof: Let $f: X \to Y$ be a closed map and $V$ be a closed set in $X$. Then $f(V)$ is closed and hence $sb^*$-closed in $Y$. Thus $f$ is $sb^*$-closed.

Remark 4.2.40: The converse of the above Theorem need not be true as seen from the following example.

Example 4.2.41: Consider $X = Y = \{a, b, c\}$ with $\tau = \{\emptyset, \{a\}, X\}$ and $s = \{f, \{a\}, \{a, b\}, Y\}$ and the map $f: X \to Y$ defined by $f(a) = a$, $f(b) = f(c) = b$. This function $f$ is $sb^*$-closed but not closed as $\{b, c\} = \{b\}$ is not closed in $Y$.

Theorem 4.2.42: If a map $f: X \to Y$ is continuous and $sb^*$-closed and $A$ is an $sb^*$ - closed set of $X$, then $f(A)$ is $sb^*$-closed in $Y$.

Proof: Let $f(A) \subseteq U$, where $U$ is open set of $Y$. Since $f$ is continuous, $f^{-1}(U)$ is an open set and therefore $b$-open set containing $A$. Hence $\text{cl}(\text{int}(A)) \subseteq f^{-1}(U)$ as $A$ is $sb^*$-closed. Since $f$ is $sb^*$-closed, $f(\text{cl}(\text{int}(A)))$ is an $sb^*$-closed set contained in the $b$-open set $U$ which implies $\text{cl}(\text{int}(f(A))) \subseteq U$. So $f(A)$ is $sb^*$-closed in $Y$. 


Theorem 4.2.43: A map \( f: X \rightarrow Y \) is sb*-closed if and only if for each subset \( S \) of \( Y \) and for each open set \( U \) containing \( f^{-1}(S) \), there is an sb*-open set \( V \) of \( Y \) such that \( S \subseteq V \) and \( f^{-1}(V) \subseteq U \).

Proof: Suppose \( f \) is sb* - closed. Let \( S \) be a subset of \( Y \) and \( U \) be an open set of \( X \) such that \( f^{-1}(S) \subseteq U \). \( V = Y - f(X - U) \) is an sb* - open set containing \( S \) such that \( f^{-1}(V) \subseteq U \).

For the converse, suppose that \( F \) is a closed set of \( X \). Then \( f^{-1}(Y - f(F)) \subseteq X - F \) and \( X - F \) is open. By hypothesis, there is an sb*-open set \( V \) of \( Y \) such that \( Y - f(F) \subseteq V \) and \( f^{-1}(V) \subseteq X - F \). Therefore \( F \subseteq X - f^{-1}(V) \). Hence \( Y - V \subseteq f(F) \subseteq f(X - f^{-1}(V)) \subseteq Y - V \) which implies \( f(F) = Y - V \). Since \( Y - V \) is sb*-closed, \( f(F) \) is sb*-closed and thus \( f \) is an sb*-closed map.

Theorem 4.2.44: If a map \( f: X \rightarrow Y \) is closed and a map \( g: Y \rightarrow Z \) is sb*-closed, then \( g \circ f: X \rightarrow Z \) is sb*-closed.

Proof: Let \( V \) be a closed set in \( X \). Since \( f: X \rightarrow Y \) is closed, \( f(V) \) is a closed set in \( Y \). Since \( g: Y \rightarrow Z \) is sb* - closed, \( h(f(V)) \) is an sb* - closed set in \( Z \). Therefore \( g \circ f: X \rightarrow Z \) is an sb* - closed map.
Theorem 4.2.45: Let \( f : X \to Y \) be a bijective.

(a) If \( f \) is sb*-continuous and \( (Y, \tau_2) \) is \( T_1 \), then \( (X, \tau_1) \) is \( sb^* - T_1 \).

(b) If \( f \) is sb*-open and \( (X, \tau_1) \) is \( sb^* - T_1 \), then \( (Y, \tau_2) \) is \( sb^* - T_1 \).

Proof: Let \( f : (X, \tau_1) \to (Y, \tau_2) \) be bijective.

(a) Suppose \( f : (X, \tau_1) \to (Y, \tau_2) \) is sb*-continuous and \( (Y, \tau_2) \) is \( T_1 \). Let \( x_1, x_2 \in X \) is \( T_1 \) with \( x_1 \neq x_2 \). Since \( f \) is bijective, \( y_1 = f(x_1) \neq f(x_2) = y_2 \) for some \( y_1, y_2 \in Y \). Since \( (Y, \tau_2) \) is \( T_1 \), there exist open sets \( G \) and \( H \) such that \( y_1 \in G \) but \( y_2 \notin G \) and \( y_2 \in H \) but \( y_1 \notin H \). Since \( f \) is bijective, \( x_1 = f^{-1}(y_1) \in f^{-1}(G) \) but \( x_2 = f^{-1}(y_2) \notin f^{-1}(G) \) and \( x_2 = f^{-1}(y_2) \in f^{-1}(H) \) but \( x_1 = f^{-1}(y_1) \notin f^{-1}(H) \). Since \( f \) is sb*-continuous, \( f^{-1}(G) \) and \( f^{-1}(H) \) are sb*-open sets in \( (X, \tau_1) \). It follows that \( (X, \tau_1) \) is \( sb^* - T_1 \). This proves (a).

(b) Suppose \( f \) is sb*-open and \( (X, \tau_1) \) is \( sb^* - T_1 \). Let \( y_1 \neq y_2 \in Y \). Since \( f \) is bijective, there exist \( x_1, x_2 \in X \), such that \( f(x_1) = y_1 \) and \( f(x_2) = y_2 \) with \( x_1 \neq x_2 \). Since \( (X, \tau_1) \) is \( sb^* - T_1 \), there exist sb*-open sets \( G \) and \( H \) in \( X \) such that \( x_1 \in G \) but \( x_2 \notin G \) and \( x_2 \in H \) but \( x_1 \notin H \). Since \( f \) is sb*-open, \( f(G) \) and \( f(H) \) are sb*-open in \( Y \) such that \( y_1 = f(x_1) \in f(G) \) and \( y_2 = f(x_2) \in f(H) \). Again since \( f \) is bijective, \( y_2 = f(x_2) \notin f(G) \) and \( y_1 = f(x_1) \notin f(H) \). Thus \( (Y, \tau_2) \) is \( sb^* - T_1 \). This proves (b).
Theorem 4.2.46: Let $f: X \to Y$ be a bijection.

(a) If $f$ is sb*-open and $X$ is $T_2$, then $Y$ is $sb^* - T_2$.

(b) If $f$ is sb*-continuous and $Y$ is $T_2$, then $X$ is $sb^* - T_2$.

Proof: Let $f: X \to Y$ be a bijection.

(a) Suppose $f$ is sb*-open and $X$ is $T_2$. Let $y_1 \neq y_2 \in Y$. Since $f$ is a bijection, there exist $x_1, x_2 \in X$ such that $f(x_1) = y_1$ and $f(x_2) = y_2$ with $x_1 \neq x_2$. Since $X$ is $T_2$, there exist disjoint open sets $U$ and $V$ in $X$ such that $x_1 \in U$ and $x_2 \in V$. Since $f$ is sb*-open, $f(U)$ and $f(V)$ are sb*-open in $Y$ such that $y_1 = f(x_1) \in f(U)$ and $y_2 = f(x_2) \in f(V)$. Again since $f$ is a bijection, $f(U)$ and $f(V)$ are disjoint in $Y$. Thus $Y$ is $sb^* - T_2$.

(b) Suppose $f: X \to Y$ is sb*-continuous and $Y$ is $T_2$. Let $x_1, x_2 \in X$ with $x_1 \neq x_2$. Let $y_1 = f(x_1)$ and $y_2 = f(x_2)$. Since $f$ is one-one, $y_1 \neq y_2$. Since $Y$ is $T_2$, there exist disjoint open sets $U$ and $V$ containing $y_1$ and $y_2$ respectively. Since $f$ is sb*-continuous bijective, $f^{-1}(U)$ and $f^{-1}(V)$ are disjoint sb*-open sets containing $x_1$ and $x_2$ respectively. Thus $X$ is $sb^* - T_2$.

Proposition 4.2.47: If $f: (X, \tau) \to (Y, \sigma)$ is an sb*-continuous function and $(X, \tau)$ is a $T_{sub^*}$ space, then $f$ is continuous.

Proof: Let $V$ be a closed set in $(Y, \sigma)$. Since $f$ is an sb*-continuous function, $f^{-1}(V)$ is an sb* - closed set in $(X, \tau)$. Since $(X, \tau)$ is a $T_{sub^*}$ space, $f^{-1}(V)$ is a closed set in $(X, \tau)$. Hence $f$ is continuous.
Proposition 4.2.48: Let $f: (X, \tau) \to (Y, \sigma)$ be any topological space and $(Y, \sigma)$ be a $T_{sub}$-space. If $f: (X, \tau) \to (Y, \sigma)$ and $g: (Y, \sigma) \to (Z, \eta)$ are sb*-continuous functions, then their composition $g \circ f: (X, \tau) \to (Z, \eta)$ is an sb*-continuous function.

Proof: Let $V$ be a closed set in $(Z, \eta)$. Since $g: (Y, \sigma) \to (Z, \eta)$ is an sb*-continuous function, $g^{-1}(V)$ is an sb*-closed set in $(Y, \sigma)$. Since $(Y, \sigma)$ is a $T_{sub}$-space, $g^{-1}(V)$ is a closed set in $(Y, \sigma)$. Since $f: (X, \tau) \to (Y, \sigma)$ is an sb*-continuous function, $f^{-1}(g^{-1}(V)) = (g \circ f)^{-1}(V)$ is an sb*-closed set in $(X, \tau)$. Hence $g \circ f: (X, \tau) \to (Z, \eta)$ is an sb*-continuous function.

Theorem 4.2.49: If a map $f: X \to Y$ is sb* - closed and continuous and $A$ is an sb* - closed set of $X$, then $f_A: A \to Y$ is continuous and sb* -closed.

Proof: Let $F$ be a closed set of $A$. Then $F$ is an sb* - closed set of $X$. From Theorem (4.2.42), it follows that $f_A(F) = f(F)$ is an sb*-closed set of $Y$. Hence $f_A$ is sb* - closed and continuous.

Theorem 4.2.50: If $f: X \to Y$ is sb* -closed and $A = f^{-1}(B)$ for some closed set $B$ of $Y$, then $f_A: A \to Y$ is sb* - closed.

Proof: Let $F$ be a closed set in $A$. Then there is a closed set $H$ in $X$ such that $F = A \cap H$. Then $f_A(F) = f(A \cap H) = f(H) \cap B$. Since $f$ is sb* - closed, $f(H)$ is sb* - closed in $Y$. So $f(H) \cap B$ is sb* - closed in $Y$. Since the intersection of a closed and sb* - closed set is sb* - closed, $f_A$ is sb* - closed.
**Theorem 4.2.51:** If a map \( f : (X, \tau) \to (Y, \sigma) \) is sb* - closed and \( A \) is a closed set of \( X \), then \( f_A : (A, \tau_A) \to (Y, \sigma) \) is sb* - closed.

**Proof:** Let \( F \) be a closed set of \( A \). Then \( F = A \cap E \) for some closed set \( E \) of \( X \) and so \( F \) is a closed set of \( (X, \tau) \). Since \( f \) is sb* - closed, \( f(F) \) is an sb* - closed set in \( (Y, \sigma) \). But \( f(F) = f_A(F) \) and therefore \( f_A : (A, \tau_A) \to (Y, \sigma) \) is sb* - closed.

**Theorem 4.2.52:** For any bijection map \( f : (X, \tau) \to (Y, \sigma) \), the following statements are equivalent:

(a) \( f^{-1} : (Y, \sigma) \to (X, \tau) \) is sb* - continuous.

(b) \( f \) is an sb* - open map.

(c) \( f \) is an sb* - closed map.

**Proof:**

\((a) \Rightarrow (b): \) Let \( U \) be an open set of \( (X, \tau) \). By assumption, \( (f^{-1})^{-1}(U) = f(U) \) is sb* - open in \( (Y, \sigma) \) and so \( f \) is sb* - open.

\((b) \Rightarrow (c): \) Let \( F \) be a closed set of \( (X, \tau) \). Then \( F^c \) is an open set of \( (X, \tau) \). By assumption, \( f(F^c) \) is sb* open in \( (Y, \sigma) \). That is, \( f(F^c) = f(F)^c \) is sb* open in \( (Y, \sigma) \) and therefore \( f(F) \) is sb* - closed in \( (Y, \sigma) \). Hence \( f \) is sb* - closed.

\((c) \Rightarrow (a): \) Let \( F \) be a closed set of \( (X, \tau) \). By assumption, \( f(F) \) is sb* - closed in \( (Y, \sigma) \). But \( f(F) = (f^{-1})^{-1}(F) \) and therefore \( f^{-1} \) is sb* - continuous.
4.3 sb*- IRRESOLUTE MAPS

In this section, we introduce sb*- irresolute maps and study the relationships between sb*- open, sb*- continuous and sb*- irresolute maps.

**Definition 4.3.1:** Let $X$ and $Y$ be topological spaces. A map $f :(X, \tau) \to (Y, \sigma)$ is said to be sb*- irresolute if the inverse image of every sb*- closed set in $Y$ is an sb*- closed set in $X$.

**Theorem 4.3.2:** A map $f : X \to Y$ is sb*- irresolute if and only if the inverse image of every sb*- open set in $Y$ is an sb*- open set in $X$.

**Proof:** Assume that $f$ is sb*- irresolute. Let $B$ be any sb*- open set in $Y$. Therefore $B^c$ is an sb*- closed set in $Y$. Since $f$ is sb*- irresolute, $f^{-1}(B^c)$ is an sb*- closed set in $X$. But $f^{-1}(B^c) = X - f^{-1}(B)$ and $f^{-1}(B)$ is an sb*-open set in $X$. Therefore the inverse image of every sb*- open set in $Y$ is an sb*- open set in $X$.

Conversely assume that the inverse image of every sb*- open set in $Y$ is an sb*- open set in $X$. Let $B$ be any sb*- closed set in $Y$. Then $B^c$ is an sb*- open set in $Y$. By assumption, $f^{-1}(B^c)$ is an sb*- open set in $X$. But $f^{-1}(B^c) = X - f^{-1}(B)$ and so $f^{-1}(B)$ is an sb*- closed set in $X$. Therefore $f$ is sb*- irresolute.

**Theorem 4.3.3:** A map $f : X \to Y$ is sb*- irresolute. Then it is sb*- continuous.

**Proof:** Let $f : X \to Y$ be sb*- irresolute. Let $F$ be any closed set in $Y$. Since every closed set is sb*- closed, $F$ is an sb*- closed set in $Y$. Since $f$ is...
sb* - irresolute, $f^{-1}(F)$ is an sb* - closed set in $X$. Hence $f$ is sb* - continuous.

**Remark 4.3.4:** The converse of the above Theorem need not be true as seen from the following example.

**Example 4.3.5:** Let $X = \{p, q, r\}$ and $Y = \{a, b, c\}$ be topological spaces with the topologies  
\[ \tau = \{\emptyset, \{p, r\}, X\} \text{ and } \sigma = \{\emptyset, \{b\}, \{a, c\}, Y\}. \]

Let $f : (X, \tau) \rightarrow (Y, \sigma)$ be defined by $f(p) = a$, $f(q) = c$, $f(r) = b$. Then $f$ is sb* - continuous. However $\{c\}$ is an sb* - open set in $Y$ but $f^{-1}\{c\} = \{q\}$ is not an sb* - open in $X$. Therefore $f$ is not sb* - irresolute.

**Theorem 4.3.6:** Let $X, Y$ and $Z$ be topological spaces. For any sb* - irresolute map $f : (X, \tau) \rightarrow (Y, \sigma)$ and for any sb* - continuous map $g : (Y, \sigma) \rightarrow (Z, \eta)$, the composition $g \circ f : (X, \tau) \rightarrow (Z, \eta)$ is sb* - continuous.

**Proof:** Let $F$ be any closed set in $Z$. Since $g$ is sb* - continuous, $g^{-1}(F)$ is an sb* - closed set in $Y$. Since $f$ is sb* - irresolute, $f^{-1}(g^{-1}(F))$ is an sb* - closed set in $X$. But $f^{-1}(g^{-1}(F)) = (g \circ f)^{-1}(F)$. Hence $g \circ f : (X, \tau) \rightarrow (Z, \eta)$ is sb* - continuous.

**Theorem 4.3.7:** Let $f : (X, \tau) \rightarrow (Y, \sigma)$ and $g : (Y, \sigma) \rightarrow (Z, \eta)$ be two sb* - irresolute maps. Then their composition $g \circ f : (X, \tau) \rightarrow (Z, \eta)$ is sb* - irresolute.

**Proof:** Let $F$ be any closed set in $Z$. Let $(g \circ f)^{-1}(F) = f^{-1}(g^{-1}(F)) = f^{-1}(V)$, where $V = g^{-1}(F)$ is an sb* - closed set in $Y$, as $g$ is sb* - irresolute. Since $f$ is sb* - irresolute, $f^{-1}(V)$ is an sb* - closed set in $X$. Thus $g \circ f : (X, \tau) \rightarrow (Z, \eta)$ is sb* - irresolute.
Remark 4.3.8: The following examples show that the concept of irresolute maps and sb* - irresolute maps are independent.

Example 4.3.9: Let \( X = Y = \{a, b, c\} \) with topologies \( \tau = \{\emptyset, \{b\}, \{a, c\}\} \), \( X \) and \( \sigma = \{\emptyset, \{a, c\}, Y\} \). The map \( f : (X, \tau) \to (Y, \sigma) \) is defined by \( f(a) = b, f(b) = a, f(c) = c \). Here \( f \) is sb* - irresolute but not irresolute.

Example 4.3.10: Let \( X = Y = \{a, b, c\} \) with topologies \( \tau = \{\emptyset, \{a\}, \{b\}, \{a, b\}, X\} \) and \( \sigma = \{\emptyset, \{a, c\}, Y\} \). The map \( f : (X, \tau) \to (Y, \sigma) \) is defined by \( f(a) = a, f(b) = c, f(c) = b \). Here \( f \) is irresolute but not sb* -irresolute.

Theorem 4.3.11: Let \( f : X \to Y \), \( g : Y \to Z \) be two maps such that \( g \circ f : X \to Z \) is an sb* - closed map.

(a) If \( f \) is continuous and onto, then \( g \) is sb* - closed.

(b) If \( g \) is irresolute and one – one, then \( f \) is sb* - closed.

Proof:

(a) Let \( F \) be a closed set in \( Y \). Since \( f^{-1}(F) \) is closed set in \( X \), \( (g \circ f)(f^{-1}(F)) \) is an sb* - closed set in \( Z \). Therefore \( g(F) \) is an sb* - closed set in \( Z \). Thus \( g \) is sb* - closed.

(b) Let \( F \) be a closed set in \( X \). Then \( (g \circ f)(F) = g(f(F)) \) is an sb* - closed set in \( Z \) and \( g^{-1}(g \circ f)(F) = g^{-1}(g(f(F))) = f(F) \) is an sb* - closed set in \( Y \). Since \( g \) is one - one and \( f \) (\( F \)) is an sb*-closed set in \( Y \), \( f \) is sb* - closed.
Theorem 4.3.12: Let \( f : (X, \tau) \rightarrow (Y, \sigma) \), \( g : (Y, \sigma) \rightarrow (Z, \eta) \) be two maps. Then their composition \( g \circ f : (X, \tau) \rightarrow (Z, \eta) \) is sb* - irresolute if \( f \) is sb* - irresolute and \( g \) is \( \alpha \) - irresolute.

Proof: Let \( V \) be an \( \alpha \) - closed set in \( Z \). \( (g \circ f)^{-1}(V) = f^{-1}(g^{-1}(V)) = f^{-1}(U) \) where \( U = g^{-1}(V) \). As \( g \) is \( \alpha \) - irresolute and \( U \) is \( \alpha \)-closed set, hence sb* -closed set in \( Y \). Since \( f \) is sb*-irresolute, \( f^{-1}(U) \) is an sb*- closed set in \( X \). Thus \( g \circ f \) is sb* - irresolute.

Theorem 4.3.13: Let \( f : X \rightarrow Y \) be an sb* - irresolute, injective map. If \( Y \) is sb* -\( T_1 \), then \( X \) is sb* -\( T_1 \).

Proof: Assume that \( Y \) is sb* -\( T_1 \). Let \( x, y \in Y \) be such that \( x \neq y \). Then there exists a pair of sb*-open sets \( U, V \) in \( Y \) such that \( f(x) \in U, f(y) \in V \) and \( f(x) \notin V, f(y) \notin U \). Then \( x \in f^{-1}(U), y \notin f^{-1}(U) \) and \( y \in f^{-1}(V), x \notin f^{-1}(V) \). Since \( f \) is sb*-irresolute, \( X \) is sb* -\( T_1 \).

Theorem 4.3.14: If \( f : (X, \tau_1) \rightarrow (Y, \tau_2) \) is bijective, sb*-irresolute map and \((Y, \tau_2) \) is sb* -\( T_2 \), then \((X, \tau_1) \) is sb* -\( T_2 \).

Proof: Suppose \( f : (X, \tau_1) \rightarrow (Y, \tau_2) \) is bijective. And \( f \) is sb*-irresolute, and \((Y, \tau_2) \) is sb* -\( T_2 \). Let \( x_1, x_2 \in X \) with \( x_1 \neq x_2 \). Since \( f \) is bijective, \( y_1 = f(x_1) \neq f(x_2) = y_2 \) for some \( y_1, y_2 \in Y \). Since \((Y, \tau_2) \) is sb* -\( T_2 \), there exist disjoint sb*-open sets \( G \) and \( H \) such that \( y_1 \in G \) and \( y_2 \in H \). Again since \( f \) is bijective, \( x_1 = f^{-1}(y_1) \in f^{-1}(G) \) and \( x_2 = f^{-1}(y_2) \in f^{-1}(H) \). Since \( f \) is sb*- irresolute, \( f^{-1}(G) \) and \( f^{-1}(H) \) are sb*-open sets in \((X, \tau_1) \). Also \( f \) is bijective, \( G \cap H = \phi \) implies that \( f^{-1}(G) \cap f^{-1}(H) = f^{-1}(G \cap H) = f^{-1}(\phi) = \phi \). It follows that \((X, \tau_1)\) is sb* -\( T_2 \).
Theorem 4.3.15: Let \( f:(X,\tau)\to(Y,\sigma) \) a mapping and \((X,\tau)\) be a \( T_{sb^*} \)-space. Then \( f \) is continuous if one of the following conditions is satisfied:

(a) \( f \) is \( sb^* \)-continuous.

(b) \( f \) is \( sb^* \)-irresolute.

Proof: Obvious.

4.4 \( sb^* \)-HOMEOMORPHISMS

We define a class of \( sb^* \)-homeomorphisms and \( sb^*r \)-homeomorphisms which are generalization of homeomorphisms and we investigate some properties of these homeomorphisms.

Definition 4.4.1: A bijection \( f:(X,\tau)\to(Y,\sigma) \) is called an \( sb^* \)-homeomorphism if \( f \) is both \( sb^* \)-continuous and \( sb^* \)-open.

Theorem 4.4.2: Every homeomorphism is an \( sb^* \)-homeomorphism.

Proof: Let \( f:X\to Y \) be a homeomorphism. Then \( f \) is continuous and open. Since every continuous function is \( sb^* \)-continuous and every open map is \( sb^* \)-open, \( f \) is \( sb^* \)-continuous and \( sb^* \)-open. Therefore \( f \) is an \( sb^* \)-homeomorphism.

Remark 4.4.3: The converse of the above Theorem need not be true as seen from the following example.

Example 4.4.4: Consider \( X=\{p,q,r\} \) and \( Y=\{a,b,c\} \) with \( \tau=\emptyset, \{p,r\}, X \) and \( \sigma=\emptyset, \{b\}, \{a,c\}, Y \) and \( f \) defined by \( f(p)=a, f(q)=c, f(r)=b \). Then \( f \) is an
sb* - homeomorphism but not a homeomorphism, since the inverse image of 
\{b\} in Y is not open in X.

Theorem 4.4.5: Let \( f: (X, \tau) \to (Y, \sigma) \) be a bijective and sb* - continuous map. Then the following statements are equivalent:

(a) \( f \) is an sb* - open map.

(b) \( f \) is an sb* - homeomorphism.

(c) \( f \) is an sb* - closed map.

Proof:

\((a) \Rightarrow (b)\): By assumption, \( f \) is bijective, sb* - continuous and sb* -open. Then by definition, \( f \) is an sb* - homeomorphism.

\((b) \Rightarrow (c)\): By assumption, \( f \) is sb* - open and bijective. By Theorem (4.4.5), \( f \) is an sb* - closed map.

\((c) \Rightarrow (a)\): Obvious.

Remark 4.4.6: The following example shows that the composition of two sb* - homeomorphisms is not always an sb* - homeomorphism.

Example 4.4.7: Let \( X = Y = Z = \{a, b, c\} \) with topologies \( \tau = \{\phi, \{a, c\}, X\} \), \( \sigma = \{\phi, \{b\}, \{a, c\}, Y\} \) and \( \eta = \{\phi, \{a\}, \{b\}, \{a, b\}, \{b, c\}, Z\} \). Let two maps \( f: X \to Y \) and \( g: Y \to Z \) be defined by \( f(a) = a, f(b) = c, f(c) = b \) and \( g(a) = b, g(b) = a, g(c) = c \) respectively. Then \( f \) and \( g \) are sb* - homeomorphisms, but their composition \( g \circ f: X \to Z \) is not an sb* - homeomorphism since \( (g \circ f)(a, c) = g(f(a,c)) = g(a, b) = \{a, b\} \) is not an sb* - open set in \( Z \).
Definition 4.4.8: A bijection $f : (X, \tau) \rightarrow (Y, \sigma)$ is said to be sb*-homeomorphism if both $f$ and $f^{-1}$ are sb* - irresolute.

Theorem 4.4.9: Every sb*-homeomorphism is an sb* - homeomorphism.

Proof: Let $f : X \rightarrow Y$ be an sb* r – homeomorphism. Then $f$ is bijective, sb* - irresolute, $f^{-1}$ is sb* - irresolute. Since every sb* - irresolute function is sb* - continuous, $f$ and $f^{-1}$ are sb* - continuous and so $f$ is an sb* - homeomorphism.

Remark 4.4.10: The following example shows that the converse of the above theorem need not be true.

Example 4.4.11: Let $X = Y = \{a, b, c\}$ with topologies $
\tau = \{\emptyset, \{a\}, \{b\}, \{a, b\}, \{b, c\}, X\}$ and $\sigma = \{\emptyset, \{b\}, \{a, c\}, Y\}$. Let $f : (X, \tau) \rightarrow (Y, \sigma)$ be defined by $f(a) = b$, $f(b) = c$, $f(c) = a$. Then $f$ is an sb* - homeomorphism but not an sb*-r - homeomorphism. Since $\{a, b\}$ is sb* - open in $Y$ but $f^{-1}(a, b) = \{a, c\}$ is not sb* - open in $X$. So $f$ is not sb* - irresolute. Thus $f$ is not an sb* r – homeomorphism.

Theorem 4.4.12: Every homeomorphism is an sb*-r - homeomorphism.

Proof: Let $f : X \rightarrow Y$ be a homeomorphism. Then $f$ is both continuous and open. Since every continuous function is sb* - irresolute, $f$ and $f^{-1}$ are also sb* - irresolute and so $f$ is an sb* r - homeomorphism.

Remark 4.4.13: The following example shows that the converse of the above theorem need not be true.
Example 4.4.14: Let \( X = \{a,b,c\} \) with topologies \( \tau = \{\emptyset, \{c\}, \{a,c\}, \{b,c\}, X\} \) and \( \sigma = \{\emptyset, \{b\}, Y\} \). Let \( f:X \rightarrow Y \) be defined by \( f(a)=a, f(b)=c, f(c)=b \). Then \( f \) is an \( \text{sb}^* \) - homeomorphism but not a homeomorphism. Since \( \{a,c\} \) is open in \( X \) but \( f(\{a,c\})=\{a,b\} \) is not open in \( Y \) and so \( f \) is not a homeomorphism.

Definition 4.4.15: The family of all \( \text{sb}^* \)-homeomorphisms of a topological space \( (X,\tau) \) onto itself is denoted by \( \text{sb}^*-h(X,\tau) \).

Proposition 4.4.16: Let \( f:(X,\tau) \rightarrow (Y,\tau_2) \) and \( g:(Y,\tau_2) \rightarrow (Z,\tau_3) \) be \( \text{sb}^* \)-homeomorphisms. Then their composition \( g \circ f:(X,\tau) \rightarrow (Z,\tau_3) \) is also an \( \text{sb}^* \)-homeomorphism.

Proof: Let \( U \) be an \( \text{sb}^* \)-open set in \( (Z,\tau_3) \). Since \( g \) is \( \text{sb}^* \)-irresolute, \( g^{-1}(U) \) is an \( \text{sb}^* \)-open in \( (Y,\tau_2) \). Since \( f \) is \( \text{sb}^* \) - irresolute, \( f^{-1}(g^{-1}(U)) = (g \circ f)^{-1}(U) \) is an \( \text{sb}^* \) - open set in \( (X,\tau) \). Therefore \( g \circ f \) is \( \text{sb}^* \)-irresolute.

Also, for an \( \text{sb}^* \)-open set \( G \) in \( (X,\tau) \), we have \( (g \circ f)(G) = g(f(G)) = g(W) \), where \( W = f(G) \). By hypothesis, \( f(G) \) is \( \text{sb}^* \)-open in \( (Y,\tau_2) \) and so again by hypothesis, \( g(f(G)) \) is an \( \text{sb}^* \)-open set in \( (Z,\tau_3) \). That is, \( (g \circ f)(G) \) is an \( \text{sb}^* \)-open set in \( (Z,\tau_3) \) and therefore \( (g \circ f)^{-1} \) is \( \text{sb}^* \)-irresolute. Also \( g \circ f \) is a bijection. Hence \( g \circ f \) is an \( \text{sb}^* \) - homeomorphism.

Theorem 4.4.17: Let \( f:(X,\tau_1) \rightarrow (Y,\tau_2) \) be an \( \text{sb}^* \)-homeomorphism. Then \( f \) induces an isomorphism from the group \( \text{sb}^*-h(X,\tau_1) \) onto the group \( \text{sb}^*-h(Y,\tau_2) \).

Proof: Using the map \( f \), we define a map \( \psi_f : \text{sb}^*-h(X,\tau_1) \rightarrow \text{sb}^*-h(Y,\tau_2) \) by \( \psi_f(h) = f \circ h \circ f^{-1} \) for every \( h \in \text{sb}^*-h(X,\tau_1) \). Then \( \psi_f \) is a bijection. Further, for
all \( h_1, h_2 \in sb^* \rightarrow h(X, \tau_1) \), \( \psi_f(h_1 \circ h_2) = f \circ (h_1 \circ h_2) \circ f^{-1} \circ (f \circ h_2 \circ f^{-1}) \)

= \( \psi_f(h_1) \circ \psi_f(h_2) \).

Therefore, \( \psi_f \) is a homeomorphism and so it is an isomorphism induced by \( f \).

**Theorem 4.4.18:** The set \( sb^* \rightarrow h(X, \tau_1) \) is a group under the composition of maps.

**Proof:** Define a binary operation \( \ast : sb^* \rightarrow h(X, \tau_1) \times sb^* \rightarrow h(X, \tau_1) \rightarrow sb^* \rightarrow h(X, \tau_1) \) by \( f \ast g = g \circ f \) for all \( f, g \in sb^* \rightarrow h(X, \tau_1) \) and \( \circ \) is the usual operation of composition of maps. \( g \circ f \in sb^* \rightarrow h(X, \tau_1) \). We know that the composition of maps is associative and the identity map \( I : h(X, \tau_1) \rightarrow h(X, \tau_1) \) belonging to \( sb^* \rightarrow h(X, \tau_1) \) serves as the identity element. If \( f \in sb^* \rightarrow h(X, \tau_1) \), then \( f^{-1} \in sb^* \rightarrow h(X, \tau_1) \) such that \( f \circ f^{-1} = f^{-1} \circ f = I \) and so inverse exists for each element of \( sb^* \rightarrow h(X, \tau_1) \). Therefore \( (sb^* \rightarrow h(X, \tau_1), \circ) \) is a group under the operation of composition of maps.